Journal of Electronic Imaging 6(4), 479–483 (October 1997).

Approximation of the frame coefficients using finite dimensional methods

Peter Casazza
University of Missouri
Department of Mathematics
Columbia, Missouri 65211
E-mail: pete@casazza.math.missouri.edu

Ole Christensen
Technical University of Denmark
Department of Mathematics
Building 303
DK-2800 Lyngby, Denmark

Abstract. *A frame is a family* $\{f_i\}_{i=1}^{\infty}$ *of elements in a Hilbert space* $\mathcal{H}$ *with the property that every element in* $\mathcal{H}$ *can be written as a (infinite) linear combination of the frame elements. Frame theory describes how one can choose the corresponding coefficients, which are called frame coefficients. From the mathematical point of view this is gratifying, but for applications, it is a problem that the calculation requires inversion of an operator on* $\mathcal{H}$*. The projection method is introduced to avoid this problem. The basic idea is to consider finite subfamilies* $\{f_i\}_{i=1}^{n}$ *of the frame and the orthogonal projection* P_n *onto span* $\{f_i\}_{i=1}^{n}$*. For* $f\in\mathcal{H}$*,* P_nf *has a representation as a linear combination of* f_i*,* $i=1,2,\ldots,n$*, and the corresponding coefficients can be calculated using finite dimensional methods. We find conditions implying that those coefficients converge to the correct frame coefficients as* $n\to\infty$*, in which case, we have avoided the inversion problem. In the same spirit, we approximate the solution to a moment problem. It turns out that the class of "well-behaving frames" are identical for the two problems we consider.* © 1997 SPIE and IS&T. [S1017-9909(97)00804-0]

1 Introduction

We begin with some definitions. Let $\mathcal{H}$ be a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ linear in the first entry. All index sets are assumed to be countable.

1.1 *Definitions*

1. A family $\{f_i\}_{i\in I}\subseteq\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if $\{f_i\}_{i\in I}$ is total and there exist numbers A and $B>0$ such that

$$A\sum_{i\in I}|c_i|^2\leq\left\|\sum_{i\in I}c_if_i\right\|^2\leq B\sum_{i\in I}|c_i|^2$$

for all sequences $\{c_i\}_{i\in I}\in\ell^2(I)$.

2. A family $\{f_i\}_{i\in I}\subseteq\mathcal{H}$ is a frame for $\mathcal{H}$ if

$$\exists A,B>0{:}A\|f\|^2\leq\sum_{i\in I}|\langle f,f_i\rangle|^2\leq B\|f\|^2,\quad\forall f\in\mathcal{H}.$$

Here A and B are called frame bounds.

3. The family $\{f_i\}_{i\in I}\subseteq\mathcal{H}$ is a frame sequence if $\{f_i\}_{i\in I}$ is a frame for its closed span.

Note that a frame $\{f_i\}_{i\in I}$ gives rise to a decomposition of the underlying space $\mathcal{H}$; if we define the frame operator by

$$S{:}\mathcal{H}\to\mathcal{H},\quad Sf=\sum_{i\in I}\langle f,f_i\rangle f_i,$$

then S is bounded and invertible, and

$$f=SS^{-1}f=\sum_{i\in I}\langle f,S^{-1}f_i\rangle f_i,\quad\forall f\in\mathcal{H}.\tag{1}$$

Note that $\{\langle f,S^{-1}f_i\rangle\}$ are called the frame coefficients.

Equation (1) can be viewed as a generalization of the well-known representation of f using an orthonormal basis $\{e_i\}_{i\in I}$:

$$f=\sum_{i\in I}\langle f,e_i\rangle e_i,\quad f\in\mathcal{H}.\tag{2}$$

The main difference is that the coefficients $\{\langle f,e_i\rangle\}$ in Eq. (2) are unique, while there might exist coefficients $\{c_i\}_{i\in I}$ other than the frame coefficients that satisfy $f=\Sigma_{i\in I}c_if_i$ for a chosen element f. If $\{c_i\}_{i\in I}$ has this property, it is well known[1] that

Paper IST-08 received Dec. 27, 1996; revised manuscript received May 27, 1997; accepted for publication May 29, 1997. This paper is a revision of a paper presented at the SPIE conference on Wavelet Applications in Signal and Image Processing IV, Aug. 1996, Denver, CO. The paper presented there appears (unrefereed) in SPIE Proceedings Vol. 2825.
1017-9909/97/$10.00

$$\sum_{i\in I}|c_i|^2=\sum_{i\in I}|\langle f,S^{-1}f_i\rangle|^2+\sum_{i\in I}|c_i-\langle f,S^{-1}f_i\rangle|^2,$$

i.e., the frame coefficients have minimal ℓ^2-norm among the sequences that can be used to represent f.

A frame is a Riesz basis if and only if the elements f_i are ω-independent in the sense that

$$\{c_i\}_{i\in I}\in\ell^2(I),\quad \sum_{i\in I}c_if_i=0\Rightarrow c_i=0,\ \forall i\in I.$$

By definition, every subfamily of a Riesz basis is a Riesz basis for its closed span. But in general, not every subfamily of a frame is a frame sequence. If every subfamily $\{f_i\}_{i\in J}$ of the frame $\{f_i\}_{i\in I}$ is a frame sequence with bounds that are common for all those frames, then we call $\{f_i\}_{i\in I}$ a Riesz frame. Clearly one only has to check the existence of a common lower bound, and it is easy to check that $\{f_i\}_{i\in I}$ is a Riesz frame if this condition is satisfied for all finite index sets $J\subseteq I$.

It turns out that the problems we consider here have very satisfying solutions for Riesz frames, so it is interesting to notice that they were introduced in a very different context. One of the main problems in Ref. 2 was to find conditions implying that a frame $\{f_i\}_{i\in I}$ contains a Riesz basis, and it was shown that every Riesz frame has this property. Later it was discovered that the same is true under the weaker condition "every subfamily of $\{f_i\}_{i\in I}$ is a frame sequence." [3]

2 Approximation of the Frame Coefficients

From the point of view of applications, the problem with calculation of the frame coefficients is to invert S. An idea to overcome this difficulty is to "truncate" the problem. Thus let $\{I_n\}_{n=1}^{\infty}$ be a family of finite subsets of I such that

$$I_1\subseteq I_2\subseteq\ldots\subseteq I_n\nearrow I.$$

Corresponding to a subfamily $\{f_i\}_{i\in I_n}$ we define the space $\mathscr{H}_n:=\text{span}\{f_i\}_{i\in I_n}$ and the frame operator

$$S_n:\mathscr{H}_n\rightarrow\mathscr{H}_n,\quad S_nf=\sum_{i\in I_n}\langle f,f_i\rangle f_i.$$

The orthogonal projection of $f\in\mathscr{H}$ onto $\mathscr{H}_n$ is given by

$$P_nf=\sum_{i\in I_n}\langle f,S_n^{-1}f_i\rangle f_i.$$

Since $P_nf\rightarrow f$ as $n\rightarrow\infty$, one can hope that the coefficients $\langle f,S_n^{-1}f_i\rangle$ converge to the frame coefficients for f, i.e., that

$$\langle f,S_n^{-1}f_i\rangle\rightarrow\langle f,S^{-1}f_i\rangle\quad\text{as } n\rightarrow\infty,\quad\forall i\in I,\quad\forall f\in\mathscr{H}. \tag{3}$$

If Eq. (3) is satisfied, we say that the projection method works. In this case, the frame coefficients can be approximated as close as we want using finite dimensional methods, i.e., linear algebra, since S_n is an operator on the finite dimensional space $\mathscr{H}_n$. This is a very important property for applications: for example, it makes it possible to use computers to approximate the frame coefficients.

If the projection method works, the next natural question is how fast the convergence in Eq. (3) is. For example, one might wish that the set of coefficients $\{\langle f,S_n^{-1}f_i\rangle\}_{i\in I_n}$ converges to the set of frame coefficients in ℓ^2 sense, i.e., that

$$\sum_{i\in I_n}|\langle f,S_n^{-1}f_i\rangle-\langle f,S^{-1}f_i\rangle|^2+\sum_{i\in I-I_n}|\langle f,S^{-1}f_i\rangle|^2 \rightarrow 0,\quad\forall f\in\mathscr{H}. \tag{4}$$

We say that the strong projection method works if Eq. (4) is satisfied. Note that this condition depends on the indexing of the elements. The second term $\Sigma_{i\in I-I_n}|\langle f,S^{-1}f_i\rangle|^2\rightarrow 0$ for every frame, $\forall f\in\mathscr{H}$, so we need to show only that $\Sigma_{i\in I_n}|\langle f,S_n^{-1}f_i\rangle-\langle f,S^{-1}f_i\rangle|^2\rightarrow 0$, $\forall f\in\mathscr{H}$.

The projection method was introduced in Ref. 1. The original presentation is slightly more general than here, since it is formulated for a family $\{f_i\}_{i\in I}$, which does not have to be a frame. In the special case of a frame, Ref. 1 contains an equivalent characterization of the projection method.

Theorem 1. Let $\{f_i\}_{i\in I}$ be a frame. Then the projection method works if and only if for any $j\in I$, there exists a constant c_j such that

$$\|S_n^{-1}f_j\|\leq c_j,\quad\forall n\text{ such that } j\in I_n.$$

Reference 1 also contains an example of a frame, where the projection method does not work. If $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis and we define the family $\{f_i\}_{i=1}^{\infty}:=\{e_1\}\cup\{e_{i-1}+(1/i)e_i\}_{i=2}^{\infty}$, then $\{f_i\}_{i=1}^{\infty}$ is a frame, but

$$\|S_n^{-1}f_1\|^2=\sum_{i=1}^{n}(i!)^2\rightarrow\infty\quad\text{for } n\rightarrow\infty.$$

As the first theoretical results concerning the strong projection method we have the following.

Theorem 2. The strong projection method works for every Riesz frame.

Proof. Let A and B denote common bounds for the Riesz frame $\{f_i\}_{i\in I}$. By Ref. 4, $\|S_n^{-1}\|\leq(1/A)$ for all $n\in N$. Now fix $f\in\mathscr{H}$. Then,

$$\sum_{i\in I_n} |\langle f,S_n^{-1}f_i\rangle-\langle f,S^{-1}f_i\rangle|^2$$
$$=\sum_{i\in I_n} |\langle P_nf,S_n^{-1}f_i\rangle-\langle f,S^{-1}f_i\rangle|^2$$
$$=\sum_{i\in I_n} |\langle S_n^{-1}P_nf-S^{-1}f,f_i\rangle|^2$$
$$\leq B\|S^{-1}f-S_n^{-1}P_nf\|^2$$
$$\leq B(\|S^{-1}f-P_nS^{-1}f\|+\|P_nS^{-1}f-S_n^{-1}P_nf\|)^2$$
$$=B\left(\|S^{-1}f-P_nS^{-1}f\| + \left\|\sum_{i\in I_n}\langle S^{-1}f,f_i\rangle S_n^{-1}f_i-S_n^{-1}P_nf\right\|\right)^2$$
$$\leq B\left(\|S^{-1}f-P_nS^{-1}f\|+\|S_n^{-1}\| \cdot\left\|\sum_{i\in I_n}\langle S^{-1}f,f_i\rangle f_i-P_nf\right\|\right)^2$$
$$\leq B\left(\|S^{-1}f-P_nS^{-1}f\| + \frac{1}{A}\left\|\sum_{i\in I_n}\langle S^{-1}f,f_i\rangle f_i-P_nf\right\|\right)^2 \to 0 \quad \text{for } n\to\infty$$

since $\Sigma_{i\in I_n}\langle S^{-1}f,f_i\rangle f_i \to f$ for $n\to\infty$.

The readers who have checked the proof might have observed that the proof does not use that all the sequences $\{f_i\}_{i\in J}$, $J\subseteq I$, are frame sequences with common bounds: Theorem 2 holds if $\{f_i\}_{i\in I}$ is a frame and $\{I_n\}_{n=1}^{\infty}$ is a family of finite index sets such that the frame sequences $\{f_i\}_{i\in I_n}$, $n\in N$, have a common lower bound, or equivalently,

$$\exists A>0\,\forall n: \sum_{i\in I_n}|\langle f,f_i\rangle|^2 \geq A\|f\|^2, \quad \forall f\in \mathscr{H}_n.$$

Let us say that $\{f_i\}_{i\in I}$ is a conditional Riesz frame (with respect to $\{I_n\}_{n=1}^{\infty}$) if the frame sequences $\{f_i\}_{i\in I_n}$, $n\in N$, have common frame bounds. In the case of a frame indexed by the natural numbers we always use the convention $I_n=\{1,2,\ldots,n\}$. This is indeed a weaker condition. For example, if $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathscr{H}$, then

$$\{f_i\}_{i\in I}:=\{e_2,e_2+\tfrac{1}{2}e_1,e_3,e_3+\tfrac{1}{4}e_1,\ldots\} =\{e_i,e_i+\tfrac{1}{2}e_1\}_{i=2}^{\infty}$$

is a conditional Riesz frame, but not a Riesz frame. The reason for formulating Theorem 1 for Riesz frames is that Riesz frames give a good intuitive feeling for the problems. Furthermore Riesz frames have the advantage that it is not necessary to index the frame elements: the conclusions hold for every arrangement of the frame elements.

In terms of the operators S_n^{-1}, a frame $\{f_i\}_{i\in I}$ is a conditional Riesz frame if and only if $\sup_n\|S_n^{-1}\|<\infty$. Observe that the notion depends on the indexing of the frame elements. For example, if $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis then $\{e_i,(1/i)e_i\}_{i=1}^{\infty}$ is a conditional Riesz frame, but $\{(1/i)e_i,e_i\}_{i=1}^{\infty}$ is not.

Actually, the strong projection method works if and only if $\{f_i\}_{i\in I}$ is a conditional Riesz frame. This is the content of the following result, which we prove in Ref. 5.

Theorem 3. Let $\{f_i\}_{i\in I}$ be a frame. Then the following statements are equivalent:

1. The strong projection method works.
2. $\{f_i\}_{i\in I}$ is a conditional Riesz frame.
3. $S_n^{-1}P_nf\to S^{-1}f$ for all $f\in\mathscr{H}$.
4. $\langle S_n^{-1}P_nf,g\rangle\to\langle S^{-1}f,g\rangle$ for all $f,g\in\mathscr{H}$.
5. $\lim_{n\to\infty}\Sigma_{i=n+1}^{\infty}|\langle S_n^{-1}P_nf,f_i\rangle|^2=0$, $\forall f\in\mathscr{H}$.

Usually statement 3 is formulated by saying that $S_n^{-1}P_n\to S^{-1}$ in the strong operator topology, and statement 4 by saying that $S_n^{-1}P_n\to S^{-1}$ in the weak operator topology. That statements 3 and 4 are equivalent is a consequence of the fact that the involved operators are positive. Also, observe that statement 4 shows that the condition for the strong projection method to work can be formulated in a very similar way as the condition for the projection method to work, cf. Eq. (3).

3 Approximation of the Solution to a Moment Problem

We now consider a related question. Again let $\{f_i\}_{i\in I}$ be a frame for $\mathscr{H}$ and let $\{a_i\}_{i\in I}\subseteq\ell^2(I)$. We ask whether there exists $f\in\mathscr{H}$ such that

$$\langle f,f_i\rangle=a_i, \quad \forall i\in I.$$

A problem of this type is called a moment problem. For the general theory refer to Ref. 6. It is easy to construct examples where there is no solution f, but as shown in Ref. 7 there always exists a unique element in $\mathscr{H}$ minimizing $\Sigma_{i\in I}|a_i-\langle f,f_i\rangle|^2$; this element is $f=\Sigma_{i\in I}a_iS^{-1}f_i$. We call $\Sigma_{i\in I}a_iS^{-1}f_i$ the best approximation solution (BAS) to the moment problem.

Corresponding to a subset $\{a_i\}_{i\in I_n}$, the unique element in $\mathscr{H}_n$ minimizing $\Sigma_{i\in I_n}|a_i-\langle f,f_i\rangle|^2$ is $\Sigma_{i\in I_n}a_iS_n^{-1}f_i$. We say that $\Sigma_{i\in I_n}a_iS_n^{-1}f_i$ is the BAS to the truncated moment problem. In analogy with the preceding, we would like to find conditions implying that

$$\sum_{i\in I_n}a_iS_n^{-1}f_i\to\sum_{i\in I}a_iS^{-1}f_i \quad \text{for } n\to\infty, \quad \forall\{a_i\}_{i\in I}\in\ell^2(I). \tag{5}$$

Observe the connection to the projection method: if Eq. (5) is satisfied (or just for all sequences with 1 in one entry, otherwise 0), then $S_n^{-1}f_i \to S^{-1}f_i$ for $n\to\infty$, $\forall i$, so the projection method works.

Zwaan[8] has shown that Eq. (5) is satisfied if $\{f_i\}_{i\in I}$ is a Riesz basis. Here we prove that Eq. (5) is satisfied if and only if $\{f_i\}_{i\in I}$ is a conditional Riesz frames. We need the operator (sometimes called the preframe operator)

$$T:\ell^2(I)\to\mathcal{H},\quad T\{c_i\}_{i\in I}=\sum_{i\in I} c_i f_i .$$

We denote the kernel of T by N_T, and T is bounded and the adjoint operator $T^*:\mathcal{H}\to\ell^2(I)$ is given by $T^*f=\{\langle f,f_i\rangle\}_{i\in I}$.

Theorem 4. Let $\{f_i\}_{i\in I}$ be a frame. Then the following statements are equivalent:

1. $\Sigma_{i\in I_n} a_i S_n^{-1} f_i \to \Sigma_{i\in I} a_i S^{-1} f_i$ for $n\to\infty$, $\forall\{a_i\}_{i\in I}\in\ell^2(I)$.
2. $S_n^{-1}\Sigma_{i\in I_n} b_i f_i \to 0$ for $n\to\infty$ for all $\{b_i\}_{i\in I}\in\ell^2(I)$ such that $\Sigma_{i\in I} b_i f_i=0$.
3. $\{f_i\}_{i\in I}$ is a conditional Riesz frame.

Proof. First, statement 1⇔statement 2. Since $\ell^2(I)$ is the orthogonal sum of the range of T^* and the kernel of T, we can write any sequence $\{a_i\}_{i\in I}\in\ell^2(I)$ as

$$\{a_i\}_{i\in I}=\{\langle g,f_i\rangle\}_{i\in I}+\{b_i\}_{i\in I},$$

for some $g\in\mathcal{H}$, $\{b_i\}_{i\in I}\in N_T$. Now, the BAS to the truncated moment problem is

$$\sum_{i\in I_n} a_i S_n^{-1} f_i=\sum_{i\in I_n}\langle g,f_i\rangle S_n^{-1}f_i+\sum_{i\in I_n} b_i S_n^{-1} f_i$$
$$=P_n g+S_n^{-1}\sum_{i\in I_n} b_i f_i .$$

The BAS to the moment problem is

$$\sum_{i\in I} a_i S^{-1} f_i=\sum_{i\in I}\langle g,f_i\rangle S^{-1}f_i+S^{-1}\sum_{i\in I} b_i f_i=g,$$

from which the result follows.

To show statement 1⇒statement 3, define $Q:\ell^2(I)\to\mathcal{H}$ by

$$Q\{a_i\}_{i\in I}=\sum_{i\in I} a_i S^{-1} f_i ,$$

and define $Q_n:\ell^2(I)\to\mathcal{H}$ by

$$Q_n\{a_i\}_{i\in I}=\sum_{i\in I_n} a_i S_n^{-1} f_i .$$

Statement 1 states that $Q_n\to Q$ in the strong operator topology. Hence $\sup_n\|Q_n\|<\infty$. But, $Q_n^* f=\{\langle f,S_n^{-1}f_i\rangle\}_{i\in I_n}$, and $P_n f=\Sigma_{i\in I_n}\langle f,S_n^{-1}f_i\rangle f_i$, implies

$$S_n^{-1}P_n f=\sum_{i\in I_n}\langle f,S_n^{-1}f_i\rangle S_n^{-1}f_i=Q_nQ_n^* f.$$

Hence,

$$\sup_n\|S_n^{-1}\|=\sup_n\|S_n^{-1}P_n\|=\sup_n\|Q_nQ_n^*\|\le\sup_n\|Q_n\|^2<\infty.$$

The fact that statement 3⇒statement 2 follows from

$$\left\|S_n^{-1}\sum_{i\in I_n} b_i f_i\right\|\le\|S_n^{-1}\|\cdot\left\|\sum_{i\in I_n} b_i f_i\right\|.$$

4 Conditional Riesz Frames

By introducing the distance between an element $f\in\mathcal{H}$ and a subspace $V\subseteq\mathcal{H}$ we can find a condition implying that $\{f_i\}_{i=1}^\infty$ is a conditional Riesz frame. Let

$$\mathrm{dist}(f,V)=\inf_{g\in V}\|f-g\|.$$

Proposition 1. Let $\{f_i\}_{i=1}^\infty\subseteq\mathcal{H}$ satisfy the upper frame condition with bound B and suppose that there exists $\epsilon>0$ such that for all $n\in N$,

$$|\langle f,f_{n+1}\rangle|\ge\epsilon\cdot\mathrm{dist}(f,\mathcal{H}_n),\quad\forall f\in\mathcal{H}_{n+1}.$$

Choose ϵ so small that $\|f_1\|\ge\epsilon$. Then $\{f_i\}_{i=1}^\infty$ is a conditional Riesz frame with bounds ϵ^2, B.

Proof. We only need to be concerned about the lower bound. First, observe that $\mathrm{dist}(f,\mathcal{H}_n)=\|f-P_n f\|$. Let us now prove the existence of a common lower bound by induction. The assumption $\|f_1\|\ge\epsilon$ immediately implies that $\{f_1\}$ is a frame for span$\{f_1\}$ with lower bound ϵ^2, so assume that $\{f_i\}_{i=1}^k$ has the same lower bound for some $k>1$. For $f\in\mathcal{H}_{k+1}$ we then have

$$\sum_{i=1}^{k+1}|\langle f,f_i\rangle|^2=\sum_{i=1}^{k}|\langle P_k f,f_i\rangle|^2+|\langle f,f_{k+1}\rangle|^2$$
$$\ge\epsilon^2\|P_k f\|^2+\epsilon^2\ \mathrm{dist}\ (f,\mathcal{H}_k)$$
$$=\epsilon^2\|P_k f\|^2+\epsilon^2\|(I-P_k)f\|^2$$
$$=\epsilon^2\|f\|^2,$$

as desired. Now we finish by showing that ϵ^2 also is a lower bound for $\{f_i\}_{i=1}^\infty$. But if $f\in\mathcal{H}$, then

$$\epsilon^2\|P_n f\|^2\le\sum_{i=1}^{n}|\langle P_n f,f_i\rangle|^2\le\sum_{i=1}^{\infty}|\langle f,f_i\rangle|^2,$$

from which the result follows by letting $n\to\infty$.

5 Examples

The most important frames are Weyl-Heisenberg frames and wavelet frames. A Weyl-Heisenberg frame with lattice parameters $a,b>0$ is a family $\{f_{m,n}(x)\}_{(m,n)\in Z^2}$ of functions of the form

$$f_{m,n}(x):=\exp(i2\pi mbx)f(x-na),$$

where $f\in L^2(R)$ is fixed. A wavelet frame consists of scaled and translated versions of a single function $f\in L^2(R)$; here one choose parameters $a>1$, $b>0$ and define

$$\{f_{m,n}(x)\}_{(m,n)\in Z^2}=\left\{\frac{1}{|a|^{n/2}}f\left(\frac{x}{a^n}-mb\right)\right\}_{(m,n)\in Z^2}.$$

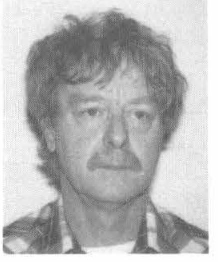
There exist easy verifiable conditions implying that such families are frames.[9] The approximation questions considered here are important for such frames, so it would be interesting to answer the question When is a wavelet frame or a Weyl-Heisenberg frame a (conditional) Riesz frame?

This question seems to be difficult. At present, we know the answer only in some special cases, namely,

1. It is well known that a Weyl-Heisenberg frame is in fact a Riesz basis if $ab=1$. Therefore we have a Riesz frame in this case.
2. Consider the Gaussian $g(x)=e^{-x^2}$. The Weyl-Heisenberg family corresponding to this function and the lattice parameters $a=1$ and $b=1/2$ is a frame. But the subfamily $\{g_{m,n}\}_{m\in 2Z,n\in Z}$ is not a frame. Thus $\{g_{m,n}\}_{(m,n)\in Z^2}$ is not a Riesz frame.

References

1. O. Christensen, "Frames and the projection method," *Appl. Comp. Harm. Anal.* **1**, 50–53 (1993).
2. O. Christensen, "Frames containing a Riesz basis and approximation of the frame coefficients using finite dimensional methods," *J. Math. Anal. Appl.* **199**, 256–270 (1996).
3. P. G. Casazza and O. Christensen, "Hilbert space frames containing a Riesz basis and Banach spaces which have no subspace isomorphic to c_0," *J. Math. Anal. Appl.* **202**, 940–950 (1996).
4. O. Christensen, "Frames and pseudo-inverses," *J. Math. Anal. Appl.* **195**, 401–414 (1995).
5. P. G. Casazza and O. Christensen, "Riesz frames and approximation of the frame coefficients," submitted for publication.
6. R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York (1980).
7. O. Christensen, "Moment problems and stability results for frames with applications to irregular sampling and Gabor frames," *Appl. Comp. Harm. Anal.* **3**, 82–86 (1996).
8. M. Zwaan, "Approximation of the solution to the moment problem in a Hilbert space," *Numer. Funct. Anal. Optim.* **11**, 601–608 (1990).
9. C. Heil and D. Walnut, "Continuous and discrete wavelet transforms," *SIAM Rev.* **31**, 628–666 (1989).



Peter Casazza received his PhD in 1972 from the University of Iowa and has been at the University of Missouri since 1983. His research is in functional analysis. He has published over 50 papers including a memoir of the AMS and a Springer Lecture Note and has delivered 220 invited addresses in over 100 universities and institutes in 35 countries of the world. He was a National Academy of Sciences Fellow to the former USSR for a year and spent a year at the Hebrew University of Jerusalem, a year at Odense University, Denmark, and 2 years at the University of Cambridge, England. His research has been supported for the last 18 years by the National Science Foundation, the Danish Natural Sciences Research Foundation, and the University of Missouri Research Board.

Ole Christensen received his MS in mathematics at the University of Aarhus, Denmark, in 1990, and his PhD in mathematics from the University of Vienna, Austria, in 1993. He was with the University of Agriculture, Vienna, and is now an assistant professor at the Technical University of Denmark.