QUEUE MODELS WITH MARKOV PROCESSES

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Abstract

In this research paper two classes of Markov processes have been discussed that naturally arise in queueing models with multiple waiting lines, and it has been concluded that a detailed analysis is possible for that.
Keywords: - Markov processes, Queueing System, Poisson distribution, quasi-birth-and-death processes.

1.1 INTRODUCTION

Many queueing systems can be modeled by a Markov process, the state space of which is given by a semi-infinite strip of states (m, n) where m ranges from 0 to s and n from 0 to ∞. In these systems, typically m denotes the state of the service facility or of the arrival process, and n denotes the number of jobs waiting in the system. m and n could also denote the number of type- 1 and type –2 customers, the waiting room for type – 1customers being finite. Often, there is a threshold N, such that the transition rates out of state (m, n) do not depend on n when $n \geq N$.

1.2 MARKOV PROCESS ON A SEMI-INFINITE STRIP

The equilibrium probabilities $p_{m,n}$ can be determined using three alternative methods. They are briefly described below. A comparison of the three methods can be found in the survey work by Mitrani [16].

METHOD 1: THE MATRIX-GEOMETRIC METHOD

In this approach the row vectors of equilibrium probabilities $\bar{p}_n = (p_{0,n}, p_{1,n}, \ldots, p_{s,n})$ are expressed as

$$\bar{p}_n = \bar{p}_N R^{n-N}, \qquad n \geq N, \qquad \ldots (1.1)$$

where the so-called rate matrix R is the minimal non- negative solution of a non-linear matrix equation. For the special class of quasi-birth-and-death processes, i.e., processes where for each state (m, n) outgoing transitions are restricted to state (k, l) with $|l-n| \leq 1$, Ramaswami and Latouche [14] have developed a highly efficient algorithm to solve the matrix equation for the rate matrix R.

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METHOD 2: THE GENERATING FUNCTION METHOD

This approach uses generating functions to solve the set of equilibrium equations. By introducing the vector generating function $\overline{g}(z) = \{g_0(z), g_1(z), \ldots, g_x(z)\}$ where

$$g_m(z) = \sum_{n=X}^{\infty} p(m,n)\, z^n ,$$

the equilibrium equations can be transformed to a matrix equation for $\overline{g}(z)$ of the form

$$\overline{g}(z)\, A(z) \doteq \overline{b}(z)$$

The vector $\overline{b}(z)$ involves a number of unknown boundary probabilities. These probabilities are determined by exploiting the zeros of the determinant of the matrix A(z) inside or no the unit circle. The generating function approach is, for example, used by Mitrani and Avi-Itzhak [17] to analyze the M/M/s queue with service interruptions.

METHOD 3 : THE SPECTRAL EXPANSION METHOD

This method is based on reducing the equilibrium equations to a vector difference equation with constant coefficients, the solution of which can be expressed in terms of eigenvalues and eigenvectors of the associated characteristic polynomial; i.e.

$$\overline{p}_n = \sum_{i=0}^{s} C_i \overline{y}_i x_i^{n-N} , \qquad n \geq N, \qquad \ldots (1.2)$$

where the geometric factor x_i are the s + 1 eigenvalues inside the unit circle, the vector $\overline{y}_i$ are the corresponding eigenvectors and the coefficients C_i follow from the boundary equations and the normalization equation. The difficult step in this approach is the computation of the eigenvalues, which are the s + 1 roots inside the unit circle of a determinant equation. A direct approach of finding these roots is inefficient for large s and therefore not recommended. However, in special cases, particular properties may be exploited to simplify the eigenvalue problem. Elwalid et al., [10]. For a class of systems with rates linear in m or s – m, Adan and Resing [2] showed that the roots can be determined very efficiently. They use a generating function technique to reduce the single equation for the s + 1 roots inside the unit circle to s + 1 equations for a single root in the interval (–1, 1). This considerably simplifies the determination of the roots, also because the computations can now be restricted to the real domain. Queueing models included in this class of systems are, e.g., the multi-server queue with poisson arrivals and E_2, H_2 or C_2 distributed service times, the M/M/s queue with service interruptions, the $\sum$ 1PP/M/1 queue and a multi-server queue with locking.

Remark 1.1 Result (1.2) may be linked to the (modified) matrix-geometric representation (1.1) of the equilibrium distribution. Clearly, when R is diagonalizable, the factors x_m in the form (1.2) are the eigenvalue of R and the row vector y_m are the associated row eigenvectors.

Remark 1.2 De Smit [18] surveys the application of matrix Wiener-Hopf factorizations to the analysis of waiting times in multidimensional queues. The class of models that this method can handle shows considerable overlap with the class which can be solved by matrix-geometric methods. One of De Smit's key examples is a semi-Markov queue. Dukhovny [9] also considers semi- Markov queues. He surveys the application of vector Riemann-Hilbert boundary value problems to the analysis of queue lengths in multidimensional queues.

1.2 MARKOV PROCESS ON THE LATTICE IN THE FIRST QUADRANT:

An interesting class of two-dimensional random walks on the lattice in the first quadrant has been studied by Fayolle, King and Mitrani [12]. They assume that there exist positive integers N_1 and N_2, such that the transition rates out of state (n_1, n_2) do not depend on n_1 for $n_1 \geq N_1$ and not on n_2 for $n_2 \geq N_2$. One example is an M/M/1 queue with two classes of customers and a restricted processor sharing discipline : Up to N_1 jobs of class 1 and up to N_2 jobs of class 2 are allowed to share the processor at any time, and the remaining jobs must wait. Fayolle et al. [12] reduce the determination of the bivariate generating function to the problem of solving a Riemann-Hilbert boundary value problem on a circle.

A quite general class of two-dimensional random walks on the lattice in the first quadrant has been analysed by Cohen and Boxma [8], where the solution is again obtained via transformation to a boundary value problem. One-step transitions to the West, South-West and South can only go to the nearest neighbor, but one-stop transitions in other directions may be more general. The functional equation for the unknown generating function f (x, y) of its equilibrium solution is of the following type : For $|x| \leq 1, |y| \leq 1$,

$$K(x, y) f(x, y) = A_{10}(x, y) f(x, 0) + A_{01}(x, y) f(0, y) + A_{00}(x, y) f(0, 0) + B(x, y) \quad \ldots (1.3)$$

The kernel K (x, y) contains all the information concerning the structure of the random walk in the interior of its state space. The boundedness of the probability generating function f (x, y) for $|x| \leq 1, |y| \leq 1$ again leads to an inspection of the zeros of the kernel, K (x, y) in this product of unit circles. For each of those zeros, the right-hand side of (1.3) should be zero. Furthermore f (x, 0) should be analytic in x for $|x| < 1$ and continuous in x for $|x| \leq 1$, similarly for f (0, y). The structure of the problem of determining f (x, 0) and f (0, y) that satisfy these conditions resembles that of a Riemann-type boundary value problem: The determination of analytic functions in prescribed domains, these functions moreover satisfying a linear relation. Cohen and Boxma [8] the above problem is indeed transformed into a Reimann-type boundary value problem. Using the extensive theory of Riemann-type boundary value problems the above-described two-dimensional random walk is in principal solved. The solution does involve the determination of some conformal mappings, which can be accomplished via the solution of singular integral equations. In most cases, this requires numerical analysis.

The above-sketched approach is surveyed in much more detail by Cohen [4]. From a numerical point of view, an interesting approach is also the transformation of the functional equation into a Fredholm integral equation by Cohen [4]; standard techniques are available to solve such an integral equation numerically.

Various matters simplify when one restricts oneself, within the class of random walks on the lattice in the first quadrant, to the subclass of nearest-neighbour random walks (only transitions to immediate neighbours may occur). In that case, the kernel K (x, y) is a biquadratic function of x and y. A pioneering study of such random walks is the one of Malyshev [15]. Together with Fayolle and Iasnogorodski [11], he has recently developed a new analytic approach for nearest-neighbour random walks in the quarter plane. Like in the above approach, they consider the (elliptic) curve $K(x, y) = 0$. A key idea of them is to use Galois automorphisms on this algebraic curve. They prove that the unknown functions f (x, 0) and f (0, y), while in general not being meromorphic functions, can be 'lifted' as meromorphic functions onto the 'universal covering' of some Riemann surface that corresponds to the algebraic curve $K(x, y) = 0$. Cohen has made several important contributions to the theory of nearest – neighbour random walks. In the monograph [5], he extensively discusses ergodicity conditions, entrance times into the boundaries, and entrance points. He relates the zero-tuples of the kernel to the distributions of the entrance times into the boundary points of the state space, by a very elegant identity. When only a few one-step transitions are allowed in a nearest-neighbour random walk, further simplifications may occur.

Cohen [6] considers the semi-homogeneous nearest-neighbour random walk without transitions to the East, North-East and East. This is the class of random walks studied by Adan [1] via the compensation method. Cohen proves that the bivariate generating function of the stationary distribution of such two-dimensional random walks in the first quadrant can be represented by meromorphic functions. Subsequently he exposes the construction of those meromorphic functions; this construction is based on the iterative calculation of poles and their residues. Cohen [6] observes that the bivariate generating function for this class of random walks may also be obtained using the boundary value method, even when one-step transitions to the North, East and North-East are allowed; but he remarks that when such transitions are excluded, then the construction of the meromorphic function via the iterative calculation of poles and residues is simpler because it avoids the explicit calculation of a conformal mapping.

Cohen [7] considers a nearest – neighbour random walk with transitions to the South-West, North and East. The generating functions f (x, 0) and f (0, y) are here not meromorphic functions. However, the bivariate generating function f (x, y) is shown to be fairly simple algebraic function that can be explicitly determined. Cohen uses the uniformization technique of Flatto and Hahn [13] and Wright [19], who studies a fork-join queue. The underlying model of this random walk study actually is a queueing model with one server, two Poisson classes of customers and 'paired service': As soon as service has been completed, a new service is started if there are customers present. In general, a couple of customers of different type is simultaneously served (after which they leave simultaneously). If only customers of one type are present after a service completion, then one customer of that type is served. If a service leaves the system empty, then the server starts serving as soon as a customer has arrived. Cohen [7] assumed exponential service time distributions. Blanc [3] allows a general service time distribution that is the same in each of the above cases. He obtains several performance measures, including the joint queue-length distribution at service completion epochs and at an arbitrary time. He accomplishes this by formulating the problem as Riemann-Hilbert or a Hilbert boundary value problem. Its solution requires a conformal mapping that in general cannot be obtained explicitly and must determined numerically.

CONCLUSION: - In this work Markov processes have been considered on a semi-infinite strip. Three methods for determining the equilibrium probabilities are briefly described and next section is concerned with tractable two-dimensional random walks on the lattice in the first quadrant.

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