Graphs whose Vertex-Neighborhoods are Anti-Sperner

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**In memory of my beloved sister, Susan Porter Hermann, who passed away June 19, 2004. Sue-Bear never cared much for mathematics, but she did appreciate it when I explained to her my discovery that 10! equals exactly the number of seconds in six weeks.*

Abstract

In this note we study graphs whose family of open vertex-neighborhoods are anti-Sperner. We exhibit properties, show constructions, and characterize the case for regular graphs.

1 Introduction

We use the standard notation as found in e.g., [7]. We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. Consider a graph G with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. For a vertex $x \in V(G)$ denote by $N_G(x)$, or just $N(x)$ when G is understood, to be the open vertex neighborhood of x, i.e., $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$. The graphs under consideration are without loops, so $x \notin N(x)$. Let $\mathcal{F} = \{N(1), N(2), \ldots, N(n)\}$ be the family of vertex-neighborhoods in G. The set system $\mathcal{F}$ is *anti-Sperner* if every member of $\mathcal{F}$ is a subset of some other, i.e., for all $i \in V(G)$, there exists a $j \neq i$ where $N(i) \subseteq N(j)$. If $\mathcal{F}$ is anti-Sperner we say that G is an anti-neighborhood-Sperner (ANS) graph. The results here apply to finite or infinite ANS graphs, however they are always locally finite, i.e., $|N(x)| < \infty$. Since having multiple edges doesn't affect the ANS property, the graphs considered here are simple. We list some properties of ANS graphs that were shown in [6]. Let $g(G)$ denote the girth of a graph G, i.e., the length of the smallest cycle in G.

Theorem 1.1 ([6]). *If G is a connected ANS graph then $g(G) \leq 4$.* □

Theorem 1.2 ([6]). *If G is a finite ANS graph then there must exist two vertices x and y with $N(x) = N(y)$.* $\square$

As an example of a finite ANS graph, the complete multipartite graph $G = K_{|A_1|,|A_2|,\dots,|A_m|}$, $m \geq 2$ is ANS if $|A_i| \geq 2$ for $i = 1, \dots, m$. Here $A_1 \cup \dots \cup A_m = V(G)$, $A_i \cap A_j = \emptyset$ for $i \neq j$. Also, a connected spanning subgraph $H \subseteq K_{|A_1|,\dots,|A_m|}$ is ANS if for each partite set A_i there exists $x, y \in A_i$ with $N_H(x) = N_H(y) = V \setminus A_i$. For $q \geq 2$, $K_{q,q}$ is an example of a q-regular ANS graph. We remark $K_{3,3} - e$ is ANS.

A more general procedure uses the tensor product of two ANS graphs, G and H, to generate an ANS graph. The *tensor product* of two graphs G and H, denoted $G \otimes H$ is defined as $V(G \otimes H) = V(G) \times V(H)$ and $(g, h)(g', h')$ is an edge in $G \otimes H$ iff $gg' \in E(G)$ and $hh' \in E(H)$.

For general properties and applications of the tensor product, please consult [1, 2, 4].

Theorem 1.3 ([6]). *If G and H are ANS graphs (finite or infinite), then $G \otimes H$ is an ANS graph.* $\square$

2 Further constructions and properties

We first list some properties of ANS graphs concerning maximum/minimum vertex degrees and vertex-connectivity. We use $\Delta(G)$, resp., $\delta(G)$, to denote the maximum, resp., minimum degree of a vertex in G.

Theorem 2.1. *If G is an ANS graph of order n, then $\Delta(G) \leq n - 2$.*

Proof. Assume otherwise, i.e., that there exists some vertex x where $d(x) \geq n-1$, so $N(x) = V \setminus \{x\}$. Now, since G is ANS there exists a vertex y where $N(x) \subseteq N(y)$. But $y \in V(G) \setminus \{x\}$ and then necessarily also in $N(y)$. But $y \in N(y)$ contradicts that we do not allow loops in G. $\square$

Theorem 2.2. *If G is a connected ANS graph, then $\delta(G) \geq 2$.*

Proof. Assume otherwise, i.e., that there exists a pendant vertex x with $d(x) = 1$. Let $N(x) = \{y\}$, i.e., $xy \in E(G)$; then $x \in N(y)$. Since G is ANS there exists a vertex z where $N(y) \subseteq N(z)$. Since $x \in N(y)$ this implies xz is also an edge of G. But xy and xz being edges in G contradicts that $d(x) = 1$. $\square$

Theorem 2.3. *If G is a connected ANS graph, then it is 2-connected.*

Proof. Assume otherwise, i.e., that there exists a cut-vertex x in G. Then $G - x = G_1 \cup G_2 \cup \dots \cup G_\omega$ has at least two components, i.e., $\omega \geq 2$. Since G is ANS, there exists some $y \in V(G)$ where $N_G(x) \subseteq N_G(y)$. This vertex y is in some component in $G - x$, w.l.o.g. say $y \in V(G_1)$. Since x is a cut-vertex it has neighbors in $V(G_1) \cap V(G_2)$, hence $N_G(x) \cap V(G_2) \neq \emptyset$.

Let $S = N_G(x) \cap V(G_2)$. Then since $S \subseteq N_G(x) \subseteq N_G(y)$, we have that in G, y is adjacent to every vertex in S. But this implies then that G_1 and G_2 are not disjoint components in $G - x$. $\square$

If one has an r-regular ANS graph with $r \geq 2$, then we can replace the containment '$\subseteq$' symbol in the definition of ANS graphs with equality '$=$'. This follows since if $i \in V(G)$ for some ANS graph G, then by definition there exists a $j \in V(G)$, $j \neq i$, where $N(i) \subseteq N(j)$. But $|N(i)| = r = |N(j)|$, hence $N(i) = N(j)$. We refer to this as the following lemma.

Lemma 2.4. *If G is an r-regular, $r \geq 2$, ANS graph, then $i, j \in V(G)$ with $N(i) \subseteq N(j)$ implies $N(i) = N(j)$.* $\square$

For the cases of $r = 2, 3$, r-regular connected ANS graphs, there is only one, namely; $C_4 \cong K_{2,2}$, resp., $K_{3,3}$.

Theorem 2.5. *The only connected 2-regular ANS graph is C_4.*

Proof. Any connected 2-regular graph is a cycle. One may check to see that C_3 is not ANS, also that C_4 is ANS. By Thm. 1.1 any other cycle has too large a girth. $\square$

Theorem 2.6. *Let G be any ANS graph. If $x, y \in V(G)$ with $N(x) \subseteq N(y)$, then $xy \notin E(G)$.*

Proof. On the contrary, assume $N(x) \subseteq N(y)$ and $xy \in E(G)$. Then $y \in N(x)$, and hence $y \in N(y)$, but this contradicts that G has no loops, i.e. $y \notin N(y)$. $\square$

Theorem 2.7. *The only connected 3-regular ANS graph (finite or infinite) is $K_{3,3}$.*

Proof. Let G be a connected 3-regular ANS graph. Let $x, y \in V(G)$ be vertices with $N(x) = N(y)$. These vertices exist by the definition of ANS and Lemma 2.4. Also, by Thm. 2.6, $xy \notin E(G)$. Let $N(x) = N(y) = \{a, b, c\}$, then $\{x, y\} \subset N(a) \cap N(b) \cap N(c)$. Now, since $|N(a)| = 3$, define $z \in V(G)$ where $N(a) = \{x, y, z\}$. Now by definition of ANS there must exist an $i \in V(G)$ with $N(i) = N(a)$, we show $i = (b \text{ or } c)$. With $N(i) = \{x, y, z\}$ we have $i \in N(x) \cap N(y) = \{a, b, c\}$, consequently $i = (b \text{ or } c)$. Without loss of generality let $i = b$, then $N(a) = N(b) = \{x, y, z\}$. Now, $\{x, y\} \subset N(c)$, define $w \in V(G)$ where $N(c) = \{x, y, w\}$. We show $z = w$. Since G is ANS there exists some $d \in V(G)$ where $N(d) = N(c) = \{x, y, w\}$. Then $d \in N(x) \cap N(y) = \{a, b, c\}$, consequently $d = (a \text{ or } b)$. Without loss of generality let $d = a$, then $\{x, y, z\} = N(a) = N(d) = N(c) = \{x, y, w\}$ hence $z = w$.

So we have $N(a) = N(b) = N(c) = \{x, y, z\}$ and $N(x) = N(y) = N(z) = \{a, b, c\}$. Since G is connected and 3-regular there are no other vertices in G. Also, by Thm. 2.6, $\{a, b, c\}$ is an independent set of vertices in G, likewise for $\{x, y, z\}$. So $G \cong K_{3,3}$, where one partite set is $\{a, b, c\}$ and the other $\{x, y, z\}$. $\square$

The following construction gives us our characterization of regular ANS graphs.

We take a host graph G, with $V(G) = \{1, 2, \ldots, n\}$. Let $H_1, H_2, \ldots, H_n$ be a collection of n graphs. The graph $I(H_1, H_2, \ldots, H_n : G)$ is defined to be the graph obtained by replacing vertex i with a copy of H_i, and if $ij \in e(G)$, then in I we connect *all* vertices in H_i to *all* vertices in H_j. So, if $ij \in E(G)$, we join H_i to H_j in I. The *join* of two graphs G and H denoted $G \vee H$, is the graph obtained from the disjoint union of $G + H$ by adding the edges $\{xy \mid x \in V(G), y \in V(H)\}$. So, for examples; $K_{2,2,2} \cong I(\overline{K}_2, \overline{K}_2, \overline{K}_2 : K_3)$; $K_{|A_1|,\ldots,|A_m|} \cong I(\overline{K}_{|A_1|}, \overline{K}_{|A_2|}, \ldots, \overline{K}_{|A_m|} : K_m)$, $C_4 \otimes C_4 \cong I(\overline{K}_4, \overline{K}_4, \overline{K}_4, \overline{K}_4 : 2K_2)$, etc. We remark that the graph $C_4 \otimes C_4$ is an example of a disconnected ANS graph, however we are primarily interested in connected graphs here. Also, $I(G, H : K_2) \cong G \vee H$, i.e., the usual definition of the join '$\vee$' between two graphs can be alternatively defined via the graph I. For the default case, where $G = \overline{K}_{|A|}$ with $|A| \geq 2$, then each vertex in G has the empty set neighborhood, we do also include this as an ANS graph.

Theorem 2.8. *Let G be any connected graph of order $n \geq 2$, with $V(G) = \{1, 2, \ldots, n\}$, then $I(\overline{K}_{|A_1|}, \ldots, \overline{K}_{|A_n|} : G)$ where $|A_i| \geq 2$ for $i \in [n]$, is an ANS graph.*

Proof. For each $i \in [n]$, let $A_i = \{a_{1,i}, a_{2,i}, \ldots, a_{|A_i|,i}\}$, then $V(I) = A_1 \cup A_2 \cdots \cup A_n$. Let $x \in V(I)$ be any vertex in I, then $x \in A_i$ for some i, let $y \in A_i$ with $x \neq y$. Then $N_I(x) = N_I(y)$, and we then have by definition that I is ANS. □

We remark that the same argument above is also valid when the vertex set of G is countably infinite. That is, in the definition of I we allow the host graph G to have countably infinite number of vertices. More formally, let $V(G) = \{1, 2, \ldots\}$ and let $H_1, H_2, \ldots$ be an infinite collection of graphs. Then $I(H_1, H_2, \ldots, : G)$ is defined analogous to the finite version.

Corollary 2.9. *Let G be any connected infinite graph with $V(G) = \{1, 2, \ldots\}$, then $I(\overline{K}_{|A_1|}, \overline{K}_{|A_2|}, \ldots : G)$ where $|A_i| \geq 2$ for all $i \in \{1, 2, \ldots\}$ is an example of an infinite ANS graph.*

Proof. The proof is the same as in Thm. 2.8. □

For the specific case; $I(H_1, H_2, \ldots, : G)$ where $H_i \cong H_j$ for all i, j say $H_i \cong H$, then we use shorthand and write $I(H_1, H_2, \ldots, : G) = I(H; G)$. So, for example, $I(\overline{K}_q; P_\infty)$, $q \geq 2$ is an example of a $2q$ regular infinite ANS graph. (Here P_∞ denotes the infinite path.)

Also, we use the name *Mirror of G* denoted $\mathrm{Mir}(G)$, for any graph G (finite or infinite) where $\mathrm{Mir}(G) \cong I(\overline{K}_2; G)$, since any two mates $x, \overline{x}$ in a $\overline{K}_2$ in $\mathrm{Mir}(G)$ have the similar left/right switching as in a mirror reflection.

So, $\mathrm{Mir}(K_n) \cong K_{\underbrace{2,2,\ldots,2}_{n\text{-times}}}$ is the hyperoctahedral graph. Also, for the hypercube Q_n, $\mathrm{Mir}(Q_n)$ is a $2n$-regular ANS graph on 2^{n+1} vertices.

Theorem 2.10. *For any $r \geq 4$, there exists an r-regular infinite ANS graph.*

Proof. For the case where r is even, the graph $I(\overline{K}_{r/2}; P_\infty)$ is an example. For the case $r = 2m+1$ is odd, let $V(P_\infty) = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and for each vertex $i \in \mathbb{Z} = V$ we replace it with a copy of H_i, where the H_i's are described as:

$$\begin{array}{ccccccccccccc} \ldots, & H_{-4}, & H_{-3}, & H_{-2}, & H_{-1}, & H_0, & H_1, & H_2, & H_3, & H_4, & H_5, & \ldots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \ldots, & \overline{K}_m, & \overline{K}_m, & \overline{K}_{m+1}, & \overline{K}_{m+1}, & \overline{K}_m, & \overline{K}_m, & \overline{K}_{m+1}, & \overline{K}_{m+1}, & \overline{K}_m, & \overline{K}_m, & \ldots \end{array}$$

Then, $I(\ldots, H_{-2}, H_{-1}, H_0, H_1, H_2, \ldots : P_\infty)$ is a $(2m+1)$-regular ANS graph. $\square$

We can generalize Thm. 2.8.

Theorem 2.11. *If $H_1, H_2, \ldots, H_n$ is a collection of ANS graphs, and G is any graph with $V(G) = [n]$, then $I(H_1, \ldots, H_n : G)$ is an ANS graph.*

Proof. Let x be any vertex in $I(H_1, \ldots, H_n : G)$, then $x = x_i$ where x_i is a vertex in some H_i. Now, since H_i is ANS there exists a vertex $y_i \in V(H_i)$ where $N_{H_i}(x_i) \subseteq N_{H_i}(y_i)$. By the definition of I we then have $N_I(x_i) \subseteq N_I(y_i)$. $\square$

The above argument also holds for infinite graphs G.

Corollary 2.12. *Let $V(G) = \{1, 2, \ldots\}$ and let $H_1, H_2, \ldots$ be an infinite collection of ANS graphs. Then $I(H_1, H_2, \ldots : G)$ is ANS.*

Proof. The proof is the same as in Thm. 2.11. $\square$

The graphs in Figures 1, 2 illustrate Thm. 2.11.

3 A characterization of regular ANS graphs

We now give a characterization of regular ANS graphs. Let Q be a finite connected r-regular ANS graph. We will show that $Q \cong I(\overline{K}_{|A_1|}, \overline{K}_{|A_2|}, \ldots, \overline{K}_{|A_n|} : G\}$ for some graph G of order n, and $|A_1| + \cdots + |A_n| = |V(Q)|$.

For each vertex $v \in V(Q)$ *define* $P(v) = \{w \in V(Q) \mid N_Q(v) = N_Q(w)\}$. We remark that $v \in P(v)$. We list some properties for reference pertaining to an ANS graph Q and its associated sets $P(v)$.

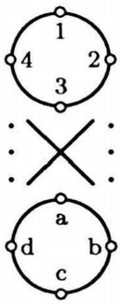

Figure 1: $I(C_4, C_4 : K_2)$ is an example of an ANS graph with 8 vertices and 24 edges.

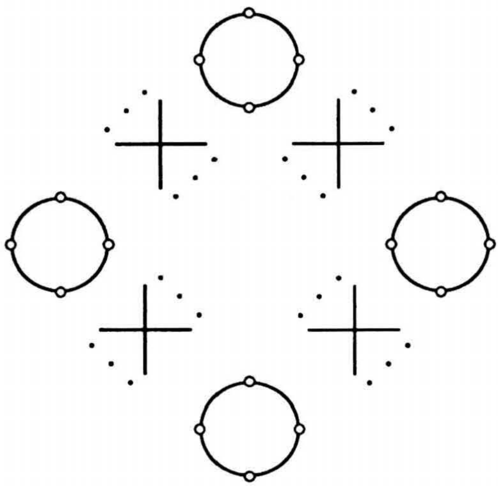

Figure 2: $I(C_4, C_4, C_4, C_4 : C_4)$ is an example of an ANS graph with 16 vertices and 80 edges.

Lemma 3.1. $|P(v)| \geq 2$ *for all* $v \in V(Q)$.

Proof. We have $v \in P(v)$, and since Q is a regular ANS graph we have by definition of ANS and Lemma 2.4 that there exists a vertex $y \neq v$, where $N(y) = N(v)$; consequently, $y \in P(v)$. $\square$

Lemma 3.2. *For each vertex* $v \in V(Q)$ *the set of vertices in* $P(v)$ *form an independent set in* Q.

Proof. This is a consequence of Thm. 2.6. $\square$

Lemma 3.3. *If Q is an r-regular connected ANS graph then $|P(v)| \leq r$ for all $v \in V(Q)$.*

Proof. Let $P(v) = \{w_1, w_2, \ldots, w_\ell\}$, for some $v \in V(Q)$. By contrary, assume $\ell > r$. By the definition of $P(v)$, $N(w_1) = N(w_2) = \cdots = N(w_\ell)$. Also, since Q is connected $N(w_i) \neq \phi$. Let $y \in N(w_1) \cap N(w_2) \cdots \cap N(w_\ell)$, then $\deg(y) \geq \ell > r$, which contradicts $\deg(y) = r$. $\square$

By Lemma 3.2, the induced subgraph $Q[P(v)]$ in Q is isomorphic to $\overline{K}_{|P(v)|}$. Let $P(v) = \{w_1, w_2, \ldots, w_{|P(v)|}\}$, then by the definition of P, we have $P(w_1) = P(w_2) = P(w_{|P(v)|})$. Consequently, the definition of $P(v)$ partitions the set of vertices in Q into equivalent classes where two vertices x, y are equivalent, $x \sim y$ iff $P(x) = P(y)$. Let $P_{v_1}, P_{v_2}, \ldots, P_{v_n}$ denote a labelling of the set of equivalent classes of $V(Q)$. These n sets will be the vertex set of our host graph G. We define $V(G) = \{P_{v_1}, \ldots, P_{v_n}\}$. Let $P_{v_i} = \{w_{1,i}, w_{2,i}, \ldots, w_{|P_{v_i}|,i}\}$, and suppose $xy \in E(Q)$. By Lemma 3.2, $x \in P_{v_i}$ for some i and $y \in P_{v_j}$ for some j, with $i \neq j$. We show the join $P_{v_i} \vee P_{v_j}$ is a subgraph of Q.

Theorem 3.4. *With the notation above, the join $P_{v_i} \vee P_{v_j}$ is a subgraph of Q.*

Proof. Let $xy \in E(Q)$ with $x \in P_{v_i}$ and $y \in P_{v_j}$ for some $i \neq j$. Since $y \in N(x)$ and $N(x) = N(w)$ for all $w \in P_{v_i}$ we have $y \in N(w)$ for all $w \in P_{v_i}$. Consequently the joint $\{y\} \vee P_{v_i}$ is a subgraph of Q. Likewise, since $x \in N(y)$ and $N(y) = N(z)$ for all $z \in P_{v_j}$ we have $x \in N(z)$ for all $z \in P_{v_j}$. Consequently the join $\{x\} \vee P_{v_j}$ is a subgraph in Q.

To finish the proof we show that for any vertex $w \in P_{v_i}$ that the join $\{w\} \vee P_{v_j}$ is a subgraph in Q; likewise, $\{z\} \vee P_{v_i}$ is a subgraph for all $z \in P_{v_j}$. Consequently, the induced subgraph

$$Q[P_{v_i} \cup P_{v_j}] \cong \overline{K}_{|P_{v_i}|} \vee \overline{K}_{|P_{v_j}|}.$$

Let w be any vertex in P_{v_i}. Above, we have established that $\{y\} \vee P_{v_i}$ is a subgraph in Q, consequently $w \in N(y)$. Since $N(z) = N(y)$ for all z in P_{v_j} we have then that $w \in N(z)$, so $\{w\} \vee P_{v_j}$ is a subgraph of Q. Using the above established subgraph $\{x\} \vee P_{v_j}$, an analogous argument gives $\{z\} \vee P_{v_i}$ is a subgraph for all $z \in P_{v_j}$. $\square$

So, combining Lemmas 3.1, 3.2, 3.3, and Thm. 3.4 gives us our characterization of r-regular ANS graphs. Given an r-regular ANS graph Q, we partition the vertex set of Q into equivalent classes, $P_{v_1}, P_{v_2}, \ldots, P_{v_n}$. We then define our host graph G, where $V(G) = \{P_{v_1}, \ldots, P_{v_n}\}$ and $P_{v_i}P_{v_j}$ is an edge in G iff $P_{v_i} \vee P_{v_j}$ is a subgraph of Q. We then have

$$Q \cong I\left(\{\overline{K}_{|Pv_i|}\}_{i\in V(G)} : G\right).$$

We refer to this as:

Theorem 3.5. *If Q is an r-regular ANS graph, then $Q \cong I(\{\overline{K}_{n_i}\}_{i\in V(G)} : G)$ for some graph G.*

To illustrate this, consider the graph $I(C_4, C_4 : K_2) = Q$ shown in Fig. 1. We have $P(1) = P(3)$, $P(2) = P(4)$, $P(a) = P(c)$, $P(b) = P(d)$. Hence there are four equivalent classes. Let $P_{v_1} = \{1,3\}$, $P_{v_2} = \{2,4\}$, $P_{v_3} = \{a,b\}$, and $P_{v_4} = \{b,d\}$. Then $V(G) = \{P_{v_1}, P_{v_2}, P_{v_3}, P_{v_4}\}$ and we see $P_{v_i} \vee P_{v_j}$ is a subgraph in Q for all i,j, $i \neq j$. Hence $I(C_4, C_4 : K_2) \cong I(\overline{K}_2, \overline{K}_2, \overline{K}_2, \overline{K}_2 : K_4) = \text{Mir}(K_4)$.

Also, it is straightforward to check that for the ANS graph in Fig. 2 we have $I(C_4; C_4) \cong \text{Mir}(\overline{C}_8)$.

Similar to Thms. 2.5 and 2.7 we can charcterize 4-regular ANS graphs.

Theorem 3.6. *If Q is a finite connected 4-regular ANS graph, then $Q \cong$ Mir(C_n) for some $n \geq 3$, or $Q = K_{4,4}$.*

Proof. By Thm. 3.5, we have $Q \cong I(\overline{K}_{|P_1|}, \ldots, \overline{K}_{|P_n|} : G)$. Let $P_{v_1}, P_{v_2}, \ldots, P_{v_n}$ be the equivalent classes of $V(Q)$. We have by Lemmas 3.1 and 3.3 that $|Pv_i| \in \{2,3,4\}$. We first show $|P_{v_i}| \neq 3$ for all i.

Assume for some i, $|P_{v_i}| = 3$. Then we have, by Thm. 3.4, that the join $P_{v_i} \vee P_{v_j}$ is a subgraph of Q for some P_{v_j}. Since Q is 4-regular, there must be a P_{v_k} where $P_{v_j} \vee P_{v_k}$ is a subgraph of Q with $|P_{v_k}| = 4 - 3 = 1$. But this contradicts Lemma 3.1.

So we have $|P_{v_i}| = 2$ or 4. Suppose $|P_{v_i}| = 4$ for some i. Then the join $P_{v_i} \vee P_{v_j}$ is a subgraph of Q for some P_{v_j}. Now $|P_{v_j}| = 2$ or 4, if $|P_{v_j}| = 4$ then $P_{v_i} \vee P_{v_j} \cong K_{4,4}$ and since Q is 4-regular connected this is then all of Q. Otherwise $|P_{v_j}| = 2$, but then since Q is 4-regular there exists a $|P_{v_k}| = 2$ with $P_{v_i} \vee P_{v_k}$ a subgraph of Q. But then the induced graph $Q[P_{v_i} \cup P_{v_j} \cup P_{v_k}] \cong K_{4,4} = Q$, one partite set is P_{v_i}, the other partite set $P_{v_j} \cup P_{v_k}$.

Finally, the remaining possibility is $|P_{v_i}| = 2$ for all i. But since Q is 4-regular, P_{v_i} is joined in Q to exactly two other classes, say, P_{v_j} and P_{v_k} with $|P_{v_j}| = |P_{v_k}| = 2$. But from (1) our host graph G is then 2-regular, i.e. a cycle, consequently $Q \cong \text{Mir}(C_n)$ for some $n \geq 3$. $\square$

As an immediate consequence of Thm. 3.5 we have that any finite 4-regular ANS has an even number of vertices.

Corollary 3.7. *If G is a connected 4-regular ANS graph then G has an even number of vertices.*

Proof. $K_{4,4}$ has 8 vertices, and Mir(C_n) has $2n$ vertices. $\square$

The developments in Section 3 can be immediately extended to infinite r-regular ANS graphs as well, the proofs are identical. So we have if Q is an infinite r-regular ANS graph then $Q \cong I(\overline{K}_{|A_1|}, \overline{K}_{|A_2|}, \ldots : G)$. Also, analogous to Thm. 3.5, the only 4-regular connected infinite ANS graph is $\mathrm{Mir}(P_\infty)$.

The present work here was motivated by previous study of the converse problem. That is, graphs whose set of vertex neighborhoods are Sperner. These graphs were shown to have applications to the self-clique graph problem in [3,4].

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