LOCALLY WELL-COVERED GRAPHS

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ABSTRACT. A graph G is called *locally well-covered* if there exists a vertex $v \in G$ such that each maximal stable set which contains v is a maximum stable set.
We prove that every graph G which is not locally well-covered contains at least one of graphs $G_1, G_2, \ldots, G_6$ (Figure 1) as an induced subgraph. Hence the maximal hereditary subclass $\mathcal{HLOCWELL}$ of locally well-covered graphs is characterized by the set $\{G_1, G_2, \ldots, G_6\}$ of minimal forbidden induced subgraphs. The class $\mathcal{HLOCWELL}$ is polynomial-time recognizible and there is a polynomial-time algorithm for finding a maximum stable set, which is valid for every graph in the class $\mathcal{HLOCWELL}$.
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1. Introduction

A set $S \subseteq V(G)$ in a graph G is *stable* or *independent* if no vertices of S are adjacent. A *maximal* stable set is an inclusion-wise maximal set that is stable. A stable set S is *maximum* if $|S| \geq |S'|$ for each stable set S' of the graph. Plummer [3] defined a graph G to be *well-covered* if every maximal stable set in G is a maximum stable set. The class $\mathcal{WELL}$ of all well-covered graphs is widely studied, see, for example, Hartnell [2], Plummer [4], Ravindra [5], Staples [7], and Zverovich [8]. Chvátal and Slater [1] and Sankaranarayana and Stewart [6] independently proved that recognizing well-covered graphs is an co-NP-complete problem.

Definition 1. *We define a graph G as* locally well-covered *if there exists a vertex $v \in G$ such that every maximal stable set which contains v is a maximum stable set. We denote by $\mathcal{LOCWELL}$ the class of all locally well-covered graphs.*

Clearly, $\mathcal{WELL} \subseteq \mathcal{LOCWELL}$.

Proposition 1. *There exists a polynomial-time algorithm for finding a maximum stable set, which is valid for every graph in the class $\mathcal{HLOCWELL}$.*

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Proof. Let $G \in \mathcal{LOCWELL}$. For every vertex $v \in V(G)$, we construct a maximal stable set I_v which contains v and choose a maximum set among them. □

We show that the maximal hereditary subclass $\mathcal{HLOCWELL}$ of locally well-covered graphs is characterized by a finite set of minimal forbidden induced subgraphs. Therefore the class $\mathcal{HLOCWELL}$ is polynomial-time recognizible, and there is a polynomial-time algorithm for finding a maximum stable set within $\mathcal{HLOCWELL}$.

2. Minimal non-locally well-covered graphs

A *non-locally well-covered graph* is a graph $G \notin \mathcal{LOCWELL}$.

Theorem 1. *Every non-locally well-covered graph G contains at least one of the graphs $G_1, G_2, \ldots, G_6$, see Figure 1, as an induced subgraph.*

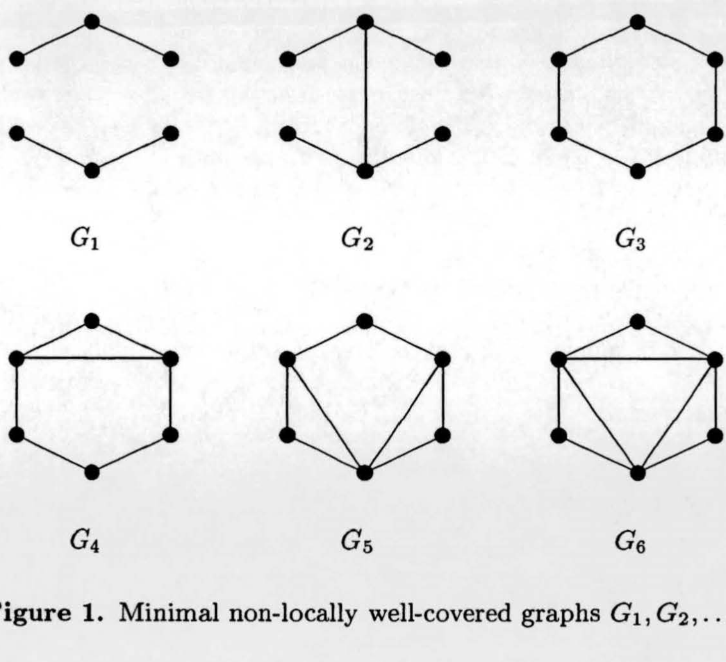

Figure 1. Minimal non-locally well-covered graphs $G_1, G_2, \ldots, G_6$.

Proof. Suppose that there exists a graph $G \notin \mathcal{LOCWELL}$ without induced subgraphs $G_1, G_2, \ldots, G_6$. Let $I = \{u_1, u_2, \ldots, u_k\}$ be a maximum stable set in G. Since $G \notin \mathcal{LOCWELL}$, there exists a maximal stable set J_i that contains a vertex $u_i \in I$, $i = 1, 2, \ldots, k$, and which is not a maximum stable set. Clearly, $|J_i \setminus I| < |I \setminus J_i|$.

Since J_i is a maximal stable set, for every vertex $u \in I \setminus J_i$ there exists a vertex $v \in J_i$ that is adjacent to u. Since $u \in I$ and I is a stable set, $v \notin J_i$, i.e., $v \in J \setminus J_i$. It follows from $|J_i \setminus I| < |I \setminus J_i|$, that there exists a vertex $v_i \in J_i \setminus I$ that is adjacent to at least two vertices of $I \setminus J_i$.

We fix such a vertex v_i for every J_i, $i = 1, 2, \ldots, k$, and denote $V = \{v_1, v_2, \ldots, v_k\}$. Note that the vertices $v_1, v_2, \ldots, v_k$ are not necessarily pairwise distinct.

Let $t = \max\{r$: for every subset $W \subseteq V$ of cardinality $|W| \leq r$ there exists a vertex in I that is adjacent to all vertices of $W\}$.

Claim 1. $0 < t < |V|$.

Proof. Every vertex $v \in V$ is adjacent to a vertex of I, therefore $t \geq 1$. Every vertex $u_i \in I$ is not adjacent to $v_i \in V$ [since $u_i, v_i \in J_i$], therefore $t < |V|$. □

Claim 2. $t \geq 2$.

Proof. Suppose that $t < 2$. By Claim 1, $t = 1$. By the definition of t, there exist distinct vertices $v_i, v_j \in V$ such that every vertex $u \in I$ is adjacent to at most one of them.

The vertex v_i is adjacent to at least two vertices of I, say, without loss of generality, to u_1 and u_2. Then v_j is non-adjacent to both u_1 and u_2. The vertex v_j is also adjacent to at least two vertices of I, say u_3 and u_4. Then v_i is non-adjacent to both u_3 and u_4. Thus, the set $\{v_i, v_j, u_1, u_2, u_3, u_4\}$ induces either

- G_1 [when v_i is non-adjacent to v_j] or
- G_2 [when v_i is adjacent to v_j],

a contradiction. □

The definition of t and $t < |V|$ [Claim 1] imply that there exists a set $W \subseteq V$ of cardinality $|W| = t + 1$ such that every vertex of I is not adjacent to at least one vertex of W. Without loss of generality, let $W = \{v_1, v_2, \ldots, v_{t+1}\}$. Note that the vertices in W are pairwise distinct.

We denote $W_j = W \setminus \{v_j\}$ for each $j = 1, 2, \ldots, t + 1$. Since $|W_j| = t$, there exists a vertex $u_{i_j} \in I$ that is adjacent to all vertices of W_j. Since the vertex u_{i_j} is non-adjacent to a vertex of W, u_{i_j} is not adjacent to v_j. By Claim 2, $|W| = t + 1 \geq 3$. It is easy to see that the set $\{v_1, v_2, v_3, u_{i_1}, u_{i_2}, u_{i_3}\}$ induces one of the graphs G_3, G_4, G_5 or G_6, a contradiction. □

A non-locally well-covered graph G is *minimal* if all proper induced subgraphs of G are in $\mathcal{LOCWELL}$. It is easy to check that minimal non-locally well-covered graphs are exactly $G_1, G_2, \ldots, G_6$.

Corollary 1. *The maximal hereditary subclass $\mathcal{HLOCWELL}$ in $\mathcal{LOCWELL}$ is defined by $\{G_1, G_2, \ldots, G_6\}$ as the set of all minimal forbidden induced subgraphs.*

Corollary 2. *The class $\mathcal{HLOCWELL}$ is polynomial-time recognizible.*

Proof. Indeed, by Corollary 1, $\mathcal{HLOCWELL}$ has exactly six minimal forbidden induced subgraphs. □

It would be interesting to extend our main result to wider classes of graphs.

Definition 2. *For every $k \geq 0$ we define a class $\mathcal{LOCWELL}(k)$ of k-*locally well-covered graphs *in the following way: $G \in \mathcal{LOCWELL}(k)$ if and only if there is a stable set I of G such that $|I| \leq k$ and every maximal stable set that contains I, is a maximum stable set.*

Thus, $\mathcal{WELL} = \mathcal{LOCWELL}(0)$ and $\mathcal{LOCWELL} = \mathcal{LOCWELL}(1)$.

Conjecture 1. *The maximal hereditary subclass of $\mathcal{LOCWELL}(k)$ has a finite forbidden induced subgraph characterization.*

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