ON CLIQUE AND CHROMATIC COEFFICIENTS

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Abstract

Explicit formulae are derived for the first five coefficients of the clique polynomial of a graph. From these results, explicit formulae are derived for the first five coefficients of the chromatic polynomial in the complete graph basis.

1. The Basic Ideas

The graphs considered here are all fine, and have neither loops nor multiple edges. We refer to Harary [8] for the basic definitions in Graph Theory. We denote by K_p, the complete graph with p nodes. We also call K_p, a **clique**, when it is a proper subgraph of a graph. We call K_1 a **trivial** clique and K_n a **proper** clique, when $n \geq 3$. We will sometimes refer to K_n as an **n-clique**.

Definitions

Let G be a graph. A **clique cover** of G is a spanning subgraph of G, in which every component is a clique.

Since all the covers referred to, will be clique covers, we will use the word "cover" to mean "clique cover", unless otherwise specified. Also, all indeterminates mentioned in this paper will be over the field of complex numbers.

Let F be the family of cliques. With each member α of F, let us associate an indeterminate $\mathbf{w}_\alpha$, called the **weight of** α. Let C be a cover of G, Then the **weight of C** is

$$w(C) = \prod_{a} w_a$$

where the product is taken over all the elements α in C. The **clique polynomial** of G is

$$K(G;\mathbf{w}) = \sum_{C} w(C),$$

where $\mathbf{w}$ is a vector of the indeterminates w_α; and the summation is taken over all the covers in G. This polynomial was introduced in [1]. Some of its properties are also given in [2], [3] and [4].

If we give each element α of F, with r nodes, the weight w_r, then the resulting clique polynomial is called the **general clique polynomial** of G. If we give each element of a cover the (same) weight w, then the resulting polynomial in w, is called the **simple clique polynomial** of G -and is denoted by K(G;w). Therefore K(G;w) is obtained from K(G;**w**), by putting $w_r = w$, for all r. If G has p nodes, we will write

$$K(G;w) = \sum_{k=0}^{p-1} a_k(G) w^{p-k}$$

where $a_k(G)$ is the number of covers of G with cardinality p-k.

Throughout this paper, we will denote the node and edge sets of a graph G, by **V(G)** and **E(G)** respectively. Also, we will assume that $|V(G)| = p$ and that $|E(G)| = q$ unless otherwise specified. If H is a subgraph of G, the notation **G-H** will used for the graph obtained from G, by removing all the nodes of H.

2. The Coefficients of the Simple Clique Polynomial

It is not difficult to see that for any graph G with p nodes, there is only one possible cover with p components; and that is, the empty graph with p nodes (We call this cover, trivial). Therefore $a_0(G) = 1$.

It is also easy to see that the only possible covers with p-1 components are those covers that consist of a component edge; together with p-2 component nodes. Therefore we have

$a_1(G) = q$.

The following lemma is taken from [5] (Lemma 1). It will be useful for deriving the results in this section.

Lemma 1

Let G be a graph with p nodes and p-n components. Then G consists of p-c component nodes, together with c-n non-trivial components, where $n \le c \le 2n$. If $0 < n < p-1$, then $n < c \le 2n$.

We now consider the case , when n = 2. From the lemma the cover C will have p-c component nodes and c-2 non-trivial components, where $2 < c \le 4$.

c = 3

The only admissible graph with 3 nodes and 1 component is the triangle K_3.

c = 4

The only admissible graph with 4 nodes and 2 components is a pair of independent edges i.e. $K_2 \cup K_2$.

Hence, the graphs which contribute to the coefficient of w^{p-2} are the triangle and $K_2 \cup K_2$. Let us denote the number of triangles in G, by $N_{K_3}(G)$. The number of pairs of independent edges in G can be counted by first choosing any pair of edges, and then omitting any chain of length 2 -denoted by P_3. We therefore get

$$a_2(G) = N_{K_3}(G) + \binom{q}{2} - N_{P_3}(G). \qquad \ldots (1)$$

We can obtain a chain with 3 nodes in G, by choosing any pair of edges at any node in G. Therefore

$$N_{P_3}(G) = \sum_{i=1}^{p} \binom{d_i}{2},$$

where d_i is the valency of node i in G.

Hence we obtain the following result.

Theorem 1

$$a_2(G) = \binom{q}{2} + N_{K_3}(G) - \sum_{i=1}^{p} \binom{d_i}{2}.$$

The following corollary shows that the third coefficient a_2 characterizes a star graph (a tree with p nodes and containing p-1 nodes of valency 1).

Corollary 1.1

A graph G is a star if and only if $a_2(G) = 0$.

Proof

Suppose that G is a star. Then G has p-1 edges. So q = p-1. Also G will contain one node of valency p-1 and p-1 nodes of valency 1. Therefore we will have

$$\sum_{i=1}^{p} \binom{d_i}{2} = \binom{p-1}{2}.$$

Clearly, G has no triangles. Therefore $N_{K_3}(G) = 0$. From the theorem, we get $a_2(G) = 0$.

Conversely, suppose that $a_2(G) = 0$. Recall that $a_2(G)$ is the number of covers consisting of p-3 component nodes and a triangle plus the number of covers consisting of p-4 component nodes and a pair of independent edges. This means that G has no triangles. Also, G does not have a pair of independent edges. Then, all the edges in G, have a node in common. Therefore G is a star.

Hence the result follows. □

We now consider the case, in which n = 3. From Lemma 1, the cover C will have p-c component nodes and c-3 non-trivial components, where $3 < c \leq 6$.

c = 4

The only admissible graph with 4 nodes and one component is the graph K_4 .

The number of such subgraphs is $N_{K_4}(G)$.

c = 5

The only admissible graph with 5 nodes and two components is the triangle K_3 together with a component edge .i.e $K_3 \cup K_2$.

Let us denote by T_{a_i,b_i,c_i} a triangle joining nodes a_i, b_i, and c_i. Then the number of graphs consisting of T_{a_i,b_i,c_i} together with a component edge is $|E(G - T_{a_i,b_i,c_i})|$. Hence, the number of covers of this type is

$$\sigma = \sum_{i=1} |E(G - T_{a_i,b_i,c_i})|. \qquad \ldots (2)$$

where the summation is taken over all triangles T_{a_i,b_i,c_i} in G.

c = 6

The only admissible graph with 6 nodes and three components is the graph consisting of three component edges i.e.$K_2 \cup K_2 \cup K_2$.

The number of such covers is taken from [6] (Theorem 1). It is

$$\tau = \binom{q}{3} - (q-2)\sum_{i=1}^{p}\binom{d_i}{2} + 2\sum_{i=1}^{p}\binom{d_i}{3} + \sum_{ij}(d_i-1)(d_j-1) - N_{K_3}(G). \quad \ldots(3)$$

Therefore we obtain the following result.

Theorem 2

$$a_3(G) = N_{K_4}(G) + \sigma + \tau.$$

We now consider the case, in which n = 4. From Lemma 1, the cover C will have p-c component nodes and c-4 non-trivial components, where $4 < c \le 8$.

$c = 5$

The only admissible graph with 5 nodes and one component is the graph K_5.

The number of such subgraphs is $N_{K_5}(G)$.

$c = 6$

The only admissible graphs with 6 nodes and two components are (i) a pair of triangles i.e. $K_3 \cup K_3$ and (ii) $K_4 \cup K_2$.

Let G_k denote the graph obtained from G by removing the k triangles T_{a_i,b_i,c_i}, where

$i = 1, 2, \ldots, k$. Then the number of graphs of type $K_3 \cup K_3$ is

$$\alpha = \sum_{k=1} N_{K_3}(G). \quad \ldots(4)$$

The number of graphs of type $K_4 \cup K_2$ is

$$\beta = \sum_{i=1} \left|E(G - Q_{a_i,b_i,c_i,d_i})\right|, \quad \ldots(5)$$

where Q_{a_i,b_i,c_i,d_i} is a 4-clique with nodes a_i, b_i, c_i and d_i; and the summation is taken over all such 4-cliques in G.

$c = 7$

The only admissible graph with 7 nodes and three components is a triangle, together with two component edges i.e. $K_3 \cup K_2 \cup K_2$.

Let T_{a_i,b_i,c_i} be a triangle in G; and let G_i be the graph $G - T_{a_i,b_i,c_i}$. Let

$$q_i = \left|E(G - T_{a_i,b_i,c_i})\right|$$

i.e. the number of edges in $G - T_{a_i,b_i,c_i}$. Also, let $d_{i,j}$ be the valency of node j in G_i.

Then, the number of pairs of independent edges in G_i is

$$\varepsilon_i = \binom{q_i}{2} - \sum_{j=1}^{p-3}\binom{d_{i,j}}{2}.$$

Hence the number of covers of the type $K_3 \cup K_2 \cup K_2$ is

$$\gamma = \sum_i \varepsilon_i, \quad \ldots(6)$$

where the summation is taken over all graphs of type G_i .

c = 8

The only admissible graph with 8 nodes and four components is $K_2 \cup K_2 \cup K_2 \cup K_2$.

We denote the number of such subgraphs by δ.

Hence we obtain the following result.

Theorem 3

$$a_4(G) = N_{K_5}(G) + \alpha + \beta + \gamma + \delta.$$

Clearly, the analysis can be continued in order to obtain expressions for higher coefficients of K(G;w). The results will become more and more complicated. It can be observed that the graphs associated with $a_k(G)$ are defined by the various partitions of c with c-n elements, where $n+1 \le c \le 2n$. Therefore the number of different families of graphs that must be counted is the number of such partitions. Let N(i,j) be the number of partitions of the integer i with j parts, in which each part is greater than 1. Each such partition defines a unique family of graphs. Therefore, the number of different families of graphs (or covers) is equinumerous with the number of different partitions. The number of different families is therefore

$$\sum_{c=n+1}^{2n} N(c, c-n).$$

It is not difficult to see that, for any fixed number of nodes p, the complete graph with p nodes will contain the greatest number of the different families of graphs. Therefore

$$a_k(G) \le a_k(K_p).$$

for any graph G with p nodes. The following result is taken from [1].

Lemma 2

$$K(K_p;w) = \sum_{k=0}^{p-1} S(p, p-k) w^{p-k},$$

where S(n,k) is the Stirling number of the second kind.

Hence, we have the following result, which gives a useful upper bound on the coefficients of K(G;w).

Theorem 4

For all graphs G with p nodes,

$$a_k(G) \le S(p, p-k).$$

The following result is added for completeness.

Theorem 5

G is a complete graph if and only $a_{p-1}(G) = 1$.

Suppose that G has no triangles (i.e. G "triangle free"), then, the only cliques in G, will be nodes and edges. Therefore the expressions for the coefficients easily reduce to the results given in the following theorem.

Theorem 6

Let G be a triangle-free graph. Then

(i) $a_0(G) = 1$,

(ii) $a_1(G) = q$,

(iii) $a_2(G) = \binom{q}{2} - \sum_{i=1}^{p}\binom{d_i}{2}$,

(iv) $a_3(G) = \tau$,

and (v) $a_4(G) = \delta$.

3. Examples

We now illustrate the results given in the above theorems.

Example 1

Let G be the following graph.

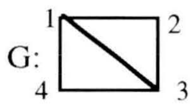


Figure 1

Then $a_0(G) = 1$ and $a_1(G) = 5$. We will use Theorem 1, to find $a_2(G)$. In this case,

$N_{K_3}(G) = 2$. $\binom{q}{2} = \binom{5}{2} = 10$ and

$$\sum_{i=1}^{p}\binom{d_i}{2} = \binom{3}{2} + \binom{3}{2} + \binom{2}{2} + \binom{2}{2} = 8.$$

$\Rightarrow$ $a_2(G) = 10 + 2 - 8 = 4$.

Since G has only 4 nodes, $a_k(G) = 0$, for $k \geq 3$. Hence we get

$$K(G;w) = w^4 + 5w^3 + 4w^2.$$

Example 2

Let G be the following graph.

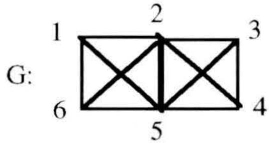


Figure 2

Clearly, $a_0(G) = 1$ and $a_1(G) = 11$. G has 8 triangles. From Theorem 1, we get

$$a_2(G) = \binom{q}{2} + N_{K_3}(G) - \sum_{i=1}^{p}\binom{d_i}{2} = \binom{11}{2} + 8 - \left[4\binom{3}{2} + 2\binom{5}{2}\right]$$

$$= 55 + 8 - 32 = 31.$$

From the graph, it can be seen that $N_{K_4}(G) = 2$. From Equation (2),

$$\sigma = \sum_{i=1} \left|E(G - T_{a_i,b_i,c_i})\right| = 4.3 + 4.1 = 16.$$

It can be easily verified that G has 5 sets of three independent edges. Therefore $\tau = 5$.

From Theorem 2, we have

$$a_3(G) = N_{K_4}(G) + \sigma + \tau = 2 + 16 + 5 = 23.$$

G has no 5-clique. From Equation (4),

$$\alpha = \sum_{k=1} N_{K_3}(G) = 2.$$

From Equation (5) (using the complete graphs Q_{1256} and Q_{2345}, we get

$$\beta = \sum_{i=1} \left|E(G - Q_{a_i,b_i,c_i,d_i})\right| = 2.$$

Since G has only 6 nodes, $\gamma = \delta = 0$. From Theorem 3,

$$a_4(G) = N_{K_5}(G) + \alpha + \beta + \gamma + \delta = 0 + 2 + 2 + 0 + 0 = 4.$$

Hence we get

$$K(G;w) = w^6 + 11w^5 + 31w^4 + 23w^3 + 4w^2.$$

4. The Coefficients of the Chromatic Polynomial

The following definitions are well known; and have been added for completeness.

Definitions

Let G be a graph. A **proper colouring** of G is an assignment of colours to the nodes of G, in such a way that adjacent nodes are coloured differently. A **λ-colouring** of G is a proper colouring of G with λ colours. This is equivalent to a partitioning of the nodes of G into λ colour classes, such that nodes in the same class are non-adjacent. When λ is an indeterminate, the number of λ-colourings of G, is a polynomial in G, called the **chromatic polynomial** of G. This polynomial will be denoted by **$P(G;\lambda)$**.

We refer to Read [10], for the basic properties of chromatic polynomials. Formulae for some of the coefficients of $P(G;\lambda)$ have been given in [5].

Let us denote the product $\lambda(\lambda-1)(\lambda-2)...(\lambda-n+1)$ by $(\lambda)_n$ $(n \geq 1)$ (referred to as, " λ falling factorial n"). We define $(\lambda)_0$ as 1. Then clearly, the chromatic polynomial of K_p is $(\lambda)_p$. We can write the chromatic polynomial of a graph G with p nodes in **the complete graph basis (or "falling factorial form")**, as

$$P(G;\lambda) = \sum_{k=0}^{p} b_k (\lambda)_{p-k}.$$

We can also write the chromatic polynomial of G, in **the null basis,** as

$$P(G;\lambda) = \sum_{k=0}^{p} c_k \lambda^{p-k}. \qquad ...(7)$$

The following result was established in [3]. It gives the connection between clique an chromatic polynomials.

Lemma 3

Let G be a graph. Then the chromatic polynomial of $\overline{G}$ is the polynomial obtained from the simple clique polynomial of G, by replacing w^k with $(\lambda)_k$.

The result of the theorem can be written as

$$P(\overline{G};\lambda) = K(G;w'),$$

where w' denotes the transformation, in which w^k is replaced by $(\lambda)_k$,

An immediate result can be obtained by replacing G with $\overline{G}$.

Lemma 4

$$P(G;\lambda) = K(\overline{G};w'),$$

where w' denotes the transformation, in which w^k is replaced by $(\lambda)_k$,

Notice that the transformation described in the lemmas suggest that the product

$w^r . w^s = w^{r+s}$ in K(G;w) is equivalent to the product $(\lambda)_r . (\lambda)_s = (\lambda)_{r+s}$ in K(G;w'). This is the well-known Zykov Product (see Korfhage [9]). This product will be especially useful, for transforming polynomials of disconnected graphs.

It is clear from the lemmas, that the coefficients of K(G;w) and those of $P(\overline{G};\lambda)$, in the complete graph basis coincide. Therefore we can immediately obtain formulae for the first five coefficients of the chromatic polynomial, in falling factorial form. The following theorem give the results, which parallel the results for the coefficients c_k in Equation (7), given in [5]. As far as we know, these result are new.

Theorem 7

Let G be a graph with p nodes and q edges , and with chromatic polynomial

$$P(G;\lambda) = \sum_{k=0}^{p} b_k (\lambda)_{p-k} .$$

Then

(i) $b_0 = 1$

(ii) $b_1 = q$

(iii) $b_2 = \binom{q}{2} + N_{K_3}(\overline{G}) - \sum_{i=1}^{p} \binom{d_i}{2}$

(iv) $b_3 = N_{K_4}(\overline{G}) + \sigma + \tau$,

and (v) $b_4 = N_{K_5}(\overline{G}) + \alpha + \beta + \gamma + \delta$,

where $\sigma, \tau, \alpha, \beta, \gamma$ and δ are the sums defined above, but with the graph replaced by its complement.

Another immediately corollary of Theorem 1 is the following.

Corollary 1.2

A graph G is a star if and only if $b_2 = 0$.

The following corollary of Theorem 6 is immediate.

Corollary 6.1

Let G be a graph with p nodes and q edges , and with chromatic polynomial

$$P(G;\lambda) = \sum_{k=0}^{p} b_k(\lambda)_{p-k}.$$

If $\overline{G}$ is triangle-free, then

(i) $b_0 = 1$,

(ii) $b_1 = q$,

(iii) $b_2 = \binom{q}{2} - \sum_{i=1}^{p}\binom{d_i}{2}$,

(iv) $b_3 = \tau$,

and (v) $b_4 = \delta$;

where τ and δ are the sums defined above, <u>but with the graph replaced by its complement.</u>

Notice that when G is triangle-free, all the clique covers are matchings. Therefore, the clique polynomial of G, coincides with its matching polynomial . This corollary is therefore essentially Theorem 2 of [7], when $a_k = b_k$.

An Illustration

We will find the chromatic polynomial of the graph G, shown in Figure 2 above.

$$P(G;\lambda) = \sum_{k=0}^{p} b_k(\lambda)_{p-k}$$

In this case $\overline{G} \cong H$., where H is the graph (with three components) shown below.

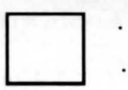
H: 

Figure 3

Now, $b_0 = 1$ and $b_1 = 4$. Clearly H is triangle-free. Therefore Corollary 6.1 can be used to find its chromatic polynomial. H has two pairs of independent edges. Therefore

$$b_2 = \binom{4}{2} - 4\binom{2}{2} = 2.$$

Hence from Theorem 7, we get

$$P(G;\lambda) = (\lambda)_6 + 4(\lambda)_5 + 2(\lambda)_4.$$

5. Summary

We have derived formulae for the first five coefficients of the clique polynomial; and in doing so, have simultaneously derived formulae for the first five chromatic coefficients, when the chromatic polynomial is expressed in the complete graph basis. The use of the formulae, necessitates the counting of various subgraphs of the graph. Wherever possible, we have replaced the numbers of certain subgraphs, by various sums, which are intended to make the counting more manageable. For example, if a graph has many edges, .it is more efficient to counting of the number of paths of length 2, by the summation given, than to count them directly. The summation α, for the counting of a pair of disjoint triangles seems quiet cumbersome; but it is efficient to use when the graph has many triangles. Also it is easy to implement on a computer.

When dealing with small graphs ($p \leq 8$), it might be more efficient to count subgraphs directly. For example, if a graph does not contain many triangles, it might be easier to count the number of disjoint pairs of triangles, by simply looking at the graph. The formulae are not meant to be used slavishly. Basically, the coefficients are the numbers of various types of clique covers; and this understanding should determine the form in which the theorems are applied.

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