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# Initial Coefficient Bounds Analysis for Novel Subclasses of Bi-Univalent Functions Linked with Lucas-Balancing Polynomials

**Sondekola Rudra Swamy [1], Daniel Breaz [2], Kala Venugopal [1], Mamatha Paduvalapattana Kempegowda [3], Luminita-Ioana Cotîrlă [4,*] and Eleonora Rapeanu [5]**

[1] Department of Information Science and Engineering, Acharya Institute of Technology, Bengaluru 560 107, Karnataka, India; swamy2704@acharya.ac.in (S.R.S.); kalavenugopal@acharya.ac.in (K.V.)
[2] Department of Mathematics, University of Alba Iulia, 510009 Alba-Iulia, Romania; dbreaz@uab.ro
[3] School of Mathematics, Alliance University, Central Campus, Chikkahadage Cross, Chandapura-Anekal Main Road, Bengaluru 562 106, India; mamatha.ravi@alliance.edu.in
[4] Department of Mathematics, Tehnical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania
[5] Department of Mathematics, "Mircea cel Batran", Naval Academy, 900218 Constanta, Romania; eleonora.rapeanu@anmb.ro
* Correspondence: luminita.cotirla@math.utcluj.ro

**Abstract:** We investigate some subclasses of regular and bi-univalent functions in the open unit disk that are associated with Lucas-Balancing polynomials in this work. For functions that belong to these subclasses, we obtain upper bounds on their initial coefficients. The Fekete–Szegö problem is also discussed. Along with presenting some new results, we also explore pertinent connections to earlier findings.

**Keywords:** bi-univalent functions; regular functions; subordination; Lucas-balancing polynomials

**MSC:** 30C45; 33C45

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## 1. Introduction

An open unit disk $\{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$ is represented by $\mathfrak{U}$, where $\mathbb{C}$ signifies a set of complex numbers. The sets of real and natural numbers are $\mathbb{R}$ and $\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \backslash \{0\}$, respectively. The set of regular functions $g$ in $\mathfrak{U}$ is denoted by $\mathcal{A}$ with the following form:

$$g(\varsigma) = \varsigma + \sum_{j=2}^{\infty} d_j \varsigma^j, \qquad (\varsigma \in \mathfrak{U}), \tag{1}$$

where $g(0) = g'(0) - 1 = 0$, and $\mathcal{S}$ is a subset of $\mathcal{A}$ that is made up of univalent functions in $\mathfrak{U}$. In accordance with Koebe's result (see [1]), every function $g$ in $\mathcal{S}$ has an inverse, which is given by

$$\hbar(w) = g^{-1}(w) = w - d_2 w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2 d_3 + d_4)w^4 + \ldots, \tag{2}$$

satisfying $\varsigma = \hbar(g(\varsigma))$ and $w = g(\hbar(w))$, $|w| < r_0(g)$, $1/4 \leq r_0(g)$, $\varsigma$, $w \in \mathfrak{U}$.

If $g$ and $g^{-1} = \hbar$ are both univalent in $\mathfrak{U}$ and $\mathfrak{U} \subset g(\mathfrak{U})$, then a function $g$ of $\mathcal{A}$ is bi-univalent in $\mathfrak{U}$. $\sigma$ represents the set of bi-univalent functions in $\mathfrak{U}$ that are identified by (1). $\frac{1}{2} log\left(\frac{1+\varsigma}{1-\varsigma}\right)$, $\frac{\varsigma}{1-\varsigma}$, and $-log(1-\varsigma)$ are few functions in the $\sigma$ family. However, the Koebe function is not a member of the $\sigma$ family. Functions $\varsigma - \frac{\varsigma^2}{2}$ and $\frac{\varsigma}{1-\varsigma^2}$, which are members of the $\mathcal{S}$ family, are not part of the $\sigma$ family.

Studies pertaining to coefficients for members of the $\sigma$ family were initiated in the 1970s. Lewin [2] stated that $|d_2| < 1.51$ for elements of $\sigma$ after examining the $\sigma$ family. It was demonstrated in [3] that for members of $\sigma$, $|d_2| < \sqrt{2}$. Tan [4] subsequently discovered coefficient-related studies for functions $\in \sigma$. In [5], the authors examined two classical subfamilies of $\sigma$. The trend over the past twenty years has been to investigate the coefficient-related estimates for elements of particular subfamilies of $\sigma$, as evidenced by papers [6–10].

The current emphasis is on functions that are subordinate to known special polynomials and belong to particular $\sigma$ subfamilies. Coefficient estimates and Fekete–Szegö functional $|d_3 - \xi d_2^2|$ for members of certain subfamilies of $\sigma$ subordinate to a known special polynomial have been found by a number of researchers. For more information on these, see [11–14]. One particular kind of these polynomials that has drawn attention recently from researchers are the Lucas-Balancing polynomials.

The Balancing numbers, denoted by $C_j$, satisfy the recurrence relation $C_{j+1} = 6C_j - C_{j-1}, j \geq 1$ with $C_0 = 1$ and $C_1 = 1$ (see [15]). The sequence $B_j = 8C_j^2 + 1, j \geq 1$ is called a Lucas-Balancing number. It satisfies the recurrence relation $B_{j+1} = 6B_j - B_{j-1}, j \geq 1$ with $B_0 = 1$ and $B_1 = 3$. These numbers have been extensively studied in the articles [16–22]. Balancing polynomials, denoted by $C_j(\varkappa), j \geq 0$, and Lucas-Balancing polynomials, denoted by $B_j(\varkappa), j \geq 0$, are natural extensions of Balancing numbers and Lucas-Balancing numbers, respectively. Balancing polynomials [23] are recursively defined by

$$C_j(\varkappa) = 6\varkappa C_{j-1}(\varkappa) - C_{j-2}(\varkappa), j \geq 2,$$

with $C_0(\varkappa) = 0$ and $C_1(\varkappa) = 1$, where $\varkappa \in \mathbb{C}$. The first few polynomials are $C_2(\varkappa) = 6\varkappa$ and $C_3(\varkappa) = 36\varkappa^2 - 1, C_4(\varkappa) = 216\varkappa^3 - 12\varkappa, \ldots$.

The Lucas-Balancing polynomials $\mathsf{B}_j(\varkappa), j \geq 0$ with $\varkappa \in \mathbb{C}$ is defined in [23]. The following is a recursive definition for these polynomials:

$$\mathsf{B}_j(\varkappa) = 6\varkappa \mathsf{B}_{j-1}(\varkappa) - \mathsf{B}_{j-2}(\varkappa) \quad \textit{with} \quad \mathsf{B}_0(\varkappa) = 1, \ \mathsf{B}_1(\varkappa) = 3\varkappa, \tag{3}$$

where $j \in \mathbb{N}\backslash\{1\}$ and $\varkappa \in \mathbb{C}$. $\mathsf{B}_2(\varkappa) = 18\varkappa^2 - 1$ and $\mathsf{B}_3(\varkappa) = 108\varkappa^3 - 9\varkappa$ are evident from (3). For further details on this field, we refer researchers to [24–26]. As stated in [17], the below-mentioned $\mathsf{B}(\varkappa, \varsigma)$ represents the generating function of the Lucas-Balancing polynomials.

$$\mathsf{B}(\varkappa, \varsigma) := \sum_{j=0}^{\infty} \mathsf{B}_j(\varkappa)\varsigma^j = \frac{1 - 3\varkappa\varsigma}{1 - 6\varkappa\varsigma + \varsigma^2}, \tag{4}$$

where $\varkappa \in [-1, 1]$ and $\varsigma \in \mathbb{C}$.

For $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{A}$ regular in $\mathfrak{U}$, $\mathfrak{z}_1$ is subordinate to $\mathfrak{z}_2$, if there is a Schwartz function $\psi(\varsigma)$ that is regular in $\mathfrak{U}$ with $\psi(0) = 0$ and $|\psi(\varsigma)| < 1$, such that $\mathfrak{z}_1(\varsigma) = \mathfrak{z}_2(\psi(\varsigma))$, $\varsigma \in \mathfrak{U}$. This subordination is symbolized as $\mathfrak{z}_1 \prec \mathfrak{z}_2$ or $\mathfrak{z}_1(\varsigma) \prec \mathfrak{z}_2(\varsigma)$, $(\varsigma \in \mathfrak{U})$. In this case, if $\mathfrak{z}_2 \in \mathcal{S}$, then

$$\mathfrak{z}_1(\varsigma) \prec \mathfrak{z}_2(\varsigma) \ \Leftrightarrow \ \mathfrak{z}_1(0) = \mathfrak{z}_2(0) \quad \textit{and} \quad \mathfrak{z}_1(\mathfrak{U}) \subset \mathfrak{z}_2(\mathfrak{U}).$$

Inspired by the previously mentioned patterns in problems involving coefficients and the Fekete–Szegö functional [27] on specific subclasses of $\sigma$, we introduce some novel subfamilies of $\sigma$ that are subordinate to Lucas-Balancing polynomials $\mathsf{H}_j(\varkappa)$ as in (3), specifically $\mathfrak{T}_\sigma^\tau(\beta, \nu, \varkappa)$, $\mathfrak{Y}_\sigma^\tau(\beta, \gamma, \mu, \varkappa)$, $\mathfrak{W}_\sigma^\tau(\beta, \nu, \varkappa)$, and $\mathfrak{O}_\sigma^\tau(\beta, \gamma, \mu, \varkappa)$.

Unless otherwise specified, this paper uses the inverse function $g^{-1}(w) = \hbar(w)$ as in (2) and the generating function $\mathsf{B}(\varkappa, \varsigma)$ as in (4).

**Definition 1.** *A function $g \in \sigma$ is said to be in the class $\mathfrak{T}_\sigma^\tau(\beta, \nu, \varkappa)$, $\tau \geq 1$, $0 \leq \nu \leq 1, \beta \in \mathbb{C}\backslash\{0\}$, and $\frac{1}{2} < \varkappa \leq 1$, if*

$$1 + \frac{1}{\beta}\left(\nu\left(\frac{[(\varsigma g'(\varsigma))']^\tau}{g'(\varsigma)}\right) + (1-\nu)\left(\frac{\varsigma(g'(\varsigma))^\tau}{g(\varsigma)}\right) - 1\right) \prec \mathsf{B}(\varkappa, \varsigma), \ \varsigma \in \mathfrak{U}, \tag{5}$$

*and*

$$1 + \frac{1}{\beta}\left(\nu\left(\frac{[(w\hbar'(w))']^\tau}{\hbar'(w)}\right) + (1-\nu)\left(\frac{w(\hbar'(w))^\tau}{\hbar(w)}\right) - 1\right) \prec \mathsf{B}(\varkappa, w), \ w \in \mathfrak{U}. \tag{6}$$

For specific choices of $\nu$ in the class $\mathfrak{T}_{\sigma}^{\tau}(\beta,\gamma,\mu,\varkappa)$, we obtain the following subfamilies of $\sigma$:

1. A function $g \in \sigma$ is in the class $\mathfrak{P}_{\sigma}^{\tau}(\beta,\varkappa) \equiv \mathfrak{T}_{\sigma}^{\tau}(\beta,0,\varkappa)$, $\tau \geq 1, \beta \in \mathbb{C}\backslash\{0\}$, *and* $\frac{1}{2} < \varkappa \leq 1$, if

$$1+\frac{1}{\beta}\left(\left(\frac{\varsigma(g'(\varsigma))^{\tau}}{g(\varsigma)}\right)-1\right) \prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma \in \mathfrak{U}, \tag{7}$$

and

$$1+\frac{1}{\beta}\left(\left(\frac{w(\hbar'(w))^{\tau}}{\hbar(w)}\right)-1\right) \prec \mathsf{B}(\varkappa,w),\ w \in \mathfrak{U}. \tag{8}$$

2. A function $g \in \sigma$ is in the class $\mathfrak{Q}_{\sigma}^{\tau}(\beta,\varkappa) \equiv \mathfrak{T}_{\sigma}^{\tau}(\beta,1,\varkappa)$, $\tau \geq 1, \beta \in \mathbb{C}\backslash\{0\}$, *and* $\frac{1}{2} < \varkappa \leq 1$, if

$$1+\frac{1}{\beta}\left(\left(\frac{[(\varsigma g'(\varsigma))']^{\tau}}{g'(\varsigma)}\right)-1\right) \prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma \in \mathfrak{U}, \tag{9}$$

and

$$1+\frac{1}{\beta}\left(\left(\frac{[(w\hbar'(w))']^{\tau}}{\hbar'(w)}\right)-1\right) \prec \mathsf{B}(\varkappa,w),\ w \in \mathfrak{U}. \tag{10}$$

**Definition 2.** *We say that* $g \in \mathfrak{Y}_{\sigma}^{\tau}(\beta,\gamma,\mu,\varkappa)$, *if the following subordinations hold:*

$$1+\frac{1}{\beta}\left(\frac{\mu\varsigma^2 g''(\varsigma)+\varsigma(g'(\varsigma))^{\tau}}{\gamma\varsigma g'(\varsigma)+(1-\gamma)g(\varsigma)}-1\right) \prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma \in \mathfrak{U} \tag{11}$$

*and*

$$1+\frac{1}{\beta}\left(\frac{\mu w^2 \hbar''(w)+\omega(\hbar'(w))^{\tau}}{\gamma w\hbar'(w)+(1-\gamma)\hbar(w)}-1\right) \prec \mathsf{B}(\varkappa,w),\ w \in \mathfrak{U}, \tag{12}$$

*where* $g \in \sigma$, $\tau \geq 1$, $0 \leq \gamma \leq 1$, $\mu \geq \gamma, \beta \in \mathbb{C}\backslash\{0\}$, *and* $\frac{1}{2} < \varkappa \leq 1$.

For specific choices of $\gamma$ and $\mu$ in $\mathfrak{Y}_{\sigma}^{\tau}(\beta,\gamma,\mu,\varkappa)$, we obtain the following subfamilies of $\sigma$:

1. $\mathfrak{Y}_{\sigma}^{\tau}(\beta,0,\mu,\varkappa) \equiv \mathfrak{K}_{\sigma}^{\tau}(\beta,\mu,\varkappa), \beta \in \mathbb{C}\backslash\{0\}, \tau \geq 1, \mu \geq 0$, *and* $\frac{1}{2} < \varkappa \leq 1$ is the class of functions $g \in \sigma$ satisfying

$$1+\frac{1}{\beta}\left(\frac{\mu\varsigma^2 g''(\varsigma)+\varsigma(g'(\varsigma))^{\tau}}{g(\varsigma)}-1\right) \prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma \in \mathfrak{U} \tag{13}$$

and

$$1+\frac{1}{\beta}\left(\frac{\mu w^2 \hbar''(w)+\omega(\hbar'(w))^{\tau}}{\hbar(w)}-1\right) \prec \mathsf{B}(\varkappa,w),\ w \in \mathfrak{U}. \tag{14}$$

2. $\mathfrak{Y}_{\sigma}^{\tau}(\beta,1,\mu,\varkappa) \equiv \mathfrak{J}_{\sigma}^{\tau}(\beta,\mu,\varkappa), \beta \in \mathbb{C}\backslash\{0\}, \tau \geq 1, \mu \geq 1$, *and* $\frac{1}{2} < \varkappa \leq 1$ is the class of functions $g \in \sigma$ satisfying

$$1+\frac{1}{\beta}\left((g'(\varsigma))^{\tau-1}+\mu\left(\frac{\varsigma g''(\varsigma)}{g'(\varsigma)}\right)-1\right) \prec \mathsf{B}(\varkappa,\varsigma), \varsigma \in \mathfrak{U} \tag{15}$$

and

$$1+\frac{1}{\beta}\left((\hbar'(w))^{\tau-1}+\mu\left(\frac{w\hbar''(w)}{\hbar'(w)}\right)-1\right) \prec \mathsf{B}(\varkappa,w), w \in \mathfrak{U}. \tag{16}$$

3. $\mathfrak{Y}_{\sigma}^{\tau}(\beta,\gamma,1,\varkappa) \equiv \mathfrak{L}_{\sigma}^{\tau}(\beta,\gamma,\varkappa), \tau \geq 1, \beta \in \mathbb{C}\backslash\{0\}, 0 \leq \gamma \leq 1$, *and* $\frac{1}{2} < \varkappa \leq 1$ is the class of functions $g \in \sigma$ satisfying

$$1+\frac{1}{\beta}\left(\frac{\varsigma^2 g''(\varsigma)+\varsigma(g'(\varsigma))^{\tau}}{\gamma\varsigma g'(\varsigma)+(1-\gamma)g(\varsigma)}-1\right) \prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma \in \mathfrak{U} \tag{17}$$

and

$$1+\frac{1}{\beta}\left(\frac{w^2\hbar''(w)+w(\hbar'(w))^\tau}{\gamma w\hbar'(w)+(1-\gamma)\hbar(w)}-1\right)\prec \mathsf{B}(\varkappa,w),\ \in \mathfrak{U}. \tag{18}$$

**Remark 1.** *(i)* $\mathfrak{L}_\sigma^\tau(\beta,0,\varkappa)\equiv\mathfrak{K}_\sigma^\tau(\beta,1,\varkappa)$. *(ii)* $\mathfrak{L}_\sigma^\tau(\beta,1,\varkappa)\equiv\mathfrak{I}_\sigma^\tau(\beta,1,\varkappa)$.

**Definition 3.** *A function $g\in\sigma$ is said to be in the class $\mathfrak{W}_\sigma^\tau(\beta,\nu,\varkappa)$, $\tau\geq 1$, $0\leq\nu\leq 1,\beta\in\mathbb{C}\backslash\{0\}$, and $\frac{1}{2}<\varkappa\leq 1$, if*

$$1+\frac{1}{\beta}\left(\nu\left(\frac{[(\varsigma g'(\varsigma))']^\tau}{g'(\varsigma)}\right)+(1-\nu)(g'(\varsigma))^\tau-1\right)\prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma\in\mathfrak{U}, \tag{19}$$

*and*

$$1+\frac{1}{\beta}\left(\nu\left(\frac{[(w\hbar'(w))']^\tau}{\hbar'(w)}\right)+(1-\nu)(\hbar'(w))^\tau-1\right)\prec \mathsf{B}(\varkappa,w),\ w\in\mathfrak{U}. \tag{20}$$

1. For $\nu=0$ in the class $\mathfrak{W}_\sigma^\tau(\beta,\nu,\varkappa)$, we obtain the class $\mathfrak{V}_\sigma^\tau(\beta,\varkappa)\equiv\mathfrak{W}_\sigma^\tau(\beta,0,\varkappa)$, $\tau\geq 1,\beta\in\mathbb{C}\backslash\{0\}$, *and* $\frac{1}{2}<\varkappa\leq 1$ where $g\in\sigma$ satisfies

$$1+\frac{1}{\beta}((g'(\varsigma))^\tau-1)\prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma\in\mathfrak{U}, \tag{21}$$

and

$$1+\frac{1}{\beta}((\hbar'(w))^\tau-1)\prec \mathsf{B}(\varkappa,w),\ w\in\mathfrak{U}. \tag{22}$$

2. For $\tau=1$ in the class $\mathfrak{W}_\sigma^\tau(\beta,\nu,\varkappa)$, we obtain the class $\mathfrak{H}_\sigma(\beta,\nu,\varkappa)\equiv\mathfrak{W}_\sigma^1(\beta,\nu,\varkappa)$ $0\leq\nu\leq 1,\beta\in\mathbb{C}\backslash\{0\}$, *and* $\frac{1}{2}<\varkappa\leq 1$, where $g\in\sigma$ satisfies

$$1+\frac{1}{\beta}\left(\nu\left(\frac{(\varsigma g'(\varsigma))'}{g'(\varsigma)}\right)+(1-\nu)(g'(\varsigma))-1\right)\prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma\in\mathfrak{U}, \tag{23}$$

and

$$1+\frac{1}{\beta}\left(\nu\left(\frac{(w\hbar'(w))'}{\hbar'(w)}\right)+(1-\nu)(\hbar'(w))-1\right)\prec \mathsf{B}(\varkappa,w),\ w\in\mathfrak{U}. \tag{24}$$

**Remark 2.** $\mathfrak{W}_\sigma^\tau(\beta,1,\varkappa)\equiv\mathfrak{Q}_\sigma^\tau(\beta,\varkappa)$.

**Definition 4.** *A function $g\in\sigma$ is said to be in the class $\mathfrak{O}_\sigma^\tau(\beta,\gamma,\mu,\varkappa)$, $\beta\in\mathbb{C}\backslash\{0\},\tau\geq 1$, $\mu\geq\gamma$, $0\leq\gamma\leq 1$, and $\frac{1}{2}<\varkappa\leq 1$, if*

$$1+\frac{1}{\beta}\left(\frac{\mu\varsigma^2 g''(\varsigma)+\varsigma(g'(\varsigma))^\tau}{\gamma\varsigma g'(\varsigma)+(1-\gamma)\varsigma}-1\right)\prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma\in\mathfrak{U} \tag{25}$$

*and*

$$1+\frac{1}{\beta}\left(\frac{\mu w^2\hbar''(w)+\omega(\hbar'(w))^\tau}{\gamma w\hbar'(w)+(1-\gamma)w}-1\right)\prec \mathsf{B}(\varkappa,w),\ w\in\mathfrak{U}. \tag{26}$$

For specific choices of $\mu$ and $\gamma$ in the family $\mathfrak{O}_\sigma^\tau(\beta,\gamma,\mu,\varkappa)$, we obtain the following subfamilies of $\sigma$:

1. $\mathfrak{O}_\sigma^\tau(\beta,0,\mu,\varkappa)\equiv\mathfrak{A}_\sigma^\tau(\beta,\mu,\varkappa),\tau\geq 1,\beta\in\mathbb{C}\backslash\{0\},\mu\geq 0$, *and* $\frac{1}{2}<\varkappa\leq 1$ is the class where $g\in\sigma$ satisfies

$$1+\frac{1}{\beta}(\mu\varsigma g''(\varsigma)+(g'(\varsigma))^\tau-1)\prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma\in\mathfrak{U} \tag{27}$$

and

$$1+\frac{1}{\beta}(\mu w\hbar''(w)+(\hbar'(w))^\tau-1)\prec \mathsf{B}(\varkappa,w),\ w\in\mathfrak{U}. \tag{28}$$

2. $\mathfrak{O}^1_\sigma(\beta,\gamma,1,\varkappa) \equiv \mathfrak{N}_\sigma(\beta,\gamma,\varkappa), \beta \in \mathbb{C}\backslash\{0\}, 0 \leq \gamma \leq 1$, *and* $\frac{1}{2} < \varkappa \leq 1$ is the class where $g \in \sigma$ satisfies

$$1 + \frac{1}{\beta}\left(\frac{\varsigma^2 g''(\varsigma) + \varsigma(g'(\varsigma))}{\gamma\varsigma g'(\varsigma) + (1-\gamma)\varsigma} - 1\right) \prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma \in \mathfrak{U} \tag{29}$$

and

$$1 + \frac{1}{\beta}\left(\frac{w^2\hbar''(w) + w(\hbar'(w))}{\gamma w\hbar'(w) + (1-\gamma)w} - 1\right) \prec \mathsf{B}(\varkappa,w),\ \in \mathfrak{U}. \tag{30}$$

3. $\mathfrak{O}^1_\sigma(\beta,\gamma,\gamma,\varkappa) \equiv \mathfrak{M}_\sigma(\beta,\gamma,\varkappa),\ \beta \in \mathbb{C}\backslash\{0\},\ 0 \leq \gamma \leq 1$, *and* $\frac{1}{2} < \varkappa \leq 1$ is the class where $g \in \sigma$ satisfies

$$1 + \frac{1}{\beta}\left(\frac{\gamma\varsigma^2 g''(\varsigma) + \varsigma g'(\varsigma)}{\gamma\varsigma g'(\varsigma) + (1-\gamma)\varsigma} - 1\right) \prec \mathsf{B}(\varkappa,\varsigma),\ \varsigma \in \mathfrak{U} \tag{31}$$

and

$$1 + \frac{1}{\beta}\left(\frac{\gamma w^2\hbar''(w) + \omega\hbar'(w)}{\gamma w\hbar'(w) + (1-\gamma)w} - 1\right) \prec \mathsf{B}(\varkappa,w),\ w \in \mathfrak{U}. \tag{32}$$

**Remark 3.** *$\mathfrak{Y}^\tau_\sigma(\beta,1,\mu,\varkappa) \equiv \mathfrak{O}^\tau_\sigma(\beta,1,\mu,\varkappa)$, as can be seen.*

For functions in the classes $\mathfrak{T}^\tau_\sigma(\beta,\mu,\varkappa)$, $\mathfrak{Y}^\tau_\sigma(\beta,\gamma,\mu,\varkappa)$, $\mathfrak{W}^\tau_\sigma(\beta,\nu,\varkappa)$, and $\mathfrak{O}^\tau_\sigma(\beta,\gamma,\mu,\varkappa)$, we find estimates for $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$ in Section 2. Presentations of intriguing outcomes of these classes and links to the established results are in Section 3.

## 2. Main Results

We find the coefficient-related estimates for $g \in \mathfrak{T}^\tau_\sigma(\beta,\nu,\varkappa)$, the class mentioned in Section 1.

**Theorem 1.** *Let $\beta \in \mathbb{C}\backslash\{0\}, \xi \in \mathbb{R}, \frac{1}{2} < \varkappa \leq 1, \tau \geq 1$, and $0 \leq \nu \leq 1$. If $g \in \sigma$ is assigned to the class $\mathfrak{T}^\tau_\sigma(\beta,\nu,\varkappa)$, then*

$$|d_2| \leq 3|\beta|\varkappa\sqrt{\frac{3\varkappa}{|9(\tau(\tau-1)(6\nu+1)+\nu+\tau^2)\beta\varkappa^2 - (2\tau-1)^2(\nu+1)^2(18\varkappa^2-1)|}}, \tag{33}$$

$$|d_3| \leq \frac{27|\beta|^2\varkappa^3}{[9(\tau(\tau-1)(6\nu+1)+\nu+\tau^2)\beta\varkappa^2 - (2\tau-1)^2(\nu+1)^2(18\varkappa^2-1)]} + \frac{3|\beta|\varkappa}{(3\tau-1)(2\nu+1)}. \tag{34}$$

*and*

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{3|\beta|\varkappa}{(3\tau-1)(2\nu+1)} & ; |1-\xi| \leq \mathcal{J} \\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|9(\tau(\tau-1)(6\nu+1)+\nu+\tau^2)\beta\varkappa^2-(2\tau-1)^2(\nu+1)^2(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{J}, \end{cases} \tag{35}$$

*where*

$$\mathcal{J} = \left|\frac{(\tau(\tau-1)(6\nu+1)+\nu+\tau^2)9\beta\varkappa^2 - (2\tau-1)^2(\nu+1)^2(18\varkappa^2-1)}{(3\tau-1)(2\nu+1)9\beta\varkappa^2}\right|. \tag{36}$$

**Proof.** Let $g \in \mathfrak{T}^\tau_\sigma(\beta,\nu,\varkappa)$. Then, from (5) and (6), we have

$$1 + \frac{1}{\beta}\left(\nu\left(\frac{[(\varsigma g'(\varsigma))']^\tau}{g'(\varsigma)}\right) + (1-\nu)\left(\frac{\varsigma(g'(\varsigma))^\tau}{g(\varsigma)}\right) - 1\right) = \mathsf{B}(\varkappa,\mathfrak{u}(\varsigma)),\ \varsigma \in \mathfrak{U} \tag{37}$$

and

$$1+\frac{1}{\beta}\left(\nu\left(\frac{[(w\hbar'(w))']^{\tau}}{\hbar'(w)}\right)+(1-\nu)\left(\frac{w(\hbar'(w))^{\tau}}{\hbar(w)}\right)-1\right)=\mathsf{B}(\varkappa,\mathfrak{v}(w)),\ w\in\mathfrak{U}. \tag{38}$$

where

$$\mathfrak{u}(\varsigma)=\sum_{j=1}^{\infty}\mathfrak{u}_j\varsigma^j,\ \ and\ \ \mathfrak{v}(w)=\sum_{j=1}^{\infty}\mathfrak{v}_jw^j \tag{39}$$

are some analytic functions with the property (see [1])

$$|\mathfrak{u}_i|\le 1,\ and\ |\mathfrak{v}_i|\le 1\,(i\in\mathbb{N}). \tag{40}$$

It is clear by using (4) and (37)–(39) that

$$1+\frac{1}{\beta}\left(\nu\left(\frac{[(\varsigma g'(\varsigma))']^{\tau}}{g'(\varsigma)}\right)+(1-\nu)\left(\frac{\varsigma(g'(\varsigma))^{\tau}}{g(\varsigma)}\right)-1\right)= \tag{41}$$

$$1+\mathsf{B}_1(\varkappa)\mathfrak{u}_1\varsigma+\left[\mathsf{B}_1(\varkappa)\mathfrak{u}_2+\mathsf{B}_2(\varkappa)\mathfrak{u}_1^2\right]\varsigma^2+\dots$$

and

$$1+\frac{1}{\beta}\left(\nu\left(\frac{[(w\hbar'(w))']^{\tau}}{\hbar'(w)}\right)+(1-\nu)\left(\frac{w(\hbar'(w))^{\tau}}{\hbar(w)}\right)-1\right)= \tag{42}$$

$$1+\mathsf{B}_1(\varkappa)\mathfrak{v}_1w+\left[\mathsf{B}_1(\varkappa)\mathfrak{v}_2+\mathsf{B}_2(\varkappa)\mathfrak{v}_1^2\right]w^2+\dots.$$

Therefore, by comparing the respective coefficients in (41) and (42), we arrive at

$$(2\tau-1)(\nu+1)d_2=\beta\mathsf{B}_1(\varkappa)\mathfrak{u}_1, \tag{43}$$

$$(3\tau-1)(2\nu+1)d_3+(2\tau^2-4\tau+1)(3\nu+1)d_2^2=\beta[\mathsf{B}_1(\varkappa)\mathfrak{u}_2+\mathsf{B}_2(\varkappa)\mathfrak{u}_1^2], \tag{44}$$

$$-(2\tau-1)(\nu+1)\,d_2=\beta\mathsf{B}_1(\varkappa)\mathfrak{v}_1 \tag{45}$$

and

$$(3\tau-1)(2\nu+1)(2d_2^2-d_3)+(2\tau^2-4\tau+1)(3\nu+1)d_2^2=\beta[\mathsf{B}_1(\varkappa)\mathfrak{v}_2+\mathsf{B}_2(\varkappa)\mathfrak{v}_1^2]. \tag{46}$$

From (43) and (45), we obtain

$$\mathfrak{u}_1=-\mathfrak{v}_1 \tag{47}$$

and also

$$2(2\tau-1)^2(\nu+1)^2d_2^2=\beta^2(\mathfrak{u}_1^2+\mathfrak{v}_1^2)(\mathsf{B}_1(\varkappa))^2. \tag{48}$$

In order to obtain the bound on $|d_2|$, we add (44) and (46).

$$2(\tau(\tau-1)(6\nu+1)+\nu+\tau^2)d_2^2=\beta\mathsf{B}_1(\varkappa)(\mathfrak{u}_2+\mathfrak{v}_2)+\beta\mathsf{B}_2(\varkappa)(\mathfrak{u}_1^2+\mathfrak{v}_1^2). \tag{49}$$

The value of $\mathfrak{m}_1^2+\mathfrak{n}_1^2$ from (48) is substituted into (49) to obtain

$$d_2^2=\frac{\beta^2\mathsf{B}_1^3(\varkappa)(\mathfrak{u}_2+\mathfrak{v}_2)}{2\left[(\tau(\tau-1)(6\nu+1)+\nu+\tau^2)\beta\mathsf{B}_1^2(\varkappa)-(\nu+1)^2(2\tau-1)^2\mathsf{B}_2(\varkappa)\right]}. \tag{50}$$

We obtain (33) by applying (40) for $\mathfrak{u}_2$ and $\mathfrak{v}_2$.

From (44), we subtract (46) to obtain the bound on $|d_3|$:

$$d_3=d_2^2+\frac{\beta\mathsf{B}_1(\varkappa)(\mathfrak{u}_2-\mathfrak{v}_2)}{2(2\nu+1)(3\tau-1)}. \tag{51}$$

This leads to the following inequality:

$$|d_3|\le|d_2|^2+\frac{|\beta\mathsf{B}_1(\varkappa)||\mathfrak{u}_2-\mathfrak{v}_2|}{2(2\nu+1)(3\tau-1)}. \tag{52}$$

We obtain (34) from (33) and (52) by applying (40) for $\mathfrak{u}_2$ and $\mathfrak{v}_2$.

Finally, we compute the bound on $|d_3 - \xi d_2^2|$ using the values of $d_2^2$ and $d_3$ from (50) and (51), respectively. Consequently, we have

$$|d_3 - \xi d_2^2| = \frac{|\beta||\mathsf{B}_1(\varkappa)|}{2}\left|\left(\frac{1}{(3\tau-1)(2\nu+1)} + \mathcal{F}(\xi,\varkappa)\right)\mathfrak{u}_2 - \left(\frac{1}{(3\tau-1)(2\nu+1)} - \mathcal{F}(\xi,\varkappa)\right)\mathfrak{v}_2\right|,$$

where

$$\mathcal{F}(\xi,\varkappa) = \frac{(1-\xi)\beta\mathsf{B}_1^2(\varkappa)}{\left[(\tau(\tau-1)(6\nu+1)+\nu+\tau^2)\beta\mathsf{B}_1^2(\varkappa) - (2\tau-1)^2(\nu+1^2)\mathsf{B}_2(\varkappa)\right]}.$$

Clearly

$$|d_3 - \xi d_2^2| \le \begin{cases} \frac{|\beta||\mathsf{B}_1(\varkappa)|}{(2\nu+1)(3\tau-1)} & ;\ |\mathcal{F}(\xi,\varkappa)| \le \frac{1}{(2\nu+1)(3\tau-1)} \\ |\beta||\mathsf{B}_1(\varkappa)||\mathcal{F}(\xi,\varkappa)| & ;\ |\mathcal{F}(\xi,\varkappa)| \ge \frac{1}{(2\nu+1)(3\tau-1)}. \end{cases} \tag{53}$$

We derive (35) from (53), where $\mathcal{J}$ is the same as in (36). □

**Remark 4.** *From Theorem 1, we can derive Theorems 1 and 2 in [28] by letting $\beta = 1$ and $\tau = 1$.*

For functions in the class $\mathfrak{Y}_\sigma^\tau(\beta,\gamma,\mu,\varkappa)$ that were discussed in Section 1, the coefficient estimates and Fekete–Szegö inequalities are given here.

**Theorem 2.** *Let $\beta \in \mathbb{C}\backslash\{0\}, \xi \in \mathbb{R}, \frac{1}{2} < \varkappa \le 1, \tau \ge 1,\ \mu \ge \gamma$, and $0 \le \gamma \le 1$. If $g \in \sigma$ is assigned to the class $\mathfrak{Y}_\sigma^\tau(\gamma,\mu,\varkappa)$, then*

$$|d_2| \le \sqrt{\frac{27|\beta|^2\varkappa^3}{|9(\gamma^2 - 2\gamma(\tau+\mu) + \tau(2\tau-1) + 4\mu)\beta\varkappa^2 - (2(\tau+\mu)-\gamma-1)^2(18\varkappa^2-1)|}}, \tag{54}$$

$$|d_3| \le \frac{27|\beta|^2\varkappa^3}{|9(\gamma^2 - 2\gamma(\tau+\mu) + \tau(2\tau-1) + 4\mu)\beta\varkappa^2 - (2(\tau+\mu)-\gamma-1)^2(18\varkappa^2-1)|} + \frac{3|\beta|\varkappa}{3(2\mu+\tau)-2\gamma-1} \tag{55}$$

*and*

$$|d_3 - \xi d_2^2| \le \begin{cases} \frac{3|\beta|\varkappa}{3(2\mu+\tau)-2\gamma-1} & ;\ |1-\xi| \le \mathcal{Q} \\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|9(\gamma^2-2\gamma(\tau+\mu)+\tau(2\tau-1)+4\mu)\beta\varkappa^2-(2(\tau+\mu)-2\gamma-1)^2(18\varkappa^2-1)|} & ;\ |1-\xi| \ge \mathcal{Q}, \end{cases} \tag{56}$$

*where*

$$\mathcal{Q} = \left|\frac{9(\gamma^2 - 2\gamma(\tau+\mu) + \tau(2\tau-1) + 4\mu)\beta\varkappa^2 - (2(\tau+\mu)-\gamma-1)^2(18\varkappa^2-1)}{(3(2\mu+\tau)-2\gamma-1)9\beta\varkappa^2}\right|.$$

**Proof.** Let $g \in \mathfrak{Y}_\sigma^\tau(\gamma,\mu,\varkappa)$. Then, from (11) and (12), we obtain

$$1 + \frac{1}{\beta}\left(\frac{\mu\varsigma^2 g''(\varsigma) + \varsigma(g'(\varsigma))^\tau}{\gamma\varsigma g'(\varsigma) + (1-\gamma)g(\varsigma)} - 1\right) = \mathsf{B}(\varkappa, \mathfrak{u}(\varsigma)),\ \varsigma \in \mathfrak{U} \tag{57}$$

and

$$1 + \frac{1}{\beta}\left(\frac{\mu w^2\hbar''(w) + \omega(\hbar'(w))^\tau}{\gamma w\hbar'(w) + (1-\gamma)\hbar(w)} - 1\right) = \mathsf{B}(\varkappa, \mathfrak{v}(w)),\ w \in \mathfrak{U}, \tag{58}$$

where $\mathfrak{u}(\varsigma)$ and $\mathfrak{v}(w)$ satisfy (39) and (40).

From (57) and (58) using (4) and (39), it is evident that

$$1+\frac{1}{\beta}\left(\frac{\mu\varsigma^2 g''(\varsigma)+\varsigma(g'(\varsigma))^\tau}{\gamma\varsigma g'(\varsigma)+(1-\gamma)g(\varsigma)}-1\right)=1+\mathsf{B}_1(\varkappa)\mathfrak{u}_1\varsigma+\left[\mathsf{B}_1(\varkappa)\mathfrak{u}_2+\mathsf{B}_2(\varkappa)\mathfrak{u}_1^2\right]\varsigma^2+\ldots \quad (59)$$

and

$$1+\frac{1}{\beta}\left(\frac{\mu w^2\hbar''(w)+\omega(\hbar'(w))^\tau}{\gamma w\hbar'(w)+(1-\gamma)\hbar(w)}-1\right)=1+\mathsf{B}_1(\varkappa)\mathfrak{v}_1 w+\left[\mathsf{B}_1(\varkappa)\mathfrak{v}_2+\mathsf{B}_2(\varkappa)\mathfrak{v}_1^2\right]w^2+\ldots. \quad (60)$$

The corresponding coefficients in (59) and (60) can therefore be compared to obtain

$$(2(\tau+\mu)-\gamma-1)d_2=\beta\mathsf{B}_1(\varkappa)\mathfrak{u}_1, \quad (61)$$

$$(3(\tau+2\mu)-2\gamma-1)d_3+((\gamma+1)(\gamma+1-2(\tau+\mu))+2\tau(\tau-1))d_2^2=\beta[\mathsf{B}_1(\varkappa)\mathfrak{u}_2+\mathsf{B}_2(\varkappa)\mathfrak{u}_1^2], \quad (62)$$

$$-(2(\tau+\mu)-\gamma-1)\,d_2=\beta\mathsf{B}_1(\varkappa)\mathfrak{v}_1 \quad (63)$$

and

$$(3(\tau+2\mu)-2\gamma-1)(2d_2^2-d_3)+((\gamma+1)(\gamma+1-2(\tau+\mu))+2\tau(\tau-1))d_2^2=\beta[\mathsf{B}_1(\varkappa)\mathfrak{v}_2+\mathsf{B}_2(\varkappa)\mathfrak{v}_1^2]. \quad (64)$$

By using the same method as in Theorem 2 with regard to (43)–(46), the results (54)–(56) of this theorem now follow from (61)–(64). □

We now provide the coefficient estimates and discuss the Fekete–Szegö issue for functions in the class $\mathfrak{W}_\sigma^\tau(\beta,\nu,\varkappa)$.

**Theorem 3.** *Let $\beta\in\mathbb{C}\backslash\{0\},\xi\in\mathbb{R},\frac{1}{2}<\varkappa\le 1,\tau\ge 1$, and $0\le\nu\le 1$. If $g\in\sigma$ is assigned to the class $\mathfrak{W}_\sigma^\tau(\beta,\nu,\varkappa)$, then*

$$|d_2|\le 3|\beta|\sqrt{\frac{3\varkappa^3}{|9\beta\varkappa^2(2\tau(\tau-1)(3\nu+1)+3\tau-\nu(2\tau-1))-4(\nu(\tau-1)+\tau)^2(18\varkappa^2-1)|}}, \quad (65)$$

$$|d_3|\le\frac{27|\beta|^2\varkappa^3}{[9\beta\varkappa^2(2\tau(\tau-1)(3\nu+1)+3\tau-\nu(2\tau-1))-4(\nu(\tau-1)+\tau)^2(18\varkappa^2-1)]}+\frac{|\beta|\varkappa}{\nu(2\tau-1)+\tau}. \quad (66)$$

*and*

$$|d_3-\xi d_2^2|\le\begin{cases}\frac{|\beta|\varkappa}{\nu(2\tau-1)+\tau} & ;|1-\xi|\le\mathcal{Z}\\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|9\beta\varkappa^2(2\tau(\tau-1)(3\nu+1)+3\tau-\nu(2\tau-1))-4(\nu(\tau-1)+\tau)^2(18\varkappa^2-1)|} & ;|1-\xi|\ge\mathcal{Z},\end{cases} \quad (67)$$

*where*

$$\mathcal{Z}=\left|\frac{9\beta\varkappa^2(2\tau(\tau-1)(3\nu+1)+3\tau-\nu(2\tau-1))-4(\nu(\tau-1)+\tau)^2(18\varkappa^2-1)}{27(\nu(2\tau-1)+\tau)\beta\varkappa^2}\right|.$$

**Proof.** Let $g\in\mathfrak{W}_\sigma^\tau(\beta,\nu,\varkappa)$. Then, from (19) and (20), we obtain

$$1+\frac{1}{\beta}\left(\nu\left(\frac{[(\varsigma g'(\varsigma))']^\tau}{g'(\varsigma)}\right)+(1-\nu)(g'(\varsigma))^\tau-1\right)=\mathsf{B}(\varkappa,\mathfrak{u}(\varsigma)),\ \varsigma\in\mathfrak{U} \quad (68)$$

and

$$1+\frac{1}{\beta}\left(\nu\left(\frac{[(w\hbar'(w))']^\tau}{\hbar'(w)}\right)+(1-\nu)(\hbar'(w))^\tau-1\right)=\mathsf{B}(\varkappa,\mathfrak{v}(w)),\ w\in\mathfrak{U}, \quad (69)$$

where $\mathfrak{u}(\varsigma)$ and $\mathfrak{v}(w)$ satisfy (39) and (40).

It is clear from (68) and (69) in combination with (4) and (39) that

$$1+\frac{1}{\beta}\left(\nu\left(\frac{[(\varsigma g'(\varsigma))']^{\tau}}{g'(\varsigma)}\right)+(1-\nu)(g'(\varsigma))^{\tau}-1\right)= \tag{70}$$

$$1+\mathsf{B}_1(\varkappa)\mathfrak{u}_1\varsigma+\left[\mathsf{B}_1(\varkappa)\mathfrak{u}_2+\mathsf{B}_2(\varkappa)\mathfrak{u}_1^2\right]\varsigma^2+\ldots$$

and

$$1+\frac{1}{\beta}\left(\nu\left(\frac{[(w\hbar'(w))']^{\tau}}{\hbar'(w)}\right)+(1-\nu)(\hbar'(w))^{\tau}-1\right)= \tag{71}$$

$$1+\mathsf{B}_1(\varkappa)\mathfrak{v}_1 w+\left[\mathsf{B}_1(\varkappa)\mathfrak{v}_2+\mathsf{B}_2(\varkappa)\mathfrak{v}_1^2\right]w^2+\ldots.$$

The corresponding coefficients in (70) and (71) can therefore be compared to obtain

$$2(\nu(\tau-1)+\tau)d_2=\beta\mathsf{B}_1(\varkappa)\mathfrak{u}_1, \tag{72}$$

$$3(\nu(2\tau-1)+\tau)d_3-2(2\nu(2\tau-1)-\tau(\tau-1)(3\nu+1))d_2^2=\beta[\mathsf{B}_1(\varkappa)\mathfrak{u}_2+\mathsf{B}_2(\varkappa)\mathfrak{u}_1^2], \tag{73}$$

$$-2(\nu(\tau-1)+\tau)\,d_2=\beta\mathsf{B}_1(\varkappa)\mathfrak{v}_1 \tag{74}$$

and

$$3(\nu(2\tau-1)+\tau)(2d_2^2-d_3)-2(2\nu(2\tau-1)-\tau(\tau-1)(3\nu+1))d_2^2=\beta[\mathsf{B}_1(\varkappa)\mathfrak{v}_2+\mathsf{B}_2(\varkappa)\mathfrak{v}_1^2]. \tag{75}$$

Using the same method as in Theorem 2 with regard to (43)–(46), the outcomes (65)–(67) of this theorem now follow from (72)–(75). □

The Fekete–Szegö inequality and coefficient estimates for functions $g\in\mathfrak{O}_{\sigma}^{\tau}(\beta,\gamma,\mu,\varkappa)$ are obtained in the following theorem.

**Theorem 4.** *Let $\beta\in\mathbb{C}\backslash\{0\},\xi\in\mathbb{R},\tau\geq 1,\ \mu\geq\gamma, 0\leq\gamma\leq 1$, and $\frac{1}{2}<\varkappa\leq 1$. If $g\in\sigma$ is assigned to the class $\mathfrak{O}_{\sigma}^{\tau}(\gamma,\mu,\varkappa)$, then*

$$|d_2|\leq\sqrt{\frac{27|\beta|^2\varkappa^3}{|9(3(2\mu+\tau-\gamma)-2(2\gamma(\tau+\mu-\gamma)-\tau(\tau-1)))\beta\varkappa^2-4(\tau+\mu-\gamma)^2(18\varkappa^2-1)|}}, \tag{76}$$

$$|d_3|\leq\frac{27|\beta|^2\varkappa^3}{|9(3(2\mu+\tau-\gamma)-2(2\gamma(\tau+\mu-\gamma)-\tau(\tau-1)))\beta\varkappa^2-4(\tau+\mu-\gamma)^2(18\varkappa^2-1)|}+\frac{|\beta|\varkappa}{2\mu+\tau-\gamma} \tag{77}$$

*and*

$$|d_3-\xi d_2^2|\leq\begin{cases}\frac{|\beta|\varkappa}{2\mu+\tau-\gamma} & ;\ |1-\xi|\leq\mathcal{X}\\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|9(3(2\mu+\tau-\gamma)-2(2\gamma(\tau+\mu-\gamma)-\tau(\tau-1)))\beta\varkappa^2-4(\tau+\mu-\gamma)^2(18\varkappa^2-1)|} & ;\ |1-\xi|\geq\mathcal{X},\end{cases} \tag{78}$$

*where*

$$\mathcal{X}=\left|\frac{9(3(2\mu+\tau-\gamma)-2(2\gamma(\tau+\mu-\gamma)-\tau(\tau-1)))\beta\varkappa^2-4(\tau+\mu-\gamma)^2(18\varkappa^2-1)}{(3(2\mu+\tau)-2\gamma-1)9\beta\varkappa^2}\right|.$$

**Proof.** Let $g\in\mathfrak{O}_{\sigma}^{\tau}(\gamma,\mu,\varkappa)$. Then, from (25) and (26), we obtain

$$1+\frac{1}{\beta}\left(\frac{\mu\varsigma^2 g''(\varsigma)+\varsigma(g'(\varsigma))^{\tau}}{\gamma\varsigma g'(\varsigma)+(1-\gamma)\varsigma}-1\right)=\mathsf{B}(\varkappa,\mathfrak{u}(\varsigma)),\ \varsigma\in\mathfrak{U} \tag{79}$$

and

$$1+\frac{1}{\beta}\left(\frac{\mu w^2\hbar''(w)+\omega(\hbar'(w))^{\tau}}{\gamma w\hbar'(w)+(1-\gamma)w}-1\right)=\mathsf{B}(\varkappa,\mathfrak{v}(w)),\ w\in\mathfrak{U}. \tag{80}$$

where $\mathfrak{u}(\varsigma)$ and $\mathfrak{v}(w)$ satisfy (39) and (40).

From (79) and (80) using (4) and (39), it is evident that

$$1+\frac{1}{\beta}\left(\frac{\mu\varsigma^2 g''(\varsigma)+\varsigma(g'(\varsigma))^\tau}{\gamma\varsigma g'(\varsigma)+(1-\gamma)\varsigma}-1\right)=1+\mathsf{B}_1(\varkappa)\mathfrak{u}_1\varsigma+\left[\mathsf{B}_1(\varkappa)\mathfrak{u}_2+\mathsf{B}_2(\varkappa)\mathfrak{u}_1^2\right]\varsigma^2+\ldots \tag{81}$$

and

$$1+\frac{1}{\beta}\left(\frac{\mu w^2\hbar''(w)+\omega(\hbar'(w))^\tau}{\gamma w\hbar'(w)+(1-\gamma)w}-1\right)=1+\mathsf{B}_1(\varkappa)\mathfrak{v}_1 w+\left[\mathsf{B}_1(\varkappa)\mathfrak{v}_2+\mathsf{B}_2(\varkappa)\mathfrak{v}_1^2\right]w^2+\ldots. \tag{82}$$

The corresponding coefficients in (81) and (82) can therefore be compared to obtain

$$2(\tau+\mu-\gamma)d_2=\beta\mathsf{B}_1(\varkappa)\mathfrak{u}_1, \tag{83}$$

$$3(\tau+2\mu-\gamma)d_3-2(2\gamma(\tau+\mu-\gamma)-\tau(\tau-1))d_2^2=\beta[\mathsf{B}_1(\varkappa)\mathfrak{u}_2+\mathsf{B}_2(\varkappa)\mathfrak{u}_1^2], \tag{84}$$

$$-2(\tau+\mu-\gamma)\,d_2=\beta\mathsf{B}_1(\varkappa)\mathfrak{v}_1 \tag{85}$$

and

$$3(\tau+2\mu-\gamma)(2d_2^2-d_3)-2(2\gamma(\tau+\mu-\gamma)-\tau(\tau-1))d_2^2=\beta[\mathsf{B}_1(\varkappa)\mathfrak{v}_2+\mathsf{B}_2(\varkappa)\mathfrak{v}_1^2]. \tag{86}$$

By using the same method as in Theorem 2 with regard to (43)–(46), the results (76)–(78) of this theorem now follow from (83)–(86). □

**Remark 5.** *From the above definitions, we can derive several subclasses of bi-univalent functions related to Lucas-Balancing polynomials for certain parameters such as $\tau,\nu,\mu,$ and $\gamma$. The corresponding results are thus derived from the results demonstrated in the paper; in the following section, we address a few of these.*

## 3. Special Cases of Main Results

The following would result from Theorem 1 when $\nu=0$:

**Corollary 1.** *Let $\beta\in\mathbb{C}\backslash\{0\},\xi\in\mathbb{R},\tau\geq 1,$ and $\frac{1}{2}<\varkappa\leq 1$. If $g\in\mathfrak{P}_\sigma^\tau(\beta,\varkappa)$, then*

$$|d_2|\leq 3|\beta|\varkappa\sqrt{\frac{3\varkappa}{|9\tau(2\tau-1)\beta\varkappa^2-(2\tau-1)^2(18\varkappa^2-1)|}},$$

$$|d_3|\leq\frac{27|\beta|^2\varkappa^3}{|9\tau(2\tau-1)^2\beta\varkappa^2-(2\tau-1)^2(18\varkappa^2-1)|}+\frac{3|\beta|\varkappa}{3\tau-1}$$

*and*

$$|d_3-\xi d_2^2|\ \leq\begin{cases}\frac{3|\beta|\varkappa}{(3\tau-1)} & ;\,|1-\xi|\leq\mathcal{J}_1\\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|(2\tau(\tau-1))9\beta\varkappa^2-(2\tau-1)^2(18\varkappa^2-1)|} & ;\,|1-\xi|\geq\mathcal{J}_1,\end{cases}$$

*where*

$$\mathcal{J}_1=\left|\frac{(2\tau(\tau-1))9\beta\varkappa^2-(2\tau-1)^2(18\varkappa^2-1)}{(3\tau-1)9\beta\varkappa^2}\right|.$$

**Remark 6.** *Allowing $\beta=1$ and $\tau=1$ in Corollary 1, we obtain Corollary 1 in [28].*

We deduce the following when $\nu=1$ in Theorem 1.

**Corollary 2.** *Let $\beta\in\mathbb{C}\backslash\{0\},\xi\in\mathbb{R},\tau\geq 1,$ and $\frac{1}{2}<\varkappa\leq 1$. If $g\in\mathfrak{Q}_\sigma^\tau(\beta,\varkappa)$, then*

$$|d_2|\leq 3|\beta|\varkappa\sqrt{\frac{3\varkappa}{|9(8\tau^2-7\tau+1)\beta\varkappa^2-4(2\tau-1)^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{27|\beta|^2\varkappa^3}{|9(8\tau^2 - 7\tau + 1)\beta\varkappa^2 - 4(2\tau - 1^2)(18\varkappa^2 - 1)|} + \frac{|\beta|\varkappa}{3\tau - 1}$$

*and*

$$|d_3 - \xi d_2^2| \quad \leq \begin{cases} \frac{|\beta|\varkappa}{(3\tau-1)} & ; |1-\xi| \leq \mathcal{J}_2 \\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|9(8\tau^2-7\tau+1)\beta\varkappa^2-4(2\tau-1)^2(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{J}_2, \end{cases}$$

*where*

$$\mathcal{J}_2 = \left|\frac{9\beta\varkappa^2(8\tau^2 - 7\tau + 1) - 4(18\varkappa^2 - 1)(2\tau - 1)^2}{27(3\tau - 1)\beta\varkappa^2}\right|.$$

**Remark 7.** *Using $\beta = 1$ and $\tau = 1$ in Corollary 2, we obtain Corollary 2 in [28].*

The following would result from Theorem 2 when $\gamma = 0$:

**Corollary 3.** *Let $\xi \in \mathbb{R}, \beta \in \mathbb{C}\backslash\{0\}, \frac{1}{2} < \varkappa \leq 1, \tau \geq 1,$ and $\mu \geq 0$. If $g \in \mathfrak{K}_\sigma^\tau(\beta, \mu, \varkappa)$, then*

$$|d_2| \leq 3|\beta|\varkappa\sqrt{\frac{3|\beta|\varkappa}{|9(\tau(2\tau - 1) + \mu)\beta\varkappa^2 - (2(\mu + \tau) - 1)^2(18\varkappa^2 - 1)|}},$$

$$|d_3| \leq \frac{27|\beta|^2\varkappa^3}{|9(\tau(2\tau - 1) + \mu)\beta\varkappa^2 - (2(\mu + \tau) - 1)^2(18\varkappa^2 - 1)|} + \frac{3|\beta|\varkappa}{3(2\mu + \tau) - 1}$$

*and*

$$|d_3 - \xi d_2^2| \quad \leq \begin{cases} \frac{3|\beta|\varkappa}{3(2\mu+\tau)-1} & ; |1-\xi| \leq \mathcal{Q}_1 \\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|9(\tau(2\tau-1)+\mu)\beta\varkappa^2-(2(\mu+\tau)-1)^2(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{Q}_1, \end{cases}$$

*where*

$$\mathcal{Q}_1 = \left|\frac{9(\tau(2\tau - 1) + \mu)\beta\varkappa^2 - (2(\mu + \tau) - 1)^2(18\varkappa^2 - 1)}{9(3(2\mu + \tau) - 1)\beta\varkappa^2}\right|.$$

**Remark 8.** *Taking $\mu = 0, \beta = 1,$ and $\tau = 1$ in Corollary 3, we obtain Corollary 1 in [28].*

The following would result from Theorem 2 when $\gamma = 1$:

**Corollary 4.** *Let $\tau \geq 1, \mu \geq 1, \xi \in \mathbb{R},$ and $\frac{1}{2} < \varkappa \leq 1$. If $g \in \mathfrak{J}_\sigma^\tau(\beta, \mu, \varkappa)$, then*

$$|d_2| \leq 3|\beta|\varkappa\sqrt{\frac{3\varkappa}{|9(2\tau^2 - 3\tau + 1 + 2\mu)\beta\varkappa^2 - 4(\mu + \tau - 1)^2(18\varkappa^2 - 1)|}},$$

$$|d_3| \leq \frac{27|\beta|^2\varkappa^3}{|9(2\tau^2 - 3\tau + 1 + 2\mu)\beta\varkappa^2 - 4(\tau + \mu - 1^2)(18\varkappa^2 - 1)|} + \frac{|\beta|\varkappa}{2\mu + \tau - 1}$$

*and*

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\beta|\varkappa}{2\mu+\tau-1} & ; |1-\xi| \leq \mathcal{Q}_2 \\ \frac{27|\beta|^2\varkappa|^3\,|1-\xi|}{|9(2\tau^2-3\tau+1+2\mu)\beta\varkappa^2-4(\mu+\tau-1)^2(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{Q}_2, \end{cases}$$

*where* $\mathcal{Q}_2 = \left|\frac{(2\tau^2 - 3\tau + 2\mu + 1)9\beta\varkappa^2 - 4(\mu + \tau - 1)^2(18\varkappa^2 - 1)}{(2\mu + \tau - 1)9\beta\varkappa^2}\right|.$

**Remark 9.** *If we permit $\mu = 1, \tau = 1,$ and $\beta = 1$ in Corollary 4, we obtain the outcome Corollary 2 [28].*

Theorem 2 would yield the following in the case where $\mu = 1$:

**Corollary 5.** *Let $\beta \in \mathbb{C}\backslash\{0\}, \xi \in \mathbb{R}, \tau \geq 1, 0 \leq \gamma \leq 1$, and $\frac{1}{2} < \varkappa \leq 1$. If $g \in \mathfrak{L}_{\Sigma}^{\tau}(\beta,\gamma,\varkappa)$, then*

$$|d_2| \leq 3|\beta|\varkappa\sqrt{\frac{3\varkappa}{|9((1-\gamma)^2+2\tau(\tau-\gamma)+3-\tau)\beta\varkappa^2-(2\tau-\gamma+1)^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{27|\beta|^2\varkappa^3}{|9((1-\gamma)^2+2\tau(\tau-\gamma)+3-\tau)\beta\varkappa^2-(2\tau-\gamma+1)^2(18\varkappa^2-1)|}+\frac{3|\beta|\varkappa}{3\tau-2\gamma+5}$$

*and*

$$|d_3-\xi d_2^2| \leq \begin{cases} \frac{3|\beta|\varkappa}{3\tau-2\gamma+5} & ; |1-\xi| \leq \mathcal{Q}_3 \\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|9((1-\gamma)^2+2\tau(\tau-\gamma)+3-\tau)\beta\varkappa^2-(2\tau-\gamma+1)^2(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{Q}_3, \end{cases}$$

*where* $\mathcal{Q}_3 = \left|\frac{((1-\gamma)^2+2\tau(\tau-\gamma)+3-\tau)9\beta\varkappa^2-(2\tau+1-\gamma)^2(18\varkappa^2-1)}{(3\tau-2\gamma+5)9\beta\varkappa^2}\right|.$

Theorem 3 would yield the following in the case where $\nu = 0$:

**Corollary 6.** *Let $\beta \in \mathbb{C}\backslash\{0\}, \xi \in \mathbb{R}, \tau \geq 1$, and $\frac{1}{2} < \varkappa \leq 1$. If $g \in \mathfrak{V}_{\sigma}^{\tau}(\beta,\varkappa)$, then*

$$|d_2| \leq 3|\beta|\sqrt{\frac{3\varkappa^3}{|9\tau(2\tau+1)\beta\varkappa^2-4\tau^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{27|\beta|^2\varkappa^3}{|9\tau(2\tau+1)\beta\varkappa^2-4\tau^2(18\varkappa^2-1)|}+\frac{|\beta|\varkappa}{\tau}$$

*and*

$$|d_3-\xi d_2^2| \quad \leq \begin{cases} \frac{|\beta|\varkappa}{\tau} & ; |1-\xi| \leq \mathcal{Z}_1 \\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|9\tau(2\tau+1)\beta\varkappa^2-4\tau^2(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{Z}_1, \end{cases}$$

*where*

$$\mathcal{Z}_1 = \left|\frac{(9\tau(2\tau+1)\beta\varkappa^2-4\tau^2(18\varkappa^2-1)}{27\tau\beta\varkappa^2}\right|.$$

Theorem 3 would yield the following in the case where $\tau = 1$:

**Corollary 7.** *Let $\xi \in \mathbb{R}, \beta \in \mathbb{C}\backslash\{0\}, \frac{1}{2} < \varkappa \leq 1$, and $0 \leq \nu \leq 1$. If $g \in \mathfrak{H}_{\sigma}(\beta,\nu,\varkappa)$, then*

$$|d_2| \leq 3|\beta|\sqrt{\frac{3\varkappa^3}{|9(3-\nu)\beta\varkappa^2-4(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{27|\beta|^2\varkappa^3}{|9(3-\nu)\beta\varkappa^2-4(18\varkappa^2-1)|}+\frac{|\beta|\varkappa}{\nu+1}$$

*and*

$$|d_3-\xi d_2^2| \quad \leq \begin{cases} \frac{|\beta|\varkappa}{\nu+1} & ; |1-\xi| \leq \mathcal{Z}_1 \\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|9(3-\nu)\beta\varkappa^2-4(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{Z}_1, \end{cases}$$

*where*

$$\mathcal{Z}_1 = \left|\frac{9(3-\nu)\beta\varkappa^2-4(18\varkappa^2-1)}{27(\nu+1)\beta\varkappa^2}\right|.$$

**Remark 10.** *In Corollary 7, if we take $\nu = 1$ and $\beta = 1$, we obtain the outcome Corollary 2 [28].*

Theorem 4 would yield the following in the case where $\gamma = 0$:

**Corollary 8.** *Let $\beta \in \mathbb{C}\backslash\{0\}, \xi \in \mathbb{R}, \frac{1}{2} < \varkappa \leq 1, \tau \geq 1$, and $\mu \geq 0$. If $g \in \mathfrak{A}_{\sigma}^{\tau}(\beta, \mu, \varkappa)$, then*

$$|d_2| \leq \sqrt{\frac{27|\beta|^2\varkappa^3}{|9(2\tau^2+\tau+6\mu)\beta\varkappa^2 - 4(\mu+\tau)^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{27|\beta|^2\varkappa^3}{|9(2\tau^2+\tau+6\mu)\beta\varkappa^2 - 4(\mu+\tau)^2(18\varkappa^2-1)|} + \frac{|\beta|\varkappa}{2\mu+\tau}$$

*and*

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\beta|\varkappa}{2\mu+\tau} & ; |1-\xi| \leq \mathcal{X}_1 \\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|9(2\tau^2+\tau+6\mu)\beta\varkappa^2 - 4(\mu+\tau)^2(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{X}_1, \end{cases}$$

*where*

$$\mathcal{X}_1 = \left|\frac{9(2\tau^2+\tau+6\mu)\beta\varkappa^2 - 4(\mu+\tau)^2(18\varkappa^2-1))}{(3(2\mu+\tau)-1)9\beta\varkappa^2}\right|.$$

The following would result from Theorem 4 when $\mu = \tau = 1$:

**Corollary 9.** *Let $\beta \in \mathbb{C}\backslash\{0\}, \xi \in \mathbb{R}, 0 \leq \gamma \leq 1$, and $\frac{1}{2} < \varkappa \leq 1$. If $g \in \mathfrak{N}_{\sigma}(\beta, \gamma, \varkappa)$, then*

$$|d_2| \leq 3|\beta|\sqrt{\frac{3|x|^3}{|9(4\gamma^2-11\gamma+9)\beta\varkappa^2 - 4(2-\gamma)^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{27|\beta|^2\varkappa^3}{|9(4\gamma^2-11\gamma+9)\beta\varkappa^2 - 4(2-\gamma)^2(18\varkappa^2-1)|} + \frac{|\beta|\varkappa}{3-\gamma}$$

*and*

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\beta|\varkappa}{3-\gamma} & ; |1-\xi| \leq \mathcal{X}_2 \\ \frac{27|\beta|^2|\varkappa|^3\,|1-\xi|}{|9(4\gamma^2-11\gamma+9)\beta\varkappa^2 - 4(2-\gamma)^2(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{X}_2, \end{cases}$$

*where* $\mathcal{X}_2 = \dfrac{1}{3-\gamma}\left|(4\gamma^2-11\gamma+9) - 4(2-\gamma)^2\left(\dfrac{18\varkappa^2-1}{9\beta\varkappa^2}\right)\right|.$

**Remark 11.** *In Corollary 9, if we take $\gamma = 1$ and $\beta = 1$, we obtain the outcome Corollary 2 [28].*

When $\tau = 1$ and $\mu = \gamma\,(0 \leq \gamma \leq 1)$ are used in Theorem 4, we obtain the following corollary.

**Corollary 10.** *Let $\beta \in \mathbb{C}\backslash\{0\}, \xi \in \mathbb{R}, 0 \leq \gamma \leq 1$, and $\frac{1}{2} < \varkappa \leq 1$. If $g \in \mathfrak{M}_{\Sigma}^{\tau}(\beta, \gamma, \varkappa)$, then*

$$|d_2| \leq 3|\beta|\sqrt{\frac{3\varkappa^3}{|9(1+2\gamma-\gamma^2)\beta\varkappa^2 - (\gamma+1)^2(18\varkappa^2-1)|}},$$

$$|d_3| \leq \frac{27|\beta|^2\varkappa^3}{|9(1+2\gamma-\gamma^2)\beta\varkappa^2 - (\gamma+1)^2(18\varkappa^2-1)|} + \frac{|\beta|\varkappa}{\gamma+1}$$

*and*

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\beta|\varkappa}{\gamma+1} & ; |1-\xi| \leq \mathcal{X}_3 \\ \frac{27|\beta|^2\varkappa^3\,|1-\xi|}{|9(1+2\gamma-\gamma^2)\beta\varkappa^2 - (\gamma+1)^2(18\varkappa^2-1)|} & ; |1-\xi| \geq \mathcal{X}_3, \end{cases}$$

*where*

$$\mathcal{X}_3 = \frac{1}{\gamma+1}\left|(1+2\gamma-\gamma^2) - (\gamma+1)^2\left(\frac{18\varkappa^2-1}{9\beta\varkappa^2}\right)\right|.$$

**Remark 12.** *We obtain the outcome Corollary 2 [28] , if we allow $\beta = 1$ and $\gamma = 1$ in Corollary 10.*

## 4. Conclusions

In the present investigation, the upper bounds of $|d_2|$ *and* $|d_3|$ for functions in the defined $\sigma$ subfamilies linked with Lucas-Balancing polynomials are determined. Furthermore, we have found the Fekete–Szegö functional $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for functions in these subfamilies. Specialization of parameters involved in our results yields new results—as stated in Section 3—that have not been previously considered. Relevant connections to the present findings are also indicated.

It might inspire many researchers to focus on a plethora of recent works based on the subclasses examined in this investigation such as subclasses of $\sigma$ linked with Lucas-Balancing polynomials using q-derivative operator, q-integral operator and operators on fractional q-calculus [29–35].

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