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Application of Backward Stochastic Differential Equation in Insurance Mathematics

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Abstract

On the condition that an insurance company is risk neutral, the paper researchs two problems as follows: one is how to arrange reinsurances in order to reduce risk and increase income; the other is how to calculate annual total reserve in order to keep a balance of terminal financial revenue and expenditure. For first problem, proportional reinsurance and excess of loss reinsurance strategies are researched by using the theory of backward stochastic differential equation. According to investment theory, a retained proportion and a retention are obtained on the basis of the explicit solution class of the linear backward stochastic differential equations established for the reinsurances. The results have a directly helpful role for insuers to design reinsurances. As for second problem, the paper integrates the calculation of annual total reserve with return on investment and establishs the linear backward stochastic differential equations of reserve.

I. Introduction

IN NON-LIFE INSURANCE business, insurers usually must consider following two problems: one is how to arrange reinsurances in order to reduce risk and increase income; the other is how to calculate annual total reserve in order to keep a balance of terminal financial revenue and expenditure. For first problem, a traditional method is to make probabilities of ruin minimum as a standard when insurers arrange reinsurances. Because the method laies particular emphasis on insurers' safety and neglects profits, the reinsurances arranged by it are often more safety and less profits. In addition, it neglects the impacts of investment and expense element etc., the results obtained by it are very difficult to play an important role in the business.

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As for second problem, there are usually two methods to solve it. One is that first seek the distribution function of claim payment for each insurance, then calculate each reserve respectively, finally add all reserves; the other is that seek the distribution function of claim payment for each insurance, then calculate the distribution function of total claim payment, finally get total reserve according to the distribution function obtained by second step. Because both methods suppose that all claims are independent of each other and all insurable distributions can be obtained well as total reserve has nothing to do with investment income, these hypothesises don't agree with reality, therefore the total reserves obtained by using them have often a gap with real demand.

A backward stochastic differential equation is mainly concerned with what the starting point of a system is for the purpose of achieving anticipated aim under the environment of a stochastic interference. Though its theoretical research is short, its advance is very quick. The main reason is that it has many very important application in economic and financial fields, besides many interesting mathematical properties. For example, famous economist Duffie and Epstein found to may describe consumption preference under an unsteady economic environment by using it, Ei Kuroui and Quenez found to may solve many important theoretical prices of derivative products (for example, options and futures) by the equation.

On the condition that an insurance company (be called company for short below) is risk neutral for first problem, proportional reinsurance and excess of loss reinsurance strategies are studied by using the theory of backward stochastic differential equation. According to investment theory, a retained proportion and a retention are obtained on the basis of the explicit solution of a special class of the linear backward stochastic differential equations established for the reinsurances. The results have a directly helpful role for insures to design reinsurances. As for second problem, the paper integrates the calculation of annual total reserve with return on investment and establishes linear backward stochastic differential equations of reserve. A calculating formula of annual total reserve is obtained on the basis of the explicit solution of a special class of its linear backward stochastic differential equations. The formula is simple and effective for insurers to calculate annual total reserve.

II. Basic Concepts and Results

Now we introduce briefly relevant basic concepts and results of a backward stochastic differential equation.

Let (Ω,F,P) be a probability space and $w(t)$ $(t\geq 0)$ be a d-dimension Wiener process in (Ω,F,P), $F_t = \sigma\,[\,w(s), s \leq t\,]$ is a σ - field $\{F_t\}$ generated by $w(t)$, where each σ-field F_t is complete. If for each $t \in [0,\infty]$, $x(t)$ is a F_t measurable random variable, then stochastic process $x(t) = x(w,t)$ is called F_t-applicable. If $E\int_0^T |x(t)|^2 dt < \infty$, where $|x(t)| = (\sum_{i=1}^{n} |x_i(t)|)^{\frac{1}{2}}$ is a Euclidian norm, then

$x(t)=x(w,t)$ is called, square integrable stochastic process. The whole of F_t-measurable and square integrable stochastic processes is written as $M(0,T,R^n)$ (Ross, 1993).

Lemma 1; (Ito's formula). *Let* $dx_i(t) = b_i(t)dt+\sigma_i(t)dw_i(t)$ $(i=1,2,3,4,......, n)$, first deriviation on x,t and second deriviation on x of function $G(x_1, ..., x_n, t)$ be continuous with respect to (x,t), where $x= (x_1, x_2, ...x_n) \in R^n$, $t \geq 0$, $w_i(t)$ $(i=1,2,....n)$ be independent Wiener processes each other, then $G(x_1(t),...,x_n(t),t)$ satisfies following stochastic differential equation.

$$dG(x(t),t) = [G_i(x(t),t) + \sum_{i=1}^{n} Gx_i(x(t),t)b_i(t) + \frac{1}{2}\sum_{i,j=1}^{n} Gx_ix_j(x(t),t)\sigma_i(t)]dt + \sum_{i=1}^{n} Gx_i(x(t),t)\sigma_i(t)dw_i(t)$$

where, the subscripts of G state the derived mumber of G respect to relevent variables (Ross, 1993).

Consider following backward stochastic differential eqaution

$$\begin{cases} -dy(t) = g(t, y(t), z(t))dt - z(t)dw(t) \\ y(T) = \xi \end{cases}$$

where, $y(t),z(t)$ are square integrable applicable processes which belong to R^n and R^{nxd} respectively, that is $(y(t),z(t)) \in M(0,T,R^n \times R^{nxd})$, $0 \leq t \leq T$.

Lemma 2. *Let* $g(.,y,z) \in M(0,T,R^n)$ and satify Lipschitz condition.

$$\left|g(t,y^1,z^1) - g(t,y^2,z^2)\right| \leq C(\left|y^1 - y^2\right| + \left|z^1 - z^2\right|, \forall t, y^1, y^2, z^1, z^2$$

then for terminal condition $\xi \in L^2(\Omega, F_T, P, R^n)$, there is an unique solution of the equation (Hull, 1997).

Lemma 3. *Let* $g(t,y(t),z(t)) = f(t)+a(t)y(t)+b(t)z(t)$, where $f(t), a(t) \in M(0,T,R)$, $b(t) \in M(0,T,R^d)$, and $a(t), b(t)$ be bounded. In addition, *let* $x(s)$ be a solution of follwing Ito's stochastic different equation.

$$\begin{cases} dx(s) = x(s)\left[a(s)ds + b(s)dw(s)\right], s \in [t,T], \\ x(t) = 1, \end{cases}$$

then for $\xi \in L^2(\Omega, F_T, P, R)$, backward stochastic differential equation

$$\begin{cases} -dy(t) = \left[a(t)y(t) + b(t)z(t) + f(t)\right]dt - z(t)dw(t), \\ y(T) = \xi, \end{cases}$$

has an unique solution, and the form of solution is (Peng, 1997).

$$y(t) = E[(\dot{\xi} x(T) + \int_t^T f(s)x(s)ds)|F_t|$$

III. Reinsurance Strategies under the impact of Investment

Now we discuss proportional reinsurance and excess-of-loss reinsurance strategies under the impact of investment as follows.

First construct their mathematical models under the impact of investment.

Suppose that there are only two sorts of assets in a financial market, where one is a no risky asset, the other is a risky asset, their prices x_0, x_1 statify following equations:

$$\begin{cases} dx_0 = r_0 x_0 dt, \\ dx_1 = r_1 x_1 dt + \sigma x_1 dw(t), \end{cases}$$

where r_0, r_1 state no risky asset yield rate and anticipated risky yield rate respectively, σ (>0) states risky asset fluctuating rate.

Suppose that an original insurer accepts the insurance that term is T and claim is a random variable ξ. In a proportional reinsurance (or an excess of loss reinsurance) contract, the original insurer bear the risk with retained proportion α (or retention M), reinsurer take surplus part (1-α)ξ (or max (0,M-ξ))). Premium is $P = (1+\theta)\lambda E(\xi)$ (or $P = (1+\theta)\lambda E(\xi)$), reinsurer permium is $P_1 = (1+\beta)(1-\alpha)\lambda E(\xi)$ (or$P_1 = (1+\beta)\lambda E[\max(0,M-\xi)]$)), where θ, β state additional premium rate and additional reinsurance premium rate respectively, $\theta \leq \beta$, λ is a claim probability. The insurance stipulates that the premium and the reinsurance premium are paid initially and claim is given finally, h is the proprotion of expense occured initially to the premium.

According to above conditions, at initial time t=0, the surplus premium is (1-h) P-P_1 after gettinig rid of the expense and the reinsurance premium. At terminal time $t = T$, the insurer will face loss $y(T) = \lambda\alpha\xi$ (or $y(T) = \lambda \min(M,\xi)$). In order to make up the loss, he will invest natuarally the surpuls premium into risky markets, so that he can get enough return.

After (1-h)P-P_1 is invested into a market at t = 0, it will change continuously as time does, denoted as $y(t)$, then $y(0)$ = (1-h)P-P_1. Suppose that the asset $y(t)$ is divided into two parts at time t, one part $y(t)l(t)$ is invested into a risky fund, the other part $y(t)[1-l(t)]$ is done into a no risky fund, where $l(t) \in [0,1]$ states the proportion invested in a risky fund. When neglecting trade expenses, taxes and dividends, we can obtain that the asset $y(t)$ satisfy following stochastic differential equation by Ito's formula

$$\begin{cases} dy(t) = [r_0 + (r_1 - r_0)l(t)]y(t)dt + \sigma l(t)y(t)dw(t), \\ y(0) = (1-h)P - P_1. \end{cases}$$

Let $z(t) = \sigma\, l(t)y(t)$, then the equation is turned into

$$\begin{cases} dy(t) = [r_0 y(t) + \dfrac{r_1 - r_0}{\sigma} z(t)]dt + z(t)dw(t), \\ y(0) = (1-h)P - P_1 \end{cases}$$

When α (or M) changes, y(T) also change in the investment, thus following condition must be satified at terminal time T

$$y(T) = \lambda\alpha\xi \qquad \text{(or } y(T) = \lambda \min(M,\xi)).$$

From above analysis, we can obtain following forward-backward stochastic differential equations

$$\begin{cases} dx(t) = x(t)[r_1 dt + \sigma dw(t)], & x(0) = [(1-h)P - P_1]l(t), \\ dy(t) = [r_0 y(t) + \dfrac{r_1 - r_0}{\sigma} z(t)]dt + z(t)dw(t), & y(T) = \lambda\sigma\xi \ (y(T) = \lambda \min(M,\xi)). \end{cases}$$

When assect $y(t)$ satifies above equation, we wish to seek a rational retained proportion α (or retention M) in order to keep $y(0) = (1-h)P-P_1 = (1-h)(1+\theta)\lambda E(\xi)-(1-\beta)(1-\alpha)\lambda E(\xi)$. (or $y(0) = (1-h)P-P_1 = (1-h)(1+\theta)\lambda E(\xi)-(1-\beta)\lambda E[\max(0,M-\xi)]$).

Theorem 1. *Let* a company be risk netural under above condition, if its asset satisfy following backward stochastic differential equation

$$\begin{cases} dy(t) = [r_0 y(t) + \dfrac{r_1 - r_0}{\sigma} z(t)]dt + z(t)dw(t), \\ y(T) = \lambda\alpha\xi,\ (y(T) = \lambda \min(M,\xi)) \end{cases}$$

where w(t) ($t \geq 0$) is a stantard Wiener process, ξ and w (t) ($t \geq 0$) are independent random varibables each other, r_0 and $\sigma > 0$ are defined as above, then a rational retained proportion must be

$$\alpha = \frac{(1-\beta)-(1-h)(1+\theta)}{(1+\beta)-\exp[-r_0 T]}$$

(*A rational retention M satifies equality* : $E[\min(\xi,M)] = \dfrac{(1-\beta)E(\xi)-(1-h)(1+\theta)E(\xi)}{(1+\beta)-\exp[-r_0 T]}$).

Proof Let $x(s)$ be a solution of following Ito's stochastic differential equation:

$$\begin{cases} dx(s) = x(s)[-r_0 ds - \dfrac{r_1 - r_0}{\sigma} dw(s), \ s \in [t,T], \\ x(t) = 1. \end{cases}$$

According to lemma 3, obtain from the equation : $a(t)= -r_0$, $b(t) = -\frac{r}{\sigma}$, $f(t)=0$, then obtain from (1): $y(t) = \lambda\alpha E [\xi . x(T) \mid F_t]$ (or $y(t) = \lambda E [\min(M,\xi) . x(T) \mid F_t]$). Particularly, $y(0) = \lambda\alpha E [\xi . x(T)]$ (or $y(0) = \lambda E [\min(M,\xi) . x(T)]$).

Because ξ and $w(t)$ are independent of each other, it follows that ξ is independent with $x(t)$, thus

$$y(0) = \lambda\alpha E(\xi) E[x(T)],$$

$$(1-h)P-P_1 = \lambda\alpha E(\xi) E[x(T)],$$

$$(1-h)(1+\theta) E(\xi) - (1+\beta)(1-\alpha) E(\xi) = \alpha E(\xi) E[x(T)],$$

that is $$\alpha = \frac{(1+\beta)-(1-h)(1+\theta)}{(1+\beta)-E[x(T)]}. \quad (2)$$

or $(1-h)(1+\theta) E(\xi) - (1+\beta) E[\max(0,M-\xi)] = E[\min(M,\xi)] E[x(T)]$,

$$E[\min(\xi,M)] = \frac{(1-\beta)E(\xi)-(1-h)(1+\theta)E(\xi)}{(1+\beta)-E[x(T)]}. \quad (3)$$

Now Prove $E[x(T)] = \exp[-r_0 T]$.

Let $G = \ln x(t)$, then G obeies following Wiener process by Ito's formula

$$dG = (-r_0 - \tfrac{1}{2}(\frac{r_1 - r_0}{\sigma})^2)dt - \frac{r_1 - r_0}{\sigma} dw(t).$$

Because $r_0, r_1 - r_0$ and σ are constant, G obeies the Wiener process with drifting rate $-r_0 - \frac{1}{2}(\frac{r_1 - r_0}{\sigma})^2$ and fluctuating rate $(-\frac{r_1 - r_0}{\sigma})^2$, therefore $\ln x(T) - \ln x(t)$ follows the normal distribtion with expectation $(-r_0 - \frac{1}{2}(\frac{r_1 - r_0}{\sigma})^2(T-t)$ and variance $(-\frac{r_1 - r_0}{\sigma})^2(T-t)$, that is

$$\ln x(T) - \ln x(t) \sim \vartheta((-r_0 - \tfrac{1}{2}(\frac{r_1 - r_0}{\sigma})^2)(T-t), \frac{r_1 - r_0}{\sigma}\sqrt{(T-t)},$$

Where $\vartheta(a,b)$ indicates the normal distribution with expectation a and standard deviation b. Let $t=0$, according to $x(0) = 1$, above form becomes

$$\ln x(T) \sim \vartheta((-r_0 - \tfrac{1}{2}(\frac{r_1 - r_0}{\sigma})^2)T, \frac{r_1 - r_0}{\sigma}\sqrt{T})$$

thus $E[x(T)]=\exp[-r_0 T]$. Substitute the result into (2) (or(3)), then formula is proved.

As a application of theorem 1, we calculate the retentions of an excess of loss reinsurance for several claim distributions in common use as follows.

If $$f(\xi)=\begin{cases}\rho e^{-p\xi}, & \xi\geq 0(\ \rho\ >0),\\ 0, & \xi<0,\end{cases}\tag{3.1}$$

then $$E(\xi)=\tfrac{1}{\rho},\ E[\min(M,\xi)]=-\tfrac{1}{\rho}(e^{-\rho M}-1),$$

thus $$-\frac{1}{\rho}(e^{-\rho M}-1)=\frac{(1+\beta)E(\xi)-(1-h)(1+\theta)E(\xi)}{(1+\beta)-\exp[-r_oT]}.$$

Let $$c=\frac{(1+\beta)E(\xi)-(1-h)(1+\theta)E(\xi)}{(1+\beta)-\exp[-r_oT]},$$

then $$-\tfrac{1}{\rho}(e^{-\rho M}-1)=c.$$

If $1-\rho c>0,$ then $$M=\tfrac{1}{\rho}\,ln\,\tfrac{1}{1-\rho c}.\tag{3.2}$$

If a random variable ξ obeies Poisson's distribution $\dfrac{e^{-\rho}\rho^{\xi}}{\xi!}$ then

$$\sum_{\xi=0}^{M}\xi\cdot\frac{e^{-\rho}\rho^{\xi}}{\xi!}=\frac{(1+\beta)\rho-(1-h)(1+\theta)\rho}{(1+\beta)-\exp[-r_0T]}$$

$$\sum_{\xi=0}^{M}\frac{e^{-p}p^{\xi}}{(\xi-1)!}=\frac{(1+\beta)\rho-(1-h)(1+\theta)\rho}{(1+\beta)-\exp[-r_0T]}.\tag{3.3}$$

From the equation, retention M can be obtained.

If a random variable ξ obeies uniform distribution U (a,b), then

$$f(\xi)=\frac{1}{b-a},\qquad F(\xi)=\frac{x-a}{b-a},\qquad E(\xi)=\frac{b+a}{2}.$$

If $4b^2-4[a^2+2(b-a)\,c\,]\geq 0$, then

$$M_1=b+\sqrt{(b-a)(b+a-2c)},\ \text{(no meaning)}$$

$$M_2 = b - \sqrt{(b-a)(b+a-2c)}.$$

If $M < a$ or $M > b$, then there are not meaning numbers that are in region $[a,b]$. If $a < M < b$, then $M = b - \sqrt{(b-a)(b+a-2c)}$.

Example 1 : Suppose that an insurer accepts the insurance that term T is one year and claim ξ obeies index distribution $F(\xi) = 1 - e^{-0.1\xi}$. Additional premium rate is $\theta = 0.2$, additional reinsurance premium rate is $\beta = 0.25$. The insurance stipulates that premium and reinsurance premium are paid initially and claim is given finally, h is the proportion of expense occured initially to the premium, surplus premium is invested into a financial market. According to information, no risky asset yield rate is $r_0 = 6\%$ in the financial market.

i. For a proportional reinsurance contract, how should a rational retained proportion a be?
ii. For an excess of loss reinsurance contract, how should a rational retention M be?

Now slove the problem as follows.

i. For a proportional reinsurance contract, because β (=0.25) >θ (=0.2), according to theorem 1, we obtain

$$\alpha = \frac{(1+\beta)-(1-h)(1+\theta)}{(1+\beta)-\exp[-r_0 T]} = \frac{1.25-0.9\text{x}1.2}{1.25-\exp(-0.06)}$$

$$= \frac{0.17}{1.25-0.9418} = \frac{0.17}{0.3082} = 0.552$$

ii. For an excess of loss reinsurance contract, because claim ξ obeies index distribution $F(\xi) = 1-e^{-0.1\xi}$. thus $E(\xi) = \frac{1}{\rho} = 10$.

According to theorem 1, we obtain

$$\int_0^M \xi f(\xi)d\xi + M[1-F(M)] = \frac{(1+\beta)E(\xi)-(1-h)(1+\theta)E(\xi)}{(1+\beta)-\exp[-r_0 T]},$$

that is $0.1\int_0^M \xi e^{-0.1\xi} d\xi + M(1-1+e^{-0.1M}) = 10 \times 0.552$

$$-10(e^{0.1M}-1) = 5.52,$$

$$e^{-0.1M} = 0.448.$$

By checking tables, it follows that M=8.

IV. Calculating Annual Total Reserve

Now we discuss the calculating problem of annual total reserve.

First construct its mathematical models under the impact of investment.

Suppose that there are only two sorts of assets in a financial market, where one is a no risky asset, the other is a risky asset, their prices x0x1 statify following equations:

$$\begin{cases} dx_0 = r_0 x_0 dt, \\ dx_1 = r_1 x_1 dt + \sigma x_1 dw(t), \end{cases}$$

where, r_0, r_1 state no risky asset yield rate and anitcipated risky asset yield rate respectively, $\sigma > 0$ states risky asset fluctuating rate.

Suppose that a company accepts k different types of insurances, their premiums putted initially are $P_1, P_2, ..., P_k$, and claims are given finally. For each insurance, h is the proportion of expense occurred initially to the premium. After getting rid of the expense, surplus premium is invested into financial markets.

In order to keep a balance of financial revenue and expenditure at terminal time, suppose that U is total reserve saved initially, and *r* is its term interest rate. In addition, n_i is the insured number that insurants cover *i* type insurance, where these people are divided into m_i groups. For each group, number is n_{ij}, claim probabaility is q_{ijnij}, claim expectation is μ_{ijnij}, $i=1, ..., k;$ $j = 1, ..., m_i$.

According to above conditions, at initial time $t = 0$, after getting rid expense, surplus premium is $(1-h)(n_1P_1 + n_2P_2 + ... + n_kP_k) = (1\text{-h}) \sum_{i=1}^{k} n_i P_i$. At terminal time $t = T$, the company will face following loss where reserve terminal value is not included

$$(x_{111} + \cdots + x_{11_{n11}}) + \cdots + (x_{1_{m1}1} + \cdots x_{1m1n1m1}) + \cdots + (x_{k11} + \cdots + x_{k1n_{k1}})$$

$$+ \cdots + (x_{km_k1} + \cdots + x_{km_kn_km_k}) - \text{U}(1+r) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \sum_{l=1}^{n_{ij}} xijl - \text{U}(1+r).$$

In order to make up the loss, the insurer will invest naturally the surplus premium into risk markets, so that he can get enough return.

After (1-h) $\sum_{i=1}^{k} n_1 P_i$ is invested into a risk market at time t=0, the asset changes continuously as time does, denoted as $y(t)$, then $Y(0) = (1\text{-h}) \sum_{i=1}^{k} n_1 P_i$.

Suppose that at time t, the asset $y(t)$ is divided into two parts, one part $y(t)$ u(t) is invested in a risky fund, the other $y(t)[1-u(t)]$ is done into a no risky fund, where $u(t) \in [0,1]$ states the proportion invested in a risky fund. When neglecting trade expenses, taxes and dividends, we can obtain that the asset $y(t)$ satisfies following stochastic differential equation by Ito's formula

$$\begin{cases} dy(t) = [r_0 + (r_1 - r_0)u(t)]y(t)dt + \sigma u(t)y(t)dw(t), \\ y(0) = (1-h)\sum_{i=1}^{k} n_i P_i \end{cases}$$

Let $z(t) = \alpha u(t)y(t)$, then above equation is turned into

$$\begin{cases} dy(t) = [r_0 y(t) + \frac{r_1 - r_0}{\sigma} z(t)]dt + z(t)dw(t), \\ y(0) = (1-h)\sum_{i=1}^{k} n_i P_i \end{cases}$$

When $n_i (i=1,2,...,k)$ changes, $y(T)$ also changes in the investment, thus at terminal time following condition must be statified:

$$y(T) = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\sum_{l=1}^{n_{ij}} xijl - U(1+r).$$

From above discussion, we can obtain following forward-backward stochastic differential equations

$$\begin{cases} dx(t) = x(t)[r_1 dt + \sigma dw(t)], & x(0) = [(1-h)\sum_{i=1}^{k} n_i P_i]u(t), \\ dy(t) = [r_0 y(t) + \frac{r_{1-r_0}}{\sigma} z(t)]dt + z(t)dw(t), & y(T) = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\sum_{l=1}^{n_{ij}} xijl - U(1+r). \end{cases}$$

When the asset $y(t)$ satisfy the equations, we wish to seek a rational annual total reserve U in order to keep $y(T) = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\sum_{l=1}^{n_{ij}} xijl - U(1+r)$.

Theorem 2. *Let* a company be risk neutral under above conditions, if its asset satisfy following backward stochastic differential equation

$$\begin{cases} dy(t) = [r_0 y(t) + \frac{r_1 - r_0}{\sigma} z(t)]dt + z(t)dw(t), \\ y(T) = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\sum_{l=1}^{n_{ij}} xijl - U(1+r), \end{cases}$$

Where $w(t)$ ($t \geq 0$) is a standard Wiener process, $\sum_{i=1}^{k}\sum_{j=1}^{m_i}\sum_{l=1}^{n_{ij}} xijl - U(1+r)$ and w (t) ($t \geq 0$) are independent random variable each other, r_0 and $\sigma > 0$ are defined as above, then a rational annual total reserve must be

$$U = \frac{\exp(-r_0 T)\sum_{i=1}^{k}\sum_{j=1}^{m_i} n_{ij} q_{ijnij} \mu_{ijnij} - (1-h)\sum_{i=1}^{k} n_i p_i}{(1+r)\exp(-r_0 T)}.$$

Proof : The proof is similar to the proof given in theoram 1 and is omitted.

Example 2 : Suppose that a company accepts three different types of insurances, that are child's life insurance, medical insurance and motor insurance. Their premiums putted originally are 100, 50 and 120 all claims are given finally. For each insurance, the proporation of expense occured initially to the premium is 20%, the surplus premium is invested into a financial market. The insured numbers that insurants cover first type, second type and third type insurance are 200,150,100, where these people divided into 3.2 and 2 groups respectively. For three groups, the numbers are respectively 100,50,50; 50,100; 40,60 claim probabilities are respectively 0.2, 0.25, 0.5 ; 0.3, 0.4; 0.4, 0.25, claim expectations are respectively 1000,1100,1200; 600, 800; 1500, 2000. According to information term interest rate is r=8%, no risky asset yield rate is r_0 = 6 % in the financial market. In order to keep a balance of financial revenue and expediture at terminal time, how should annual total reserve U be?

According to known condition and theorem 2, it follows that

$$U = \frac{\exp(-r_0)(n_{11}\cdot q_{11_{n11}}\cdot \mu_{11_{n11}} + n_{12}\cdot q_{12_{n12}}\cdot \mu_{12_{n12}} + n_{13}\cdot q_{13_{n13}}\cdot \mu_{13_{n13}} + n_{21}\cdot q_{21_{n21}}\cdot \mu_{21_{n21}} + n_{22}\cdot q_{22_{n22}}\cdot \mu_{22_{n22}})}{(1+r)\exp(-r_0)}$$

$$+ \frac{\exp(-r_0)(n_{31}\cdot q_{31_{n31}}\cdot \mu_{31_{n31}} + n_{32}\cdot q_{32_{n32}}\cdot \mu_{32_{n32}}) - (1-h)(n_1 P_1 + n_2 P_2 + n_3 P_3)}{(1+r)\exp(-r_0)}$$

$$= \frac{\exp(-0.06)()100\times 0.2\times 1000 + 50\times 0.25\times 1100 + 50\times 0.5\times 1200 + 50\times 0.3\times 600 + 100\times 0.4\times 800 + 40\times 0.4\times 1500)}{1.08\times \exp(-0.06)}$$

$$+ \frac{\exp(0.06)(60\times 0.25\times 2000) - 0.8(200\times 100 + 150\times 50 + 100\times 120)}{1.08\times \exp(-0.06)}$$

$$= 115923.36$$

V. Conclusion

The investment of insurance fund is an inexorable trend, whether good or bad will affect reinsurance strategies and annual total reserve value, therefore when researching relevant problems, it is much necessary to consider the impact of investment element for making use of resources effectively and making decisions rightly.

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