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Classical and umbral moonshine: Connections and p-adic properties

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Abstract. The classical theory of *monstrous moonshine* describes the unexpected connection between the representation theory of the monster group M, the largest of the sporadic simple groups, and certain modular functions, called Hauptmoduln. In particular, the n-th Fourier coefficient of Klein's j-function is the dimension of the grade n part of a special infinite dimensional representation $V^\natural$ of the monster group. More generally the coefficients of Hauptmoduln are graded traces T_g of $g \in M$ acting on $V^\natural$. Similar phenomena have been shown to hold for the Mathieu group M_{24}, but instead of modular functions, *mock modular forms* must be used. This has been conjecturally generalized even further, to *umbral moonshine*, which associates to each of the 23 Niemeier lattices a finite group, infinite dimensional representation, and mock modular form. We use *generalized Borcherds products* to relate monstrous moonshine and umbral moonshine. Namely, we use mock modular forms from umbral moonshine to construct via generalized Borcherds products rational functions of the Hauptmoduln T_g from monstrous moonshine. This allows us to associate to each pure A-type Niemeier lattice a conjugacy class g of the monster group, and gives rise to identities relating dimensions of representations from umbral moonshine to values of T_g. We also show that the logarithmic derivatives of the Borcherds products are p-adic modular forms for certain primes p and describe some of the resulting properties of their coefficients modulo p.

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1. Introduction

Monstrous moonshine begins with the surprising connection between the coefficients of the modular function

$$J(\tau) := j(\tau) - 744 = \frac{(1+240\sum_{n=1}^{\infty}\sum_{d|n} d^3q^n)^3}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} - 744$$
$$= \frac{1}{q} + 196884q + 21493760q^2 + \ldots$$

and the representation theory of the monster group M, which is the largest of the simple sporadic groups. Here $q := e^{2\pi i\tau}$ and $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$. McKay noticed that 196884, the q^1 coefficient of $J(\tau)$, can be expressed as a linear combination of dimensions of irreducible representations of the monster group M. Namely,

$$196884 = 196883 + 1.$$

McKay saw that the same was true for other Fourier coefficients of $J(\tau)$. For example,

$$21493760 = 21296876 + 196883 + 1.$$

In [28], McKay and Thompson conjectured that the n-th Fourier coefficient of $J(\tau)$ is the dimension of the grade n part of a special infinite-dimensional graded representation $V^\natural$ of M.

This was later expanded into the full monstrous moonshine conjecture by Thompson, Conway, and Norton [11,27]. Since the graded dimension is just the graded trace of the identity element, they looked at the graded traces $T_g(\tau)$ of nontrivial elements g of M acting on $V^\natural$ and conjectured that they were all expansions of principal moduli, or Hauptmoduln, for certain genus zero congruence groups Γ_g commensurable with $\mathrm{SL}_2(\mathbb{Z})$. Note that these T_g are constant on each of the 194 conjugacy classes of M, and therefore are class functions, which automatically have coefficients which are $\mathbb{C}$-linear combinations of irreducible characters of M. Part of the task of proving monstrous moonshine was showing that they were in fact $\mathbb{Z}_{\geq 0}$-linear combinations.

By way of computer calculation, Atkin, Fong, and Smith [26] verified the existence of a virtual representation of M. Then using vertex-operator theory, Frenkel, Lepowsky, and Meurman [16] finally constructed a representation $V^\natural$ of M thereby providing a beautiful algebraic explanation for the original numerical observations of McKay and Thompson. Borcherds [1] further developed the theory of vertex-operator algebras, which he then used in [2] to prove the full conjectures as given by Conway and Norton.

Monstrous moonshine provides an example of coefficients of modular functions enjoying distinguished properties. Moreover, their values at Heegner points have also been considered important. A *Heegner point* τ of discriminant $d < 0$ is a complex number of the form $\tau = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ with $a, b, c \in \mathbb{Z}$, $\gcd(a, b, c) = 1$, and $d = b^2 - 4ac$. The values of principal moduli at such points are called *singular moduli*. As an example of their importance, it is a classical fact that the singular moduli of $j(\tau)$ generate Hilbert class fields of imaginary quadratic fields. Moreover, the other McKay-Thompson series arising in monstrous moonshine satisfy analogous properties [7]. It is natural to ask what other interesting properties the values of the Hauptmoduln $T_g(\tau)$ could possess. We show that some of these values are related to another kind of moonshine, called *umbral moonshine*.

Recently, it was shown that phenomena similar to monstrous moonshine occur for other q-series and groups. In particular, the Mathieu group M_{24} exhibits moonshine [15,19], with the role of the j-invariant played by a *mock modular form* of weight 1/2, denoted $H^{(2)}(\tau)$. A mock modular form is the holomorphic part of a *harmonic weak Maass form*. Cheng, Duncan, and Harvey conjecture in [10] that this is a special case of a more general phenomenon, which they call *umbral moonshine*. For each of the 23 Niemeier lattices X they associate a vector-valued mock modular form $H^X(\tau)$, a group G^X, and an infinite-dimensional graded representation K^X of G^X such that the Fourier coefficients of H^X encode the dimensions of the graded components of K^X.

In particular, if $c^X(n, h)$ is the n-th Fourier coefficient of the h-th component of H^X, then

$$c^X(n,h) = \begin{cases} a^X \dim_{K^X_{h,-D/4m}} & \text{if } n = -D/4m \text{ where} \\ & \quad D \in \mathbb{Z},\, D = h^2 \pmod{4m}, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $a^X \in \{1, 1/3\}$ and

$$K^X = \bigoplus_{h \pmod{2m}} \bigoplus_{\substack{D\in\mathbb{Z} \\ D=h^2 \pmod{4m}}} K^X_{h,-D/4m}.$$

For more information on umbral moonshine see Section 2 and for a definition of H^X see Section 4.

Using *generalized Borcherds products* (see [6]), we describe a connection between the mock modular forms $H^X(\tau)$ of umbral moonshine and the McKay-Thompson series $T_g(\tau)$ of monstrous moonshine. *Generalized Borcherds products* are a method to produce modular functions as infinite products of rational functions whose exponents come from the coefficients of mock modular forms, and they can be viewed as generalizations of the automorphic products in Theorem 13.3 of [3].

Table 1. Pure A-type root systems.

Root System X	Coxeter Number $m(X)$	Mock Modular Form H^X
A_1^{24}	2	$H^{(2)}(\tau)$
A_2^{12}	3	$H^{(3)}(\tau)$
A_3^{8}	4	$H^{(4)}(\tau)$
A_4^{6}	5	$H^{(5)}(\tau)$
A_6^{4}	7	$H^{(7)}(\tau)$
A_8^{3}	9	$H^{(9)}(\tau)$
A_{12}^{2}	13	$H^{(13)}(\tau)$
A_{24}^{1}	25	$H^{(25)}(\tau)$

We focus on the Niemeier lattices X whose root systems are of pure A-type according to the ADE classification. They are listed in Table 1, along with their Coxeter numbers $m(X)$ and the notation we will use for the mock modular form H^X.

Table 2 gives the monstrous moonshine dictionary for the conjugacy classes g which correspond to pure A-type cases of umbral moonshine[1]. Note that $\eta(\tau)$ is the *Dedekind eta function*, defined by

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty}(1 - q^n).$$

All of our Hauptmoduln are normalized so that they have the form $q^{-1}+O(q)$, which is why all of the η-quotients in the table have a constant added to them.

There is an evident correspondence between the pure A-type lattices X in Table 1 and the conjugacy classes g in Table 2. We give this correspondence in Table 3.

We show that for a pure A-type Niemeier lattice X and its corresponding conjugacy class $g := g(X)$, the "Galois (twisted) traces" of the CM values of the McKay-Thompson series $T_g(\tau)$ are the coefficients of the mock modular form H^X. To more precisely state this, we set up the following notation.

Let X be a pure A-type Niemeier lattice with Coxeter number $m := m(X)$ and corresponding conjugacy class $g := g(X)$. We call a pair (Δ, r) *admissible* if Δ is a negative fundamental discriminant and $r^2 \equiv \Delta \pmod{4m}$. We also let $e(a) := e^{2\pi i a}$.

[1]The case $X = A_{24}$ corresponds to $g(X) = (25Z)$, which is what Conway and Norton call a "ghost element". This means that $\Gamma_0(25)$ is the only genus zero $\Gamma_0(N)$ that does not correspond to a conjugacy class of the monster group. The parentheses are used to indicate a ghost element.

Table 2. The dictionary of monstrous moonshine.

Monster Conjugacy Class g	Congruence Subgroup Γ_g	McKay-Thomspon Series $T_g(\tau)$
2B	$\Gamma_0(2)$	$\eta(\tau)^{24}/\eta(2\tau)^{24}+24$
3B	$\Gamma_0(3)$	$\eta(\tau)^{12}/\eta(3\tau)^{12}+12$
4C	$\Gamma_0(4)$	$\eta(\tau)^{8}/\eta(4\tau)^{8}+8$
5B	$\Gamma_0(5)$	$\eta(\tau)^{6}/\eta(5\tau)^{6}+6$
7B	$\Gamma_0(7)$	$\eta(\tau)^{4}/\eta(7\tau)^{4}+4$
9B	$\Gamma_0(9)$	$\eta(\tau)^{3}/\eta(9\tau)^{3}+3$
13B	$\Gamma_0(13)$	$\eta(\tau)^{2}/\eta(13\tau)^{2}+2$
(25 Z)	$\Gamma_0(25)$	$\eta(\tau)/\eta(25\tau)+1$

Theorem 1.1. *Let $c^+(n,h)$ be the n-th Fourier coefficient of the h-th component of H^X. Let (Δ, r) be an admissible pair for X. Then the twisted generalized Borcherds product*

$$\Psi_{\Delta,r}(\tau, H^X) := \prod_{n=1}^{\infty} P_\Delta(q^n)^{c^+\left(\frac{|\Delta|n^2}{4m}, \frac{rn}{2m}\right)},$$

where

$$P_\Delta(x) := \prod_{b\in\mathbb{Z}/|\Delta|\mathbb{Z}} [1-e(b/\Delta)x]^{\left(\frac{\Delta}{b}\right)}$$

is a rational function in $T_g(\tau)$ with a discriminant Δ Heegner divisor.

Remark 1.2. We consider only the pure A-type cases, because these are the ones for which the harmonic Maass form transforms under the Weil representation. See Section 4 for more information.

The next result gives a precise description of the rational functions in Theorem 1.1. In particular, it gives a "twisted" trace function for the values of T_g at points in the divisor and the coefficients c^+ of the mock modular forms H^X. It is often the case that coefficients of automorphic forms can be expressed in terms of singular moduli (see e.g., [4,5,13,29]).

Corollary 1.3. *By Theorem 1.1, we can write*

$$\Psi_{\Delta,r}(\tau, H^X) = \prod_i (T_g(\tau) - T_g(\alpha_i))^{\gamma_i}$$

Table 3. Correspondence between umbral and monstrous moonshine.

Root System X	Conjugacy Class $g(X)$
A_1^{24}	2B
A_2^{12}	3B
A_3^{8}	4C
A_4^{6}	5B
A_6^{4}	7B
A_8^{3}	9B
A_{12}^{2}	13B
A_{24}^{1}	(25Z)

for some discriminant Δ Heegner points α_i. Then we have that

$$c^+\left(\frac{|\Delta|}{4m}, \frac{r}{2m}\right) = \frac{1}{\epsilon_\Delta}\sum_i \gamma_i \cdot T_g(\alpha_i),$$

where

$$\epsilon_\Delta = \sum_{b\in\mathbb{Z}/|\Delta|\mathbb{Z}} e(b/\Delta)\cdot\left(\frac{\Delta}{b}\right).$$

Remark 1.4. Assuming the umbral moonshine conjecture, the previous corollary implies the following "degree" formula in traces of singular moduli for classical moonshine functions:

$$\frac{1}{\epsilon_\Delta}\sum_i \gamma_i \cdot T_g(\alpha_i) = c^+\left(\frac{|\Delta|}{4m}, \frac{r}{2m}\right) = a^X \dim_{K^X_{r,|\Delta|/4m}}. \tag{2}$$

In the case where $m = 2$, the relationship between the coefficients of the mock-modular form and the dimensions of the graded components of the representation has been proven by Gannon [19], and so our work implies the following:

$$\frac{1}{\epsilon_\Delta}\sum_i \gamma_i \cdot T_{2B}(\alpha_i) = c^+\left(\frac{|\Delta|}{8}, \frac{r}{4}\right) = \dim_{K^{(2)}_{r,|\Delta|/8}}. \tag{3}$$

Example 1.5. *Let $X = A_1^{24}$, so $m(X) = 2$ and $g(X) = 2B$. Then the corresponding McKay-Thompson series is*

$$T_g(\tau) = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 24 = \frac{1}{q} + 276q + \ldots.$$

We pick the admissible pair $(\Delta, r) = (-7, 1)$. *In Section 5, we will show that*

$$\Psi_{\Delta,r}(\tau, H^X) = \frac{(T_g(\tau) - T_g(\alpha_1))^2}{(T_g(\tau) - T_g(\alpha_2))^2} = \frac{\left(T_g(\tau) - \frac{1-45\sqrt{-7}}{2}\right)^2}{\left(T_g(\tau) - \frac{1+45\sqrt{-7}}{2}\right)^2}$$
$$= 1 + 90\sqrt{-7}q + (28350 + 45\sqrt{-7})q^2 + \dots,$$

where $\alpha_1 := \frac{-1+\sqrt{-7}}{4}$ *and* $\alpha_2 := \frac{1+\sqrt{-7}}{4}$. *Note that* $T_g(\alpha_1)$ *and* $T_g(\alpha_2)$ *are algebraic integers of degree 2 which form a full set of conjugates. Their twisted trace is*

$$2[T_g(\alpha_1) - T_g(\alpha_2)] = -90\sqrt{-7},$$

which matches the q^1 *Fourier coefficient above. To check Corollary 1.3, we note that*

$$\epsilon_\Delta = \sum_{b \in \mathbb{Z}/7\mathbb{Z}} e(-b/7) \cdot \left(\frac{-7}{b}\right) = -\sqrt{-7}$$

and

$$\frac{1}{\epsilon_\Delta} \sum_i \gamma_i T_g(\alpha_i) = 90 = c^+(7/8, 1/4) = \dim_{K^{(2)}_{1,7/8}}.$$

Example 1.6. *As a second example, again consider* $X = A_1^{24}$, *so* $m(X) = 2$ *and* $g(X) = 2B$. *We pick the admissible pair* $(\Delta, r) = (-15, 1)$. *Let* $\rho_1, \rho_2, \rho_3, \rho_4$ *be the roots of*

$$x^4 - 47x^3 + 192489x^2 - 9012848x + 122529840,$$

with ρ_1, ρ_2 *having positive imaginary parts. Then*

$$\Psi_{-15,1} = \frac{(T_g(\tau) - \rho_1)^2 (T_g(\tau) - \rho_2)^2}{(T_g(\tau) - \rho_3)^2 (T_g(\tau) - \rho_4)^2}.$$

We get that

$$\epsilon_{-15} = \sqrt{-15},$$

and

$$\frac{1}{\epsilon_\Delta} \sum_i \rho_i T_g(\alpha_i) = 462 = c^+(15/8, 1/4) = \dim_{K^{(2)}_{1,15/8}}.$$

In view of this correspondence, it is clear that the mock modular forms of umbral moonshine have important properties. The congruence properties of their coefficients have just begun to be studied. For example, [12] examines the parity of the coefficients of the McKay-Thompson series for Mathieu moonshine in relation to a certain conjecture in [9], which in our case corresponds to $X = A_1^{24}$. Congruences modulo higher primes were also considered in [22].

Let $\Theta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. Given the product expansion of a generalized Borcherds product, it is natural to consider its logarithmic derivative. It turns out that this logarithmic derivative has nice arithmetic properties. This idea was also used in [6] and [23].

Theorem 1.7. *Fix a pure A-type Niemeier lattice X with Coxeter number m. Let (Δ, r) be an admissible pair. Consider the logarithmic derivative*

$$f_{\Delta,r}(\tau) = \sqrt{\Delta}\sum a_{\Delta,r}(n)q^n := \sqrt{\Delta}\sum_n \sum_{ij=n} i c^+\left(\frac{|\Delta| i^2}{4m}, \frac{ri}{2m}\right)\left(\frac{\Delta}{j}\right) q^n$$

of $\Psi_{\Delta,r}(\tau) = \Psi_{\Delta,r}(\tau, H^X)$. Then $f_{\Delta,r}(\tau)$ is a meromorphic weight 2 modular form.

When p is inert or ramified in $\mathbb{Q}(\sqrt{\Delta})$, it turns out that $f_{\Delta,r}(\tau)$ is more than just a meromorphic modular form; it is a p-adic modular form. Essentially, a p-adic modular form is a q-series which is congruent modulo any power of p to a holomorphic modular form; we refer the reader to Section 6.1 for the definition.

Theorem 1.8. *Let X be a pure A-type Niemeier lattice with Coxeter number m. Let (Δ, r) be admissible and suppose p is inert or ramified in $\mathbb{Q}(\sqrt{\Delta})$. Then $f_{\Delta,r}$ is a p-adic modular form of weight 2.*

We will use this result to study the p-divisibility of the coefficients $a_{\Delta,r}(n)$.

Corollary 1.9. *Let X, Δ, r, p be as above. Then for all $k \geq 1$ there exists $\alpha_k > 0$ such that*

$$\#\{n \leq x : a_{\Delta,r}(n) \not\equiv 0 \pmod{p^k}\} = O\left(\frac{x}{(\log x)^{\alpha_k}}\right).$$

In particular, if we let

$$\pi_{\Delta,r}(x; p^k) := \#\{n \leq x : a_{\Delta,r}(n) \equiv 0 \pmod{p^k}\},$$

then

$$\lim_{x\to\infty} \frac{\pi_{\Delta,r}(x; p^k)}{x} = 1.$$

Remark 1.10. Corollary 1.9 also applies to any constant multiple of $f_{\Delta,r}$ with integral coefficients. In the example below, we consider the coefficients of

$$\frac{f_{-7,1}(\tau)}{90\sqrt{-7}} = q + O(q^2).$$

Table 4. Divisibility of $a_{-7,1}(n)$ by $p = 2, 3$.

x	$\pi_2(x)/x$	$\pi_3(x)/x$
50	0.38	0.64
100	0.45	0.68
150	0.47	0.69
200	0.49	0.71
250	0.48	0.71
300	0.49	0.72
$\vdots$	$\vdots$	$\vdots$
∞	.5?	1

However, it is not always the case that the analogous normalization has integral coefficients.

Example 1.11. *We illustrate Corollary 1.9 for $X = A_1^{24}$, $\Delta = -7$, $r = 1$. Note that this is the same case considered in Example 1.5. The first few coefficients of the normalized logarithmic derivative are given by*

$$\frac{f_{-7,1}(\tau)}{90\sqrt{-7}} =: \sum_{n\geq 1} a_{-7,1}(n)q^n = q + q^2 - 4371q^3 + q^4 + 17773755q^5 + \ldots$$

The prime $p = 2$ is split in $\mathbb{Q}(\sqrt{-7})$, and so Theorem 1.8 and Corollary 1.9 do not apply. Therefore, we expect the coefficients $a_{-7,1}(n)$ to be equally distributed modulo 2, but cannot prove anything about them. The prime $p = 3$ is inert, so Corollary 1.9 tell us that, asymptotically, 100% of the coefficients $a_{-7,-1}(n)$ are divisible by 3. We illustrate this behavior in Table 1.11.

2. Umbral moonshine

In this section, we summarize the main objects and conjectures of umbral moonshine. However, we first briefly describe Mathieu moonshine, which umbral moonshine generalized.

2.1 Mathieu moonshine

In 2010, the study of a new form of moonshine commenced, called Mathieu moonshine. Let $\mu(z, \tau) := \mu(z, z, \tau)$ be Zwegers' famous function from his thesis [30], which is defined in the appendix. Let $H^{(2)}(\tau)$ be the q-series

$$H^{(2)}(\tau) := -8 \sum_{\omega\in\left\{\frac{1}{2},\frac{1+\tau}{2},\frac{\tau}{2}\right\}} \mu(\omega, \tau) = 2q^{-1/8}(-1 + 45q + 231q^2 + \ldots),$$

which occurs in the decomposition of the elliptic genus of a K3 surface into irreducible characters of the $N = 4$ superconformal algebra. This is a mock-modular form, and plays the role of $J(\tau)$ in Mathieu moonshine. Eguchi, Ooguri, and Tachikawa conjectured that the Fourier coefficients encode dimensions of irreducible representations of the Mathieu group M_{24} [15]. This was extended to the full Mathieu moonshine conjecture by [8,14,17,18], which included providing mock modular forms $H_g^{(2)}$ for every $g \in M_{24}$. The existence of an infinite dimensional M_{24} module underlying the mock modular forms was shown by Gannon in 2012 [19].

2.2 The objects of umbral moonshine

Cheng, Duncan, and Harvey generalized even further – conjecturing that Mathieu moonshine is but one example of a more general phenomenon which they call umbral moonshine [10].

For each of the 23 Niemeier root systems X, which are unions of irreducible simply-laced root systems with the same Coxeter number, they associate many objects, including a group G^X (playing the role of M), a mock modular form $H^X(\tau)$ (playing the role of $j(\tau)$), and an infinite dimensional graded G^X module K^X (playing the role of the M-module $V^\natural$) Table 5 gives a more complete list of the associated objects.

The ADE classification of simply laced Dynkin diagrams allows us to classify the irreducible components of the Niemeier root systems X. We will focus on the simplest cases – the root systems of pure A-type, i.e. $X = A_{m-1}^{24/(m-1)}$, where $(m - 1) \mid 24$. In these cases, the lambency ℓ is an integer and equals m, and $\Gamma^X = \Gamma_0(m)$. The case $X = A_1^{24}$ corresponds to Mathieu moonshine, with $G^X = M_{24}$ and $H^X = H^{(2)}$, as defined above. We will generally refer to H^X , S^X, ψ^X, and T^X as $H^{(m)}$, $S^{(m)}$, $\psi^{(m)}$, and j_m respectively. These are the main quantities from Table 5 that we will work with, and we will only define them for pure A-type. This is done in Section 4.

2.3 The conjectures of umbral moonshine

The main conjectures of umbral moonshine are as follows:

1. The mock modular form H^X encodes the graded super-dimension of a certain infinite-dimensional, $\mathbb{Z}/2m\mathbb{Z} \times \mathbb{Q}$-graded G^X-module K^X.
2. The graded super-characters H_g^X arising from the action of G^X on K^X are vector-valued mock modular forms with concretely specified shadows S_g^X.
3. The umbral McKay-Thompson series H_g^X are uniquely determined by an optimal growth property which is directly analogous to the genus zero property of monstrous moonshine.

Table 5. This table gives the objects associated to a Niemeier root system X.

L^X	The Niemeier lattice corresponding to X
m	The Coxeter number of all irreducible components of X
W^X	The Weyl group of X
$G^X := \mathrm{Aut}(L^X)/W^X$	The umbral group corresponding to X
π^X	The (formal) product of Frame shapes of Coxeter elements of irreducible components of X
Γ^X	The genus zero subgroup attached to X
T^X	The normalized Hauptmodul of Γ^X, whose eta-product expansion corresponds to π^X
ℓ	The lambency. A symbol that encodes the genus zero group Γ^X. Sometimes used instead of X to denote which case of umbral moonshine is being considered.
ψ^X	The unique meromorphic Jacobi form of weight 1 and index m satisfying certain conditions.
H^X	The vector-valued mock modular form of weight 1/2 whose $2m$ components furnish the theta expansion of the finite part of ψ^X. Called the umbral mock modular form.
S^X	The vector-valued cusp form of weight 3/2 which is the shadow of H^X. Called the umbral shadow.
H_g^X	The umbral McKay-Thompson series attached to $g \in G^X$. It is a vector-valued mock modular form of weight 1/2, and equals H^X when g is the identity.
S_g^X	The vector-valued cusp form *conjectured* to be the shadow of H_g^X.
K^X	The *conjectural* infinite dimensional graded G^X-module whose graded super-dimension is encoded by H^X.

3. Vector-valued modular forms

In this section, we follow [6] in giving the needed background on vector-valued modular forms, though we state results in less generality.

3.1 A lattice related to $\Gamma_0(m)$

We will define a lattice L and a dual lattice L' related to $\Gamma_0(m)$ such that the components of our vector-valued modular forms are labeled by the elements of L'/L.

We consider the quadratic space

$$V := \{X \in \mathrm{Mat}_2(\mathbb{Q}) : \mathrm{tr}(X) = 0\}$$

with the quadratic form $P(X) := m\det(X)$.[2] The corresponding bilinear form is then $(X, Y) := -m\operatorname{tr}(XY)$. Let L be the lattice

$$L := \left\{ \begin{pmatrix} b & -a/m \\ c & -b \end{pmatrix};\quad a, b, c \in \mathbb{Z} \right\}.$$

The dual lattice is then given by

$$L' := \left\{ \begin{pmatrix} b/2m & -a/m \\ c & -b/2m \end{pmatrix};\quad a, b, c \in \mathbb{Z} \right\}.$$

We will switch between viewing elements of L' as matrices and as quadratic forms, with the matrix

$$X = \begin{pmatrix} b/2m & -a/m \\ c & -b/2m \end{pmatrix}$$

corresponding to the integral binary quadratic form

$$Q = [mc, b, a] = mcx^2 + bxy + ay^2.$$

Note that then $P(X) = -\operatorname{Disc}(Q)/4m$.

We identify L'/L with $\left(\frac{1}{2m}\mathbb{Z}\right)/\mathbb{Z}$, and the quadratic form P with the quadratic form $\frac{h}{2m} \mapsto \frac{-h^2}{4m}$ on $\mathbb{Q}/\mathbb{Z}$. We will also occasionally identify $\frac{h}{2m} \in \mathbb{Q}/\mathbb{Z}$ with $h \in \mathbb{Z}/2m\mathbb{Z}$.

For a fundamental discriminant D and $r/2m \in L'/L$ with $r^2 \equiv D \pmod{4N}$, let

$$\mathcal{Q}_{D,r} := \{Q = [mc, b, a] : a, b, c \in \mathbb{Z}, \operatorname{Disc}(Q) = D, b \equiv r \pmod{2m}\}. \tag{4}$$

The action of $\Gamma_0(m)$ on this set is given by the usual action of congruence subgroups on binary quadratic forms. We will later be working with $\mathcal{Q}_{D,r}/\Gamma_0(m)$.

3.2 The Weil representation

By $\mathrm{Mp}_2(\mathbb{Z})$ we denote the integral metaplectic group. It consists of pairs (γ, ϕ), where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi^2(\tau) = c\tau + d$. The group $\widetilde{\Gamma} := \mathrm{Mp}_2(\mathbb{Z})$ is generated by $S := \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$ and $T := \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$.

We consider the *Weil representation* ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ corresponding to the discriminant form L'/L. We denote the standard basis elements of $\mathbb{C}[L'/L]$

[2] Note that this corrects a typo in [6].

by $\mathfrak{e}_h$, $h/2m \in L'/L$. Then the Weil representation ρ_L associated with the discriminant form L'/L is the unitary representation of $\widetilde{\Gamma}$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(T)\mathfrak{e}_h = e(h^2/4m)\mathfrak{e}_h,$$

and

$$\rho_L(S)\mathfrak{e}_h = \frac{e(-1/8)}{\sqrt{2m}} \sum_{h' \in \mathbb{Z}/2m\mathbb{Z}} e(hh'/2m)\mathfrak{e}_{h'}.$$

3.3 Harmonic weak maass forms

If $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is a function, we write

$$f = \sum_{h \in \mathbb{Z}/2m\mathbb{Z}} f_h \mathfrak{e}_h$$

for its decomposition into components. For $k \in \frac{1}{2}\mathbb{Z}$, let $M^!_{k,\rho_L}$ denote the space of $\mathbb{C}[L'/L]$ valued weakly holomorphic modular forms of weight k and type ρ_L for the group $\widetilde{\Gamma}$. The subspaces of holomorphic modular forms (resp. cusp forms) are denoted by M_{k,ρ_L} (resp. S_{k,ρ_L}). Now, assume that $k \leq 1$. A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a *harmonic weak Maass form* (of weight k with respect to $\widetilde{\Gamma}$ and ρ_L) if it satisfies:

1. $f(M\tau) = \phi(\tau)^{2k}\rho_L(M,\phi)f(\tau)$ for all $(M,\phi) \in \widetilde{\Gamma}$;
2. $\Delta_k f = 0$;
3. There is a polynomial

$$P_f(\tau) = \sum_{h \in \mathbb{Z}/2m\mathbb{Z}} \sum_{\substack{n \in \mathbb{Z} - \frac{h^2}{4m}, \\ -\infty \ll n \leq 0}} c^+(n,h)e(n\tau)\mathfrak{e}_h$$

such that

$$f(\tau) - P_f = O(e^{-\epsilon v})$$

for some $\epsilon > 0$ as $v \to +\infty$.

Note here that

$$\Delta_k := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

is the usual weight k hyperbolic Laplace operator, and that $\tau = u + iv$. We denote the vector space of these harmonic weak Maass forms by $\mathcal{H}_{k,\rho_L}$.

The Fourier expansion of any $f \in \mathcal{H}_{k,\rho_L}$ gives a unique decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{h \in \mathbb{Z}/2m\mathbb{Z}} \sum_{\substack{n \in \mathbb{Z} - \frac{h^2}{4m}, \\ -\infty \ll n}} c^+(n,h) e(n\tau) \mathfrak{e}_h, \tag{5}$$

$$f^-(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q}, \\ n<0}} c^-(n,h) W(2\pi n v) e(n\tau) \mathfrak{e}_h, \tag{6}$$

and $W(x) := \int_{-2x}^{\infty} e^{-t} t^{-k} dt = \Gamma(1-k, 2|x|)$ for $x < 0$. Then f^+ is called the *holomorphic part* and f^- the *nonholomorphic part* of f. The polynomial P_f is also uniquely determined by f and is called its *principal part.* We define a *mock modular form* of weight k to be the holomorphic part f^+ of a harmonic weak Maass form f of weight k which has $f^- \neq 0$. Its weight is just the weight of the harmonic weak Maass form.

Recall that there is an antilinear differential operator defined by

$$\xi_k : \mathcal{H}_{k,\overline{\rho}_L} \to S_{2-k,\rho_L}, \quad f(\tau) \mapsto \xi_k(f)(\tau) := 2iy^k \frac{\overline{\partial}}{\partial \overline{\tau}},$$

where $\overline{\rho}_L$ is the complex conjugate representation. The Fourier expansion of $\xi_k(f)$ is given by

$$\xi_k(f) = - \sum_{h \in \mathbb{Z}/2m\mathbb{Z}} \sum_{n \in \mathbb{Q}, n>0} (4\pi n)^{1-k} \overline{c^-(-n,h)} q^n \mathfrak{e}_h.$$

The kernel of ξ_k is equal to $M^!_{k,\overline{\rho}_L}$, and we have the following exact sequence:

$$0 \to M^!_{k,\overline{\rho}_L} \to \mathcal{H}_{k,\overline{\rho}_L} \to S_{2-k,\rho_L} \to 0.$$

We call $\xi_k(f)$ the *shadow* of f. Note that $\xi_k(f)$ uniquely determines f^-, but the f^+ is only determined up to the addition of a weakly holomorphic modular form.

4. Defining the umbral mock modular forms

In this section we define the mock modular forms $H^{(m)}$ from umbral moonshine, as well as their shadows $S^{(m)}$ and non-holomorphic parts. Note that we only give definitions for the pure A-type cases – see [10] for a more detailed and general definition. We also refer the reader to the appendix for definitions of $\varphi_1^{(m)}(\tau, z)$, $\mu_{m,0}(\tau, z)$, $\theta_{m,r}(\tau, z)$, and $R(u;\tau)$.

For each lambency $m \in \{2,3,4,5,7,9,13,25\}$, which correspond to the pure A-type cases, define the Jacobi form $\psi^{(m)}$ by

$$\psi^{(m)}(\tau,z) := c_m \varphi_1^{(m)}(\tau,z)\mu_{1,0}(\tau,z),$$

where $c_m = 2$ for $m = 2,3,4,5,7,13$ and $c_m = 1$ for $m = 9,25$. We can break up $\psi^{(m)}$ into a finite part $\psi_F^{(m)}$ and a polar part $\psi_P^{(m)}$. The polar part is given by

$$\psi_P^{(m)}(\tau,z) = \frac{24}{m-1}\mu_{m,0}(\tau,z).$$

Then the mock modular form $H^{(m)}$ is defined by

$$\psi_F^{(m)}(\tau,z) = \psi^{(m)}(\tau,z) - \psi_P^{(m)}(\tau,z) = \sum_{h\in\mathbb{Z}/2m\mathbb{Z}} H_h^{(m)}(\tau)\theta_{m,h}(\tau,z), \tag{7}$$

where

$$\theta_{m,h}(\tau,z) := \sum_{n\equiv h \pmod{2m}} q^{n^2/4m}y^k.$$

Note that $\psi^{(m)}$ satisfies an optimal growth condition, which is that

$$q^{1/4m}H_h^X(\tau) = O(1) \tag{8}$$

as $\tau \to i\infty$ for all $h \in \mathbb{Z}/2m\mathbb{Z}$.

We also define the shadow $S^{(m)}(\tau)$, the non-holomorphic part $F_r^{(m)}(\tau)$, and the harmonic weak Maass form $\widehat{H}^{(m)}(\tau)$ corresponding to the mock modular form $H^{(m)}$ via their components:

$$S_h^{(m)}(\tau) := \sum_{n\equiv h \pmod{2m}} nq^{n^2/4m}, \tag{9}$$

$$\begin{aligned} F_h^{(m)}(\tau) &:= \int_{-\bar\tau}^{i\infty} \frac{S_h^{(m)}(z)}{\sqrt{-i(z+\tau)}}dz \\ &= -2mq^{-(h-m)^2/4m}R\left(\frac{h-m}{2m}(2m\tau)+\frac{1}{2};2m\tau\right), \text{ and} \end{aligned} \tag{10}$$

$$\widehat{H}_h^{(m)}(\tau) := H_h^{(m)}(\tau) + F_h^{(m)}(\tau) \tag{11}$$

Note that by definition, $S_h^{(m)}(\tau) = -S_{-h}^{(m)}(\tau)$. Therefore, $S_0^{(m)} = S_m^{(m)} = 0$. The same is true of $H_h^{(m)}$. We can write this in terms of Shimura's theta functions as $S_h^{(m)}(\tau) = \theta(\tau;h,2m,2m,x)$ [25]. Then using the transformation laws for his θ-functions, we get that $S^{(m)}$ transforms as follows:

$$S_h^{(m)}(\tau+1) = e(h^2/4m)S_h^{(m)}(\tau), \text{ and}$$

$$S_h^{(m)}(-1/\tau) = \tau^{3/2}\frac{e(-1/8)}{\sqrt{2m}}\sum_{k \pmod{2m}} e(kh/2m)S_k^{(m)}(\tau).$$

Thus, we have

$$S^{(m)}(\tau+1) = \rho_L(T)S^{(m)}(\tau), \text{ and}$$
$$S^{(m)}(-1/\tau) = \tau^{3/2}\rho_L(S)S^{(m)}(\tau).$$

From these transformations, we see that $S^{(m)}(\tau) : \mathbb{H} \to \mathbb{C}[L'/L]$ is a weight 3/2 vector-valued modular form transforming under the Weil representation ρ_L, i.e. an element of the space $M_{3/2,\rho_L}$. From [10], we know that $H^{(m)}$ is a mock modular form with shadow $S^{(m)}$.[3] This gives us the following theorem.

Theorem 4.1. *We have that $\widehat{H}^{(m)}(\tau) : \mathbb{H} \to \mathbb{C}[L'/L]$ is a weight 1/2 vector-valued harmonic weak Maass form transforming under the Weil representation $\overline{\rho}_L$, i.e., it is an element of $\mathcal{H}_{1/2,\overline{\rho}_L}$. Moreover, it has shadow $S^{(m)}(\tau)$, non-holomorphic part $F^{(m)}$, and principal part $P(\tau) = -2q^{-1/4m}(\mathfrak{e}_1 - \mathfrak{e}_{2m-1})$.*

The reason we focus on the lattices of pure A-type is because this theorem is not true for the other cases – the vector-valued harmonic weak Maass forms no longer transform under the Weil representation.

5. Relating umbral and monstrous moonshine

In this section, we explain the relationship between the mock modular forms $H^{(m)}$ from umbral moonshine and the Hauptmoduln T_g from monstrous moonshine.

5.1 Twisted generalized Borcherds products

We begin by giving the theorem of Bruinier and Ono we will use.

Let $c^+(n,h)$ be the n-th Fourier coefficient of $H_h^{(m)}$. Let (Δ, r) be an admissible pair, so that Δ is a negative fundamental discriminant and $r^2 \equiv \Delta \pmod{4m}$. Let $\Psi_{\Delta,r}(\tau, \widehat{H}^m)$ be the twisted generalized Borcherds product defined in Theorem 1.1.

Theorem 5.1 (Theorem 6.1 in [6]). *We have that $\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)})$ is a weight 0 meromorphic modular function on $\Gamma_0(m)$ with divisor $Z_{\Delta,r}(\widehat{H}^{(m)})$.*

[3] In fact, it is the only vector-valued mock modular form with shadow $S^{(m)}$ satisfying the optimal growth condition in 8.

For this theorem to make sense, we need to define the twisted Heegner divisor $Z_{\Delta,r}(\widehat{H}^{(m)})$ associated to $\widehat{H}^{(m)}$. It is defined by

$$Z_{\Delta,r}(\widehat{H}^{(m)}) := \sum_{h \in \mathbb{Z}/2m\mathbb{Z}} \sum_{n<0} c^+(n,h) Z_{\Delta,r}(n,h).$$

Since the principal part of $\widehat{H}^{(m)}$ is $-2q^{-h^2/4m}(\mathfrak{e}_1 - \mathfrak{e}_{2m-1})$, this means that

$$Z_{\Delta,r}(\widehat{H}^{(m)}) = 2Z_{\Delta,r}\left(\frac{-1}{4m}, \frac{-1}{2m}\right) - 2Z_{\Delta,r}\left(\frac{-1}{4m}, \frac{1}{2m}\right).$$

Now, we just have to compute the divisors $Z_{\Delta,r}\left(\frac{-1}{4m}, \frac{h}{2m}\right)$. They are defined as follows.

$$Z_{\Delta,r}\left(\frac{-1}{4m}, \frac{h}{2m}\right) := \sum_{Q \in \mathcal{Q}_{\Delta,hr}/\Gamma_0(m)} \frac{\chi_\Delta(Q)}{w(Q)} \alpha_Q,$$

where $w(Q) = 2$ for $\Delta < -4$, χ_Δ is the generalized genus character defined in Gross-Kohnen-Zagier, and α_Q is the unique root of $Q(x, 1)$ in $\mathbb{H}$.

5.2 Proofs of Theorem 1.1 and Corollary 1.3

Proof of Theorem 1.1. Theorem 5.1 gives us that $\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)})$ is a weight 0 meromorphic modular function on $\Gamma_0(m)$ with specified divisor, which is a discriminant Δ Heegner divisor. For all of our m, $\Gamma_0(m)$ has genus zero. Therefore, $\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)})$ is a rational function in the Hauptmodul for $\Gamma_0(m)$. The normalized Hauptmodul, which we call $j_m(\tau)$, is defined by

$$j_m(\tau) := \frac{\eta(\tau)^{24/(m-1)}}{\eta(m\tau)^{24/(m-1)}} + \frac{24}{m-1}. \tag{12}$$

But using Table 1, we see that $j_m(\tau)$ is equal to $T_{g(X)}(\tau)$, the graded trace of $g(X) \in M$ on V. $\square$

Proof of Corollary 1.3. From Theorem 1.1, we have that

$$\prod_{n=1}^{\infty} P_\Delta(q^n)^{c^+\left(\frac{|\Delta| n^2}{4m}, \frac{rn}{2m}\right)} = \prod_i (T_g(\tau) - T_g(\alpha_i))^{\gamma_i}.$$

We equate the q^1 Fourier coefficients of each side, using Table 2 to get the Fourier expansion

$$T_g(\tau) = \frac{1}{q} + O(q). \qquad \square$$

Table 6. Quadratic forms needed for $m = 2$, $\Delta = -7, r = 1$ case.

Quadratic form $= Q$	α_Q	$\chi_\Delta(Q)$	$j_2(\alpha_Q)$
$Q_1 = [2, 1, 1]$	$\alpha_1 = \frac{-1+\sqrt{-7}}{4}$	1	$\gamma_1 := \frac{1+45\sqrt{-7}}{2}$
$Q_2 = [-2, 1, -1]$	$\alpha_2 = \frac{1+\sqrt{-7}}{4}$	-1	$\gamma_2 := \frac{1-45\sqrt{-7}}{2}$
$-Q_2$	α_2	1	γ_2
$-Q_1$	α_1	-1	γ_1

5.3 Examples

For each pure A-type case X with coxeter number m, we illustrate how to write $\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)})$ as a rational function in j_m. Note that here $\Delta < 0$ is a fundamental discriminant and $r \in \mathbb{Z}$ is such that $\Delta \equiv r^2 \pmod{4m}$.

First we work out an example for $m = 2$ in some detail, then list one example for each m. In Section 5.4, we explain how to find representatives of $Q_{\Delta,r}/\Gamma_0(m)$ using a method of Gross, Kohen, and Zagier.

Consider the case $m = 2$, $\Delta = -7, r = 1$. Using the method of Section 5.4, we compute that $Q_{-7,1}/\Gamma_0(2) = \{Q_1, Q_2\}$ and that $Q_{-7,-1}/\Gamma_0(2) = \{-Q_1, -Q_2\}$, where the quadratic forms Q, their Heenger points α_Q, and their generalized genus characters $\chi_\Delta(Q)$ are given in Table 6. We also include the value of j_2 at each Heegner point.

Using the table, the divisor of $\Psi_{-7,1}(\tau)$ is given by:

$$(-\alpha_1 + \alpha_2) - (\alpha_1 - \alpha_2) = 2\alpha_2 - 2\alpha_1.$$

Therefore,

$$\Psi_{-7,1}(\tau, \widehat{H}^{(2)}) = \frac{(j_2(\tau) - \gamma_2)^2}{(j_2(\tau) - \gamma_1)^2}.$$

Similarly, for each value of m corresponding to a pure A-type case, we demonstrate in Table 7 how to write $\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)})$ as a rational function in j_m for some nice choice of Δ, r. In all the examples we consider,

$$\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)}) = \frac{(j_m(\tau) - \gamma_2)^2}{(j_m(\tau) - \gamma_1)^2}$$

for some $\gamma_1, \gamma_2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{\Delta})}$. Note that $\Psi_{\Delta,r}$ will not always be a rational function of this particular form – we always picked Δ with class number 1.

5.4 Computing the elements in $Q_{\Delta,r}/\Gamma_0(m)$

In this section, we explain how to compute $Q_{\Delta,r}/\Gamma_0(m)$, following [20].

Table 7. Examples.

m	Δ	r	γ_1	γ_2
2	-7	1	$\frac{1+45\sqrt{-7}}{2}$	$\frac{1-45\sqrt{-7}}{2}$
3	-11	1	$17+8\sqrt{-11}$	$17-8\sqrt{-11}$
4	-7	3	$\frac{-15+3\sqrt{-7}}{2}$	$\frac{-15-3\sqrt{-7}}{2}$
5	-11	3	$-3+2\sqrt{-11}$	$-3-2\sqrt{-11}$
7	-19	3	$\frac{3+3\sqrt{-19}}{2}$	$\frac{3-3\sqrt{-19}}{2}$
9	-11	5	$-1+\sqrt{-11}$	$-1-\sqrt{-11}$
13	-43	3	$\frac{7+\sqrt{-43}}{2}$	$\frac{7-\sqrt{-43}}{2}$
25	-19	9	$\frac{\sqrt{-19}}{2}$	$\frac{-\sqrt{-19}}{2}$

Let $Q^0_{\Delta,r}$ be the subset of primitive forms. Then we have a $\Gamma_0(m)$-invariant bijection of sets

$$Q_{\Delta,r} = \bigcup_{\ell^2|\Delta} \left(\bigcup_{h\in S(\ell)} \ell Q^0_{\Delta/\ell^2,h} \right),$$

where $S(\ell) := \{h \in \mathbb{Z}/2m\mathbb{Z} : h^2 \equiv \Delta/\ell^2 \pmod{4m}, \ell h \equiv r \pmod{2m}\}$. Since we pick Δ to be a fundamental discriminant, the only possible prime we need to worry about is $\ell = 2$. In our examples, we always choose Δ, r such that $S(2) = \emptyset$. In this case, we just need to work with $Q^0_{\Delta,r}$.

Now, let $n := \left(m, r, \frac{r^2-\Delta}{4m}\right)$. Then for $Q = [mc, b, a] \in Q^0_{\Delta,r}$, define $n_1 := (m, b, a), n_2 := (m, b, c)$, which are coprime and have product n. We have the following result:

Lemma 5.2 (Section 1.1 of [20]). *Define n as above and fix a decomposition $n = n_1 n_2$ with n_1, n_2 positive and relatively prime. Then there is a 1:1 correspondence between the $\Gamma_0(m)$-equivalence classes of forms $[cm, b, a] \in Q^0_{\Delta,r}$ satisfying $(m, b, a) = n_1, (m, b, c) = n_2$ and the $\mathrm{SL}_2(\mathbb{Z})$ equivalence classes of forms in Q^0_Δ given by $Q = [mc, b, a] \mapsto \tilde{Q} = [cm_1, b, a\,m_2]$, where $m_1 \cdot m_2$ is any decomposition of m into coprime positive factors satisfying $(n_1, m_2) = (n_2, m_1) = 1$. In particular, $|Q^0_{\Delta,r}/\Gamma_0(m)| = 2^{\upsilon}|Q^0_\Delta/\mathrm{SL}_2(\mathbb{Z})|$, where υ is the number of prime factors of n.*

Note that $|Q^0_\Delta/\mathrm{SL}_2(\mathbb{Z})|$ equals $2h(\Delta)$ for $\Delta < 0$, where the factor of 2 arises because Q^0_Δ also contains negative semi-definite forms.

In our examples, we always choose Δ, r such that $n = 1$, so that $|Q^0_{\Delta,r}/\Gamma_0(m)| = |Q^0_\Delta/\mathrm{SL}_2(\mathbb{Z})| = 2h(\Delta)$, where $h(\Delta)$ is the class number of $\mathbb{Q}(\sqrt{\Delta})$. The theory of reduced forms allows us to easily compute $\mathcal{Q}^0_\Delta/\mathrm{SL}_2(\mathbb{Z})$.

6. p-adic properties of the logarithmic derivative

6.1 p-adic modular forms

For each $i \in \mathbb{N}$, let $f_i = \sum a_i(n)q^n$ be a modular form of weight k_i with $a_i(n) \in \mathbb{Q}$. If for each n, the $a_i(n)$ converge p-adically to $a(n) \in \mathbb{Q}_p$, then $f := \sum a(n)q^n$ is called a p-adic modular form. For $p \neq 2$, we define the weight space

$$W := \varprojlim_t \mathbb{Z}/\phi(p^t)\mathbb{Z} = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}.$$

For $p = 2$, we define

$$W := \varprojlim_t \mathbb{Z}/2^{t-2}\mathbb{Z} = \mathbb{Z}_2.$$

Then the k_i converge to an element $k \in W$, which we call the weight of f. We identify integers by their image in $\mathbb{Z}_p \times \{0\}$.

6.2 Proof of Theorem 1.7

Proof of Theorem 1.7. By Theorem 1.1, $\Psi_{\Delta,r}(\tau)$ is a meromorphic modular function, so that $\Theta(\Psi_{\Delta,r}(\tau))$ is a weight 2 meromorphic modular form on $\Gamma_0(m)$. Thus, the logarithmic derivative $\frac{\Theta(\Psi_{\Delta,r}(\tau))}{\Psi_{\Delta,r}(\tau)}$ is a weight 2 meromorphic modular form on $\Gamma_0(m)$ whose poles are simple and are supported on Heegner points of discriminant Δ. □

6.3 Proof of Theorem 1.8 and its Corollary

Proof of Theorem 1.8. We show that if (Δ, r) is an admissible pair and p is inert or ramified in $\mathbb{Q}(\sqrt{\Delta})$, that

$$f_{\Delta,r} := \frac{\Theta(\Psi_{\Delta,r}(\tau))}{\Psi_{\Delta,r}(\tau)}$$

is a p-adic modular form of weight 2. Say f has poles at $\alpha_1, \ldots, \alpha_n$, all of which are CM points of discriminant Δ. For each α_i, there is some zero β_i of E_{p-1} such that $j(\tau) - j(\alpha_i) \equiv j(\tau) - j(\beta_i)$ (see Theorem 1 of [21]). Then let

$$\mathcal{E} := E_{p-1} \prod_i \frac{(j(\tau) - j(\alpha_i))}{(j(\tau) - j(\beta_i))}.$$

This has weight $p-1$, is congruent to 1 modulo p, has zeros at $\alpha_1, \ldots, \alpha_n$, and has no poles. Let $f_t := f\mathcal{E}^{(p^t)}$. Then $f_t \equiv f \pmod{p^t}$ and is a modular form of weight $k_t = 2 + (p-1)p^t \equiv 2 \pmod{\phi(p^{t+1})}$, so f is a p-adic modular form of weight 2. □

Proof of Corollary 1.9. This corollary follows directly for the coefficients of any p-adic modular form using the following beautiful result, proven by Serre [24] using the theory of Galois representations.

Lemma 6.1 (Serre [24] Theorem 4.7 (I)). *Let K be a number field and $\mathcal{O}_K$ the ring of integers of K. Suppose $f(\tau) = \sum_{n\geq 0} a_n q^n \in \mathcal{O}_K[[q]]$ is a modular form of integer weight $k \geq 1$ on a congruence subgroup. For any prime p, let $\mathfrak{p}$ be a prime above p in $\mathcal{O}_K$. Let $m \geq 1$. Then there exists a positive constant α_m such that*

$$\#\{n \leq X : a_n \not\equiv 0 \pmod{\mathfrak{p})^m}\} = O\left(\frac{X}{(\log X)^{\alpha_m}}\right).$$

□

7. Appendix: Definitions of Jacobi forms, Theta functions, etc.

We define the Jacobi theta functions $\theta_i(\tau, z)$ as follows for $q := e(\tau)$ and $y := e(z)$.

$$\theta_2(\tau, z) := q^{1/8} y^{1/2} \prod_{n=1}^{\infty} (1-q^n)(1+yq^n)(1+y^{-1}q^{n-1})$$

$$\theta_3(\tau, z) := \prod_{n=1}^{\infty} (1-q^n)(1+yq^{n-1/2})(1+y^{-1}q^{n-1/2})$$

$$\theta_4(\tau, z) := \prod_{n=1}^{\infty} (1-q^n)(1-yq^{n-1/2})(1-y^{-1}q^{n-1/2})$$

We use them to define weight zero index $m-1$ weak Jacobi forms $\varphi_1^{(m)}$ as follows. Let

$$\varphi_1^{(2)} := 4(f_2^2 + f_3^2 + f_4^2),$$

$$\varphi_1^{(3)} := 2(f_2^2 f_3^2 + f_3^2 f_4^2 + f_4^2 f_2^2),$$

$$\varphi_1^{(4)} := 4 f_2^2 f_3^2 f_4^2,$$

$$\varphi_1^{(5)} := \frac{1}{4}(\varphi_1^{(4)}\varphi_1^{(2)} - (\varphi_1^{(3)})^2)$$

$$\varphi_1^{(7)} := \varphi_1^{(3)}\varphi_1^{(5)} - (\varphi_1^{(4)})^2$$

$$\varphi_1^{(9)} := \varphi_1^{(3)}\varphi_1^{(7)} - (\varphi_1^{(5)})^2$$
$$\varphi_1^{(13)} := \varphi_1^{(5)}\varphi_1^{(9)} - 2(\varphi_1^{(7)})^2$$

where $f_i(\tau, z) := \theta_i(\tau, z)/\theta_i(\tau, 0)$ for $i = 2, 3, 4$.

For the remaining positive integers m with $m \leq 25$, we define $\varphi_1^{(m)}$ recursively.

For $(12, m-1) = 1$ and $m > 5$ we set

$$\varphi_1^{(m)} = (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m-3)\varphi_1^{(m-2)}\varphi_1^{(3)} - 2(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)}.$$

For $(12, m-1) = 2$ and $m > 10$ we set

$$\varphi_1^{(m)} = \frac{1}{2}((12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m-3)\varphi_1^{(m-2)}\varphi_1^{(3)} - 2(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)}).$$

For $(12, m-1) = 3$ and $m > 9$, we set

$$\varphi_1^{(m)} = \frac{2}{3}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{3}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)} - (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)}.$$

For $(12, m-1) = 4$ and $m > 16$ we set

$$\varphi_1^{(m)} = \frac{1}{4}((12, m-13)\varphi_1^{(m-12)}\varphi_1^{(13)} + (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} - (12, m-9)\varphi_1^{(m-8)}\varphi_1^{(9)}).$$

For $(12, m-1) = 6$ and $m > 18$ we set

$$\varphi_1^{(m)} = \frac{1}{3}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{6}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)} - \frac{1}{2}(12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)}.$$

For $m = 25$, we set

$$\varphi_1^{(25)} = \frac{1}{2}\varphi_1^{(21)}\varphi_1^{(5)} - \varphi_1^{(19)}\varphi_1^{(7)} + \frac{1}{2}(\varphi_1^{(13)})^2.$$

See the appendix of [10] for more information on the space of weight zero Jacobi forms.

We use two versions of an Appell-Lerch sum. The first is the generalized Appell-Lerch sum $\mu_{m,0}$, defined as in [10]. It is given by

$$\mu_{m,0}(\tau, z) := -\sum_{k\in\mathbb{Z}} q^{mk^2} y^{2mk} \frac{1+yq^k}{1-yq^k},$$

and is the holomorphic part of a weight 1 index m "real-analytic Jacobi form".

Zwegers [30] uses a slightly different version of the Appell-Lerch sum. He first defines the theta function

$$\vartheta(z, \tau) := \sum_{\nu\in 1/2+\mathbb{Z}} q^{\nu^2/2} y^{\nu} e(\nu/2).$$

Then he defines

$$\mu(u, v; \tau) := \frac{e(u/2)}{\vartheta(v; \tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{(n^2+n)/2} e(nv)}{1-q^n e(u)}.$$

This is completed to a "real-analytic Jacobi form" $\tilde{\mu}(u, v; \tau)$ of weight 1/2 by letting

$$\tilde{\mu}(u, v; \tau) := \mu(u, v; \tau) + \frac{i}{2} R(u-v; \tau),$$

where

$$R(z, \tau) := \sum_{\nu\in 1/2+\mathbb{Z}} \{\operatorname{sgn}(\nu) - E(\nu + a)\sqrt{2t}\}(-1)^{\nu-1/2} q^{-\nu^2/2} y^{-\nu},$$

$t := \Im(\tau)$, $a := \frac{\Im(u)}{\Im(\tau)}$, and $E(z) := 2\int_0^z e^{-\pi u^2} du$.

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2. Actions of $\mathbb{Z}_2$ on finite dimensional real C^*-algebras

In this section, we describe all possible actions of $\mathbb{Z}_2$, the group with two elements, on finite dimensional real C^*-algebras. Our complete list is given by the following theorem.

Theorem 1. *Let $(A_\varphi, \mathbb{Z}_2, \alpha)$ be a real C^*-dynamical system with A finite dimensional. Then $(A, \mathbb{Z}_2, \alpha)$ is equivariantly isomorphic to a direct sum of C^*-dynamical $\mathbb{Z}_2$ systems of the following special forms (we describe the non-identity automorphism):*

$\alpha_{\mathbb{R}1}^{\{l,k\}} : M_n(\mathbb{R}) \to M_n(\mathbb{R})$, *given by* $x \mapsto Ad\, u(x)$, *where* $u = I_k \oplus (-I_l)$ *and* $n = k + l$;

$\alpha_{\mathbb{R}2}^{2n} : M_{2n}(\mathbb{R}) \to M_{2n}(\mathbb{R})$, *given by* $x \mapsto Ad\, D_n(x)$, *where* $D_n = \mathrm{diag}(D, \dots, D)$ *(n times) and D is the rotation matrix* $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$;

$\alpha_{\mathbb{R}\mathbb{R}}^{n} : M_n(\mathbb{R}) \oplus M_n(\mathbb{R}) \to M_n(\mathbb{R}) \oplus M_n(\mathbb{R})$, *given by* $(x, y) \mapsto (y, x)$;

$\alpha_{\mathbb{H}1}^{\{l,k\}} : M_n(\mathbb{H}) \to M_n(\mathbb{H})$, *given by* $x \mapsto Ad\, u(x)$, *where* $u = I_k \oplus (-I_l)$ *and* $n = k + l$;

$\alpha_{\mathbb{H}2}^{n} : M_n(\mathbb{H}) \to M_n(\mathbb{H})$, *given by* $x \mapsto Ad\, u(x)$, *where* $u = i\, I_n$;

$\alpha_{\mathbb{H}\mathbb{H}}^{n} : M_n(\mathbb{H}) \oplus M_n(\mathbb{H}) \to M_n(\mathbb{H}) \oplus M_n(\mathbb{H})$, *given by* $(x, y) \mapsto (y, x)$;

$\alpha_{\mathbb{C}1}^{\{l,k\}} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, *given by* $x \mapsto Ad\, u(x)$, *where* $u = I_k \oplus (-I_l)$ *and* $n = k + l$;

$\alpha_{\mathbb{C}2}^{n} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, *given by* $x \mapsto \bar{x}$, *where* $\bar{x}$ *denotes the matrix derived from x by taking the complex conjugates of all the entries;*

$\alpha_{\mathbb{C}3}^{2n} : M_{2n}(\mathbb{C}) \to M_{2n}(\mathbb{C})$, *given by* $x \mapsto Ad\, D_n(\bar{x})$, *where* D_n *is as above;*

$\alpha_{\mathbb{C}\mathbb{C}}^{n} : M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \to M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$, *given by* $(x, y) \mapsto (y, x)$.

Furthermore, the special forms described above are pairwise non-isomorphic, so the expression of a given dynamical system as a direct sum of them is unique up to reordering the summands.

Proof. Let $(A_\varphi, \mathbb{Z}_2, \alpha)$ be a real C^*-dynamical system with A finite dimensional. Then α restricts to a period two permutation of the set of minimal central projections of A. It follows that the minimal central projections are either fixed or switched in pairs. Hence we may write $A \cong A_1 \oplus \cdots \oplus A_k$, where each A_i is either a simple summand of A that is left globally invariant by α, or is a direct sum of two simple summands whose minimal central projections are interchanged by α. In the latter case, since α maps one summand to the other, the two summands must be isomorphic. We shall now determine all of the possible isomorphism classes for the sub-systems $(A_i, \mathbb{Z}_2, \alpha)$.

Suppose first that $A_i \cong M_n(\mathbb{R})$. It follows from [4] that $\alpha : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is given by $\alpha(x) = Ad\, u(x)$ for some orthogonal matrix u. Since α gives an action of $\mathbb{Z}_2$, it follows that u^2 lies in the centre of $M_n(\mathbb{R})$, so $u^2 = 1$ or $u^2 = -1$. If $u^2 = 1$, then u is conjugate within the orthogonal matrices to $I_k \oplus (-I_l)$ for some k, l with $k + l = n$, giving us the form $\alpha_{\mathbb{R}1}^{\{l,k\}}$. The equation $u^2 = -1$ can only hold for an orthogonal matrix with even dimension, and in this case u is conjugate within the orthogonal matrices to one of the form D_n, giving us form $\alpha_{\mathbb{R}2}^{2n}$.

Suppose next that $A_i \cong M_n(\mathbb{H})$. Again it follows from [4] that $\alpha : M_n(\mathbb{H}) \to M_n(\mathbb{H})$ is given by $\alpha(x) = Ad\, u(x)$ for some unitary matrix u, and, since the centre of $M_n(\mathbb{H})$ is also $\mathbb{R}I_n$, we have $u^2 = 1$ or $u^2 = -1$. It follows from the spectral theorem for quaternionic matrices (cf [3]) that u is conjugate within the unitary group of $M_n(\mathbb{H})$ to a diagonal matrix with entries in the closed upper half plane of $\mathbb{C}$. If $u^2 = 1$, then the diagonal entries are 1 or -1, and we have form $\alpha_{\mathbb{H}1}^{\{l,k\}}$. If $u^2 = -1$, then we have $i\, I_n$, and form $\alpha_{\mathbb{H}2}^{n}$.

Suppose next that $A_i \cong M_n(\mathbb{C})$. Consider the restriction of α to the centre of A_i. This is either the identity map, or complex conjugation. If it is the identity map, then it follows from [4] that $\alpha : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is given by $\alpha(x) = Ad\, u(x)$ for some unitary matrix u, and since i lies in the centre of A_i, we have only the one case, $u^2 = 1$, to consider. It follows that if the action on the centre is trivial, we have form $\alpha_{\mathbb{C}1}^{\{l,k\}}$. If the action on the centre is complex conjugation, then it follows from [4] that α is given by $\alpha(x) = Ad\, u(\bar{x})$ for some unitary matrix u. We have $\alpha^2(x) = u\overline{(u\bar{x}u^*)}u^* = (u\bar{u})x(u\bar{u})^* = x$, for all x, so $u\bar{u} = \gamma I_n$ for some $\gamma \in \mathbb{T}$. We have $\bar{u} = \gamma u^*$, and since u is unitary, $\bar{u}$ commutes with u and $u\bar{u} = \bar{u}u = \bar{\gamma} I_n$, so $\gamma = \bar{\gamma}$ and $\gamma = 1$ or $\gamma = -1$.

Consider first the case where $\gamma = 1$. In this case, $\bar{u} = u^*$, so u is a symmetric matrix. It follows that there exists a symmetric matrix w in the functional calculus of u such that $wu = uw$ and $w^2 = u$. We then have $w^* u \bar{w} = 1$. It follows that $\alpha \circ Ad\, w = Ad\, w \circ \beta$, where $\beta(x) = \bar{x}$. Thus, in this case we have form $\alpha_{\mathbb{C}2}^{n}$.

Now consider the case where $\gamma = -1$. In this case, we have $u = -u^T$, where the superscript T denotes the transpose, so considering determinants shows we must have n even. We have that u^T has the same spectral projections as u, so taking the transpose, which for projections is the same as taking their complex conjugates, induces a period two permutation of them. Since the eigenvalues are multiplied by -1, none of the projections is fixed by taking the transpose. Thus we may write $u = (a_1 P_1 - a_1 Q_1) + \cdots + (a_k P_k - a_k Q_k)$ where P_i, Q_i are the spectral projections of u, a's and $-a$'s are its eigenvalues, and $Q_i = P_i^T = \bar{P}_i$ for each i. Since taking complex conjugates is a *-automorphism, it follows that the rank of P_i is the same as that of Q_i for each i. If P_i is not of rank one, we may write $P_i = p_{i1} + \cdots + p_{ij}$ for a set of pairwise orthogonal sub-projections $p_{i1} + \cdots + p_{ij}$. If we let $q_{il} = p_{il}^T$, it follows that in our decomposition of u above, we may assume that all of the P_i, Q_i are rank one, although the a's may no longer be distinct. Assuming we have done this, consider the projection $E_i = P_i + Q_i$. We have $E_i = \bar{E}_i$, so E_i is a rank two projection in $M_n(\mathbb{R})$. It follows that there exists a real orthogonal matrix M such that $M^T u M$ is a block diagonal matrix with two by two blocks on the diagonal, $M^T u M = \operatorname{diag}(u_1, \ldots u_{(n/2)})$ say, where each u_k satisfies $u_k^T = -u_k$. For a two by two unitary matrix, it is easy to see that this implies $u_k = b_k D_2$ for some $b_k \in \mathbb{T}$. Letting $c_k \in \mathbb{T}$ be such that $c_k^2 = b_k$, and letting $C = \operatorname{diag}(c_i I_2, \ldots, c_{(n/2)} I_2)$, we have $C = C^T$ and $u = (MC) D_n (MC)^T$. Let $w = (MC)$. Then $\alpha_{\mathbb{C}3}^{2n} \circ Ad\, w^*(x) = Ad\,(D_n w^T)(\bar{x})$ and $Ad\, w^* \circ \alpha(x) = Ad\,(w^* u)(\bar{x})$, but $w^* u = D_n w^T$, so we have that our system is isomorphic to form $\alpha_{\mathbb{C}3}^{2n}$.

Suppose next that $A_i \cong M_n(\mathbb{F}) \oplus M_n(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{H}$. Then, from [4], there exist unitaries u, v in $M_n(\mathbb{F})$ such that $\alpha(x, y) = (Ad\, u(y), Ad\, v(x))$. Since $\alpha^2 = id$, it follows that $uv = 1$ or $uv = -1$, and we may take $v = u^*$. Now let $w = (v, v^2)$. We have $\alpha \circ Ad\, w(x, y) = \alpha(vxv^*, v^2 y (v^2)^*) = (uv(vyv^*)v^*u^*, v(vxv^*)v^*) = (vyv^*, (v^2)x(v^2)^*)$. Let $\beta(x, y) = (y, x)$. Then we have $Ad\, w \circ \beta(x, y) = Ad\, w(y, x) = (vyv^*, (v^2)x(v^2)^* = \alpha \circ Ad\, w(x, y)$. Thus we have the forms $\alpha_{\mathbb{R}\mathbb{R}}^n$ and $\alpha_{\mathbb{H}\mathbb{H}}^n$.

Finally, suppose next that $A_i \cong M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. We have $\alpha(x, y) = (\delta(y), \gamma(x))$, Where δ and γ are either of the form $x \mapsto uxu^*$ or of the form $x \mapsto v\bar{x}v^*$ for unitaries u, v. If both are of the form $x \mapsto uxu^*$, then the same argument as in the case of $\mathbb{R}$ or $\mathbb{H}$ shows that we have the form $\alpha_{\mathbb{C}\mathbb{C}}^n$. It is not possible to have $\alpha(x, y) = (Ad\, u(\bar{y}), Ad\, v(x))$ or $\alpha(x, y) = (Ad\, u(y), Ad\, v(\bar{x}))$, since this would result in α^2 failing to be the identity action on the centre. Thus it remains to consider $\alpha(x, y) = (Ad\, u(\bar{y}), Ad\, v(\bar{x}))$. We have that $\alpha^2 = id$ gives $u\bar{v} = \lambda 1$ for some $\lambda \in \mathbb{T}$, so replacing u with λu if necessary, we may assume $v = u^T$. Let $w = (1, u^T)$, and let $\eta(x, y) = (\bar{y}, \bar{x})$. Then $\alpha \circ Ad\, w(x, y) = (u(u^* \bar{y} u)u^*, u^T(\bar{x})(u^T)^*) = (\bar{y}, u^T(\bar{x})(u^T)^*) = Ad\, w \circ \eta(x, y)$, so α and η are inner conjugate. We complete this part of the proof by noticing that η is outer conjugate to $\alpha_{\mathbb{C}\mathbb{C}}^n$ by the outer automorphism $(x, y) \mapsto (\bar{x}, y)$.

That the above special forms are pairwise non-isomorphic follows from considering their fixed point sub-algebras. Elementary calculations show that: for $\alpha_{\mathbb{R}1}^{\{l,k\}}$, we have $A^{\mathbb{Z}_2} \cong M_l(\mathbb{R}) \oplus M_k(\mathbb{R})$; for $\alpha_{\mathbb{R}2}^{2n}$, we have $A^{\mathbb{Z}_2} \cong M_n(\mathbb{C})$; for $\alpha_{\mathbb{R}\mathbb{R}}^n$, we have $A^{\mathbb{Z}_2} \cong M_n(\mathbb{R})$; for $\alpha_{\mathbb{H}1}^{\{l,k\}}$, we have $A^{\mathbb{Z}_2} \cong M_l(\mathbb{H}) \oplus M_k(\mathbb{H})$; for $\alpha_{\mathbb{H}2}^n$, we have $A^{\mathbb{Z}_2} \cong M_n(\mathbb{C})$; for $\alpha_{\mathbb{H}\mathbb{H}}^n$, we have $A^{\mathbb{Z}_2} \cong M_n(\mathbb{H})$;

for $\alpha_{\mathbb{C}1}^{\{l,k\}}$, we have $A^{\mathbb{Z}_2} \cong M_l(\mathbb{C}) \oplus M_k(\mathbb{C})$; for $\alpha_{\mathbb{C}2}^{n}$, we have $A^{\mathbb{Z}_2} \cong M_n(\mathbb{R})$; for $\alpha_{\mathbb{C}3}^{2n}$, we have $A^{\mathbb{Z}_2} \cong M_n(\mathbb{H})$; and for $\alpha_{\mathbb{C}\mathbb{C}}^{n}$, we have $A^{\mathbb{Z}_2} \cong M_n(\mathbb{C})$. It follows that for every pair, either the algebras are non-isomorphic, or the fixed point sub-algebras are non-isomorphic. □

Remark 1. Inspecting the crossed products of the basic building blocks above shows that two of them are cocycle conjugate (exterior equivalent) if, and only if, they are isomorphic.

3. Homomorphisms between building blocks

In this section, we describe possible unital equivariant *-homomorphisms between our basic building blocks. We shall need these for the existence theorem in section 5.

We consider first the equivariant unital maps between simple building blocks isomorphic to $M_n(\mathbb{R})$ or $M_n(\mathbb{H})$. In these cases, the embedding $\psi : A \to B$ gives a tensor product decomposition of $B \cong \psi(A) \otimes_{\mathbb{R}} (A' \cap B)$, so the action on B is a tensor product action. We have the following combinations:

$$\alpha_{\mathbb{R}1}^{\{l,k\}} \otimes \alpha_{\mathbb{R}1}^{\{r,s\}} \cong \alpha_{\mathbb{R}1}^{\{lr+ks,kr+ls\}} \qquad \alpha_{\mathbb{H}1}^{\{l,k\}} \otimes \alpha_{\mathbb{H}1}^{\{r,s\}} \cong \alpha_{\mathbb{R}1}^{\{4(lr+ks),4(kr+ls)\}}$$

$$\alpha_{\mathbb{R}1}^{\{l,k\}} \otimes \alpha_{\mathbb{R}2}^{2n} \cong \alpha_{\mathbb{R}2}^{2n(k+l)} \qquad \alpha_{\mathbb{H}1}^{\{l,k\}} \otimes \alpha_{\mathbb{H}2}^{n} \cong \alpha_{\mathbb{R}2}^{4n(k+l)}$$

$$\alpha_{\mathbb{R}2}^{2n} \otimes \alpha_{\mathbb{R}2}^{2m} \cong \alpha_{\mathbb{R}1}^{\{2mn,2mn\}} \qquad \alpha_{\mathbb{H}2}^{n} \otimes \alpha_{\mathbb{H}2}^{m} \cong \alpha_{\mathbb{R}1}^{\{2mn,2mn\}}$$

$$\alpha_{\mathbb{R}1}^{\{l,k\}} \otimes \alpha_{\mathbb{H}1}^{\{r,s\}} \cong \alpha_{\mathbb{H}1}^{\{lr+ks,kr+ls\}} \qquad \alpha_{\mathbb{R}2}^{2n} \otimes \alpha_{\mathbb{H}1}^{\{l,k\}} \cong \alpha_{\mathbb{H}_2}^{n(k+l)}$$

$$\alpha_{\mathbb{R}1}^{\{l,k\}} \otimes \alpha_{\mathbb{H}2}^{n} \cong \alpha_{\mathbb{H}2}^{2n(k+l)} \qquad \alpha_{\mathbb{R}2}^{2n} \otimes \alpha_{\mathbb{H}2}^{m} \cong \alpha_{\mathbb{H}_1}^{\{mn,mn\}}$$

It should be noticed that, with these building blocks, in no case do we have more than one kind of embedding from one type of building block into another.

Next, we consider the case where the target algebra is non-simple. We have $\psi : A \to B$; $x \mapsto (\psi_1(x), \psi_2(x))$, where ψ_1 and ψ_2 are unital *-homomorphisms. Equivariance implies that $\psi_2 = \psi_1 \circ \alpha$, and there are no other restrictions.

Next, we consider the case of a non-simple building block mapped to a simple one. Suppose $\psi : A \oplus A \to B$; $(x, y) \mapsto \psi_1(x) + \psi_2(y)$. Then $\psi(x, 0) = \psi_1(x) = \psi(\alpha(0, x)) = \beta(\psi(0, x)) = \beta \circ \psi_2(x)$, so we must have $\psi_1 = \beta \circ \psi_2$. There are no other restrictions, so we will have an equivariant unital *-homomorphism from $A \oplus A \to B$ if, and only if, there exist projections $p_1, p_2 \in B$ such that $p_1 + p_2 = 1$, $\beta(p_1) = p_2$, and there exists a unital

*-homomorphism $\psi_1 : A \to p_1 B p_1$. A straightforward inspection shows that projections $p_1, p_2 \in B$ with $p_1 + p_2 = 1$ and $\beta(p_1) = p_2$ exists if, and only if, n is even and, in the case that the target algebra's action is one of $\alpha_{\mathbb{R}1}^{\{r,s\}}$, $\alpha_{\mathbb{C}1}^{\{r,s\}}$, or $\alpha_{\mathbb{H}1}^{\{r,s\}}$, we have $r = s$.

It remains to consider those cases where both building blocks are simple, and one or both of them are of the form $M_n(\mathbb{C})$.

Consider the case where both building blocks are complex matrices. If both actions restrict to complex conjugation on the centre, then the actions respect the tensor product decomposition of B as $\psi(A) \otimes_{\mathbb{C}} (\psi(A)' \cap B)$, and we have the combinations $\alpha_{\mathbb{C}2}^{n} \otimes_{\mathbb{C}} \alpha_{\mathbb{C}2}^{m} \cong \alpha_{\mathbb{C}2}^{mn}$, $\alpha_{\mathbb{C}2}^{n} \otimes_{\mathbb{C}} \alpha_{\mathbb{C}3}^{2m} \cong \alpha_{\mathbb{C}3}^{2mn}$, and $\alpha_{\mathbb{C}3}^{2n} \otimes_{\mathbb{C}} \alpha_{\mathbb{C}3}^{2m} \cong \alpha_{\mathbb{C}2}^{4mn}$. If both actions are trivial on the centre, then both must be type $\alpha_{\mathbb{C}1}^{\{l,k\}}$, and we have the complex case $\alpha_{\mathbb{C}1}^{\{l,k\}} \otimes_{\mathbb{C}} \alpha_{\mathbb{C}1}^{\{r,s\}} \cong \alpha_{\mathbb{C}1}^{\{rk+ls,sk+rl\}}$ from [4].

It is clear that there is no equivariant *-homomorphism from $\alpha_{\mathbb{C}1}^{\{1,0\}}$ to $\alpha_{\mathbb{C}2}^{1}$, however, $\alpha_{\mathbb{C}1}^{\{1,0\}}$ embeds into $\alpha_{\mathbb{R}1}^{\{2,0\}}$ with the standard embedding, and $\alpha_{\mathbb{R}1}^{\{2,0\}}$ embeds into $\alpha_{\mathbb{C}2}^{2}$, with the standard embedding. Using this, we get the following: $\alpha_{\mathbb{C}1}^{\{k,l\}} \cong \alpha_{\mathbb{R}1}^{\{l,k\}} \otimes_{\mathbb{R}} \alpha_{\mathbb{C}1}^{\{1,0\}} \to \alpha_{\mathbb{R}1}^{\{l,k\}} \otimes_{\mathbb{R}} \alpha_{\mathbb{C}2}^{2} \cong \alpha_{\mathbb{C}2}^{2(k+l)}$. Using a tensor product with one of our actions found above, we get: $\alpha_{\mathbb{C}1}^{\{l,k\}} \cong \alpha_{\mathbb{R}1}^{\{l,k\}} \otimes_{\mathbb{R}} \alpha_{\mathbb{C}1}^{\{1,0\}} \to \alpha_{\mathbb{C}3}^{2(k+l)} \otimes_{\mathbb{R}} \alpha_{\mathbb{C}1}^{\{1,0\}} \cong \alpha_{\mathbb{C}3}^{2(k+l)} \oplus \alpha_{\mathbb{C}3}^{2(k+l)}$. Thus we get an embedding from $\alpha_{\mathbb{C}1}^{\{l,k\}}$ into $\alpha_{\mathbb{C}3}^{2(k+l)}$. In the other direction, we have $\alpha_{\mathbb{C}3}^{2} \to \alpha_{\mathbb{R}1}^{\{2,2\}} \to \alpha_{\mathbb{C}1}^{\{2,2\}}$, and $\alpha_{\mathbb{C}2}^{1} \to \alpha_{\mathbb{R}2}^{2} \to \alpha_{\mathbb{R}2}^{2} \otimes_{\mathbb{R}} \alpha_{\mathbb{R}2}^{2} \cong \alpha_{\mathbb{R}1}^{\{2,2\}} \to \alpha_{\mathbb{C}1}^{\{2,2\}}$. Tensoring with a standard embedding from $M_n(\mathbb{R})$ to $M_{nl}(\mathbb{R})$, with the identity actions, gives higher multiplicities.

Consider an embedding $\psi : M_n(\mathbb{R}) \to M_{nk}(\mathbb{C})$. Here we have $B \cong \psi(A) \otimes_{\mathbb{R}} (\psi(A)' \cap B)$, and our action on B decomposes as a tensor product action. We have the following possible combinations: $\alpha_{\mathbb{R}1}^{\{l,k\}} \otimes \alpha_{\mathbb{C}1}^{\{r,s\}} \cong \alpha_{\mathbb{C}1}^{\{lr+ks,kr+ls\}}$, $\alpha_{\mathbb{R}2}^{2n} \otimes \alpha_{\mathbb{C}1}^{\{r,s\}} \cong \alpha_{\mathbb{C}2}^{2m(r+s)}$, $\alpha_{\mathbb{R}1}^{\{l,k\}} \otimes \alpha_{\mathbb{C}2}^{n} \cong \alpha_{\mathbb{C}2}^{n(l+k)}$, $\alpha_{\mathbb{R}2}^{2n} \otimes \alpha_{\mathbb{C}2}^{m} \cong \alpha_{\mathbb{C}3}^{2nm}$, $\alpha_{\mathbb{R}1}^{\{l,k\}} \otimes \alpha_{\mathbb{C}3}^{2m} \cong \alpha_{\mathbb{C}3}^{2m(l+k)}$, and $\alpha_{\mathbb{R}2}^{2n} \otimes \alpha_{\mathbb{C}3}^{2m} \cong \alpha_{\mathbb{C}2}^{4nm}$.

With the embeddding $\psi : M_n(\mathbb{H}) \to M_{2nk}(\mathbb{C})$, we again have $B \cong \psi(A) \otimes_{\mathbb{R}} (\psi(A)' \cap B)$, and our action on B decomposes as a tensor product action. We have the following possible combinations: $\alpha_{\mathbb{H}1}^{\{l,k\}} \otimes \alpha_{\mathbb{C}1}^{\{r,s\}} \cong \alpha_{\mathbb{C}1}^{2\{lr+ks,kr+ls\}}$, $\alpha_{\mathbb{H}2}^{n} \otimes \alpha_{\mathbb{C}1}^{\{r,s\}} \cong \alpha_{\mathbb{C}1}^{\{2nr,2ns\}}$, $\alpha_{\mathbb{H}1}^{\{l,k\}} \otimes \alpha_{\mathbb{C}2}^{n} \cong \alpha_{\mathbb{C}2}^{2n(l+k)}$, $\alpha_{\mathbb{H}2}^{n} \otimes \alpha_{\mathbb{C}2}^{m} \cong \alpha_{\mathbb{C}3}^{2nm}$, $\alpha_{\mathbb{H}1}^{\{l,k\}} \otimes \alpha_{\mathbb{C}3}^{2m} \cong \alpha_{\mathbb{C}3}^{4m(l+k)}$, and $\alpha_{\mathbb{H}2}^{n} \otimes \alpha_{\mathbb{C}3}^{2m} \cong \alpha_{\mathbb{C}2}^{4nm}$.

Finally, we have to consider maps from $M_n(\mathbb{C})$ into $M_{2nk}(\mathbb{R})$ and $M_{nk}(\mathbb{H})$. In these cases, we no longer have $B \cong \psi(A) \otimes_{\mathbb{R}} (\psi(A)' \cap B)$, as $\psi(A) \cap (\psi(A)' \cap B)$ contains more than the scalars.

It is straightforward to check that there exists a unital equivariant embedding from $\mathbb{C}$ with complex conjugation to $M_2(\mathbb{R})$ with action given by the unitary $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which is conjugate to type $\alpha_{\mathbb{R}1}^{1,1}$. However, a simple calculation shows that if $Ad\, u$ implements complex conjugation on the image of

$\mathbb{C}$ under the standard embedding, then the determinant of u must be -1, so there does not exist an embedding of $\mathbb{C}$ into $M_2(\mathbb{R})$ with action of type $\alpha^2_{\mathbb{R}2}$. We do get an equivariant inclusion from $\mathbb{C}$ into $M_4(\mathbb{R})$ with action $\alpha^4_{\mathbb{R}2} \cong \alpha^{\{1,1\}}_{\mathbb{R}1} \otimes_{\mathbb{R}} \alpha^2_{\mathbb{R}2}$ given by $x \mapsto st(x) \otimes 1$, where st denotes the standard embedding above. One can check that conjugation by $j \in \mathbb{H}$ gives complex conjugation on the embedded copy of $\mathbb{C}$, and that this action on $\mathbb{H}$ is conjugate to $\alpha^1_{\mathbb{H}2}$. Proceeding as above, we get the following embeddings: $\alpha^1_{\mathbb{C}2} \to \alpha^1_{\mathbb{H}2} \otimes_{\mathbb{R}} \alpha^2_{\mathbb{R}2} \cong \alpha^{\{1,1\}}_{\mathbb{H}1}$, $\alpha^2_{\mathbb{C}3} \cong \alpha^1_{\mathbb{H}2} \otimes_{\mathbb{R}} \alpha^1_{\mathbb{C}2} \to \alpha^1_{\mathbb{H}2} \otimes_{\mathbb{R}} \alpha^1_{\mathbb{H}2} \cong \alpha^{\{2,2\}}_{\mathbb{R}1}$, $\alpha^2_{\mathbb{C}3} \cong \alpha^1_{\mathbb{H}2} \otimes_{\mathbb{R}} \alpha^1_{\mathbb{C}2} \to \alpha^1_{\mathbb{H}2} \otimes_{\mathbb{R}} \alpha^{\{1,1\}}_{\mathbb{R}1} \cong \alpha^2_{\mathbb{H}2}$, $\alpha^2_{\mathbb{C}3} \cong \alpha^1_{\mathbb{H}2} \otimes_{\mathbb{R}} \alpha^1_{\mathbb{C}2} \to \alpha^1_{\mathbb{H}2} \otimes_{\mathbb{R}} \alpha^2_{\mathbb{R}2} \cong \alpha^{\{1,1\}}_{\mathbb{H}1}$, and $\alpha^2_{\mathbb{C}3} \cong \alpha^1_{\mathbb{H}2} \otimes_{\mathbb{R}} \alpha^1_{\mathbb{C}2} \to \alpha^1_{\mathbb{H}2} \otimes_{\mathbb{R}} \alpha^{\{1,1\}}_{\mathbb{H}1} \cong \alpha^4_{\mathbb{R}2}$. Tensoring with the standard embedding of $M_n(\mathbb{R})$ into $M_{nl}(\mathbb{R})$ gives embeddings $\alpha^n_{\mathbb{C}2} \to \alpha^{\{nl,nl\}}_{\mathbb{H}1}$, and so on.

That these are, up to conjugacy, all of the possibilities for the given pairs of algebras will follow from the uniqueness theorem in section 5.

4. The Invariant

In this section, we define our invariant, and describe it for each of the special building blocks from the first section.

Definition 1. *Given a real C^*-dynamical system $(A, \alpha, \mathbb{Z}_2)$, let $S(A,\alpha)$ denote the subset of $K_0(A)^+$ consisting of the classes of projections p in some matrix algebra $A \otimes_{\mathbb{R}} M_n(\mathbb{R})$ such that there exists a unital equivariant $*$-homomorphism from $\alpha^1_{\mathbb{C}2}$ to $(A \otimes_{\mathbb{R}} M_n(\mathbb{R}), \alpha \otimes id)$ taking the unit to p.*

Definition 2. *Given an action α of $\mathbb{Z}_2$ on a real C^*-algebra A_φ, our invariant for α, to be denoted* $\mathrm{Inv}(A_\varphi, \alpha)$ *shall consist of the commutative diagram*

$$\begin{array}{ccccc}
K_0(A_\varphi) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A_\varphi \otimes_{\mathbb{R}} \mathbb{H}) \\
\downarrow & & \downarrow & & \downarrow \\
K_0(A_\varphi \rtimes^{\mathbb{R}}_{\alpha} (\mathbb{Z}_2)) & \longrightarrow & K_0(A \rtimes_\alpha (\mathbb{Z}_2)) & \longrightarrow & K_0(A_\varphi \rtimes^{\mathbb{R}}_{\alpha} (\mathbb{Z}_2) \otimes_{\mathbb{R}} \mathbb{H}),
\end{array}$$

where each map is induced by the canonical inclusion, along with the the positive cones of all of the K_0 groups, the classes of the identity elements in all the K_0 groups, the class of the special element $[(1+g)/2]$, where g is the nonidentity element of $\mathbb{Z}_2$, in the middle K_0 group in the bottom row, the period two automorphisms induced by α on the top row, the period two automorphisms induced by $\hat{\alpha}$ on K_0 groups of the crossed products in the bottom row, and the distinguished sub-semigroup $S(A,\alpha)$.

Given two such systems, G and H say, a morphism of invariants from G to H shall consist of six group homomorphisms ν_r, $\nu_c \nu_h$, μ_r, μ_c, μ_h such that the following diagram

$$\begin{tikzcd}
G_1 \arrow[rr] \arrow[dr, "\nu_r"] \arrow[dd] & & G_2 \arrow[rr] \arrow[dr, "\nu_c"] \arrow[dd] & & G_3 \arrow[dr, "\nu_h"] \arrow[dd] & \\
& H_1 \arrow[rr] \arrow[dd] & & H_2 \arrow[rr] \arrow[dd] & & H_3 \arrow[dd] \\
G_4 \arrow[rr] \arrow[dr, "\mu_r"] & & G_5 \arrow[rr] \arrow[dr, "\mu_c"] & & G_6 \arrow[dr, "\mu_h"] & \\
& H_4 \arrow[rr] & & H_5 \arrow[rr] & & H_6
\end{tikzcd}$$

commutes, each of the six maps respects the positive cones, specified elements, and distinguished sub-semigroup, and all of the maps intertwine the $\mathbb{Z}_2$ actions on their domains and ranges.

Next, we describe the invariants for each of the basic forms from the preceding section. Below, $\mathbb{Z}$ always has its usual ordering, and $\mathbb{Z} \oplus \mathbb{Z}$ has positive cone $\{(n, m) \mid n, m \geq 0\}$. We only describe $S(A, \alpha)$ for the case we shall need it in for the existence theorem below.

For $\alpha_{\mathbb{R}1}^{\{l,k\}}$ we have:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z},
\end{tikzcd}$$

where each horizontal map is the identity map, and each vertical map is $x \mapsto (x, x)$. In each case, α_* is the identity and $\hat{\alpha}_*$ flips the two summands. The class of the identity in each group in the top row is n, in the bottom row it is (n, n), and the class of the special element is (k, l).

For $\alpha_{\mathbb{R}2}^{2n}$ we have:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z},
\end{tikzcd}$$

where the horizontal maps in the top row are the identity, in the bottom row they are $x \mapsto (x, x)$ and $(x, y) \mapsto x + y$, the first vertical map is the identity, the second is $x \mapsto (x, x)$, and the third is multiplication by two. In each case, α_* is the identity, and $\hat{\alpha}_*$ is the identity on the outside groups and flips the two summands on the middle one. The class of the identity in each group in the top row is $2n$, along the bottom row it is $2n$, $(2n, 2n)$ and $4n$ respectively, and the special element is (n, n). The sub-semigroup $S(A, \alpha)$ is $\{4k\,k \geq 1\}$.

For $\alpha^n_{\mathbb{R}\mathbb{R}}$ we have:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}, \end{array}$$

where each horizontal map is the identity and each vertical map is $(x, y) \mapsto x + y$. In each case, α_* flips the two summands and $\hat{\alpha}_*$ is the identity. The class of the unit in each group in the top row is (n, n), in each group in the bottom row it is $2n$, and the special element is n.

For $\alpha^{\{l,k\}}_{\mathbb{H}1}$ we have:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}, \end{array}$$

where the horizontal maps are each two times the identity map, the vertical maps are each $x \mapsto (x, x)$, the classes of the unit in the groups along the top are n, $2n$, and $4n$ respectively, and they are (n, n), $(2n, 2n)$, and $(4n, 4n)$ along the bottom row. In each case α_* is the identity and $\hat{\alpha}_*$ flips the two summands. The class of the special element is $(2k, 2l)$.

For $\alpha^n_{\mathbb{H}2}$, notice that the crossed product is isomorphic to $M_{2n}(\mathbb{C})$. We have:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}, \end{array}$$

where in the top row the maps are both multiplication by two, along the bottom the maps are $x \mapsto (x, x)$ and $(x, y) \mapsto x + y$, The first vertical map is multiplication by two, the second is $x \mapsto (x, x)$, and the third is the identity. The classes of the units in the top row are n, $2n$, and $4n$, while in the bottom row they are $2n$, $(2n, 2n)$ and $4n$. The special element is (n, n), and $\hat{\alpha}_*$ is the identity on the outside terms and flips the two summands on the middle one. In each case, α_* is the identity.

For $\alpha^n_{\mathbb{H}\mathbb{H}}$ we have:

$$\begin{array}{ccccc} \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}, \end{array}$$

where the horizontal maps are each two times the identity, the vertical maps are each $(x, y) \mapsto x + y$, the classes of the identity in the top row are (n, n), $(2n, 2n)$, and $(4n, 4n)$, while on the bottom they are $2n$, $4n$, and $8n$. In each case α_* flips the two summands, and $\hat{\alpha}_*$ is the identity. The special element is $2n$.

For $\alpha_{\mathbb{C}1}^{\{l,k\}}$ we have:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z},
\end{tikzcd}$$

where the horizontal maps along the top are $x \mapsto (x, x)$ and $(x, y) \mapsto x + y$; along the bottom they are $(x, y) \mapsto (x, y, x, y)$ and $(a, b, c, d) \mapsto (a + b, c + d)$; the vertical maps are $x \mapsto (x, x)$, $(x, y) \mapsto (x, x, y, y)$, and $x \mapsto (x, x)$; the classes of the identity in the top row are n, (n, n), and n; in the bottom row they are (n, n), (n, n, n, n), and (n, n); the α_* maps are the identity on the outside groups and the flip in the middle; the maps $\hat{\alpha}_*$ are the flip, $(a, b, c, d) \mapsto (b, a, d, c)$; and the flip respectively, and the special element is (l, k, l, k).

For $\alpha_{\mathbb{C}2}^{n}$ we have:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z},
\end{tikzcd}$$

where the maps along the top row are $x \mapsto (x, x)$ and $(x, y) \mapsto x + y$; along the bottom row they are the identity; the vertical maps from left to right are multiplication by two, $(x, y) \mapsto x + y$, and the identity; the classes of the identity are n, (n, n) and $2n$ along the top row, and $2n$ in each case along the bottom; the α_* maps are the identity, the flip, and the identity; $\hat{\alpha}_*$ is the identity in each case; and the special element is n.

For $\alpha_{\mathbb{C}3}^{2n}$ we have:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z},
\end{tikzcd}$$

where the maps along the top row are $x \mapsto (x, x)$ and $(x, y) \mapsto x + y$; along the bottom row they are multiplication by two; the vertical maps from left to right are the identity, $(x, y) \mapsto x + y$, and multiplication by two; the classes of the identity are n, (n, n) and $2n$ along the top row, n, $2n$, and $4n$ along the bottom; the α_* maps are the identity, the flip, and the identity; $\hat{\alpha}_*$ is the identity in each case; and the special element is n.

For $\alpha^n_{\mathbb{CC}}$ we have:

$$\begin{tikzcd}
\mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}\oplus\mathbb{Z} \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

where the maps along the top row are $(x, y) \mapsto (x, y, x, y)$ and $(a, b, c, d) \mapsto (a+b, c+d)$; along the bottom row they are $x \mapsto (x, x)$ and $(x, y) \mapsto x+y$; the vertical maps are $(x, y) \mapsto x + y$, $(a, b, c, d) \mapsto (a + b, c + d)$, and $(x, y) \mapsto x + y$; the classes of the units along the top are (n, n), (n, n, n, n), and (n, n); along the bottom they are $2n$, $(2n, 2n)$, and $4n$; the α_* maps are the flip, $(a, b, c, d) \mapsto (b, a, d, c)$ and the flip; $\hat{\alpha}_*$ is the identity on the outside terms and the flip in the middle; and the special element is (n, n).

Remark 2. The diagrams above are able to distinguish between the types of basic building blocks, so that if we cut down one of our basic types by an invariant projection, the resulting sub-system is of the same type.

5. Existence and uniqueness theorems

In this section, we prove the existence and uniqueness theorems we shall need for our intertwining argument in the next section. We begin with the existence theorem.

Theorem 2. *Let $(A_\varphi, \mathbb{Z}_2, \alpha)$ and $(B_\varphi, \mathbb{Z}_2, \beta)$ be two finite dimensional real C^*-dynamical systems and let $M = (\nu_r, \nu_c, \nu_h, \mu_r, \mu_c, \mu_h)$ be a morphism of invariants from* $\mathrm{Inv}(A, \alpha)$ *to* $\mathrm{Inv}(B, \beta)$. *Then there exists a unital equivariant *-homomorphism $\eta : A_\varphi \to B_\varphi$ such that $K_0(\eta) = \nu_r$, $K_0(c\eta) = \nu_c$, $K_0(h\eta) = \nu_h$, $K_0(\tilde{\eta}) = \mu_r$, $K_0((\widetilde{c\eta})) = \mu_c$, and $K_0((\widetilde{h\eta})) = \mu_h$, where $\tilde{}$ denotes the canonical extension to the crossed product.*

Proof. Since the invariant respects the decomposition of B_φ into a direct sum of our basic building blocks, it will suffice to consider the case where B_φ is a single one of these. Furthermore, since the invariant is compatible with direct sums of homomorphisms, it will suffice to consider the case where A_φ is a single building block as well.

We consider the possible combinations of building blocks in the same order as in section three, beginning with those where both A and B are isomorphic to $M_n(\mathbb{R})$ or $M_n(\mathbb{H})$. With A of type $\alpha^{\{l,k\}}_{\mathbb{R}1}$ and B of type $\alpha^{\{m,n\}}_{\mathbb{R}1}$, we have the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr] \arrow[dr, "\nu_r"] \arrow[dd] & & \mathbb{Z} \arrow[rr] \arrow[dr, "\nu_c"] \arrow[dd] & & \mathbb{Z} \arrow[dr, "\nu_h"] \arrow[dd] & \\
& \mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z} \arrow[dd] \\
\mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\mu_r"] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\mu_c"] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[dr, "\mu_h"] & \\
& \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] & & \mathbb{Z}\oplus\mathbb{Z}
\end{tikzcd}$$

Our conditions imply $\nu_r(x) = tx$, where $m + n = t(l + k)$; $\nu_c(x) = tx$; $\nu_h(x) = tx$; μ_r is given by a matrix of the form $\left(\begin{smallmatrix} r & s \\ s & r \end{smallmatrix}\right)$, with r and s positive integers with $lr + ks = m$, $ls + kr = n$and $t = r + s$; and μ_c and μ_h are given by the same matrix as μ_r. This is exactly the invariant of the embedding $x \mapsto x \otimes 1_{r+s}$ in the decomposition $\alpha_{\mathbb{R}1}^{\{l,k\}} \otimes \alpha_{\mathbb{R}1}^{\{r,s\}} \cong \alpha_{\mathbb{R}1}^{\{lr+ks,kr+ls\}}$, so the desired $*$-homomorphism exists. The other fifteen combinations with these building blocks are handled similarly. We leave verifying the details to the reader. Notice that in these cases we do not need to use the sub-semigroup S.

Next, we consider the case where the target algebra is non-simple. Suppose $\psi_* : \mathrm{Inv}(A, \alpha) \to \mathrm{Inv}(B, \beta)$ is a morphism of invariants, with $\psi_* = (\nu_r, \nu_c, \nu_h, \mu_r, \mu_c, \mu_h)$. Projecting onto the first summand along the top row of $\mathrm{Inv}(B, \beta)$, we see that $(\pi_1 \circ \nu_r, \pi_1 \circ \nu_c, \pi_1 \circ \nu_h)$ is a morphism for the invariant for finite dimensional real C^*-algebras from [9], so there exists a unital real $*$-homomorphism $\psi^1 : A \to \pi_1(B)$ giving rise to it. Let $\psi : A \to B$ be given by $\psi(x) = (\psi^1(x), \psi^1 \circ \alpha(x))$. Then ψ is a unital equivariant $*$-homomorphism, and it is straightforward to verify that $\mathrm{Inv}(\psi) = \psi_*$. In these cases, we also do not need to use the sub-semigroup S.

Next, we examine the case of a non-simple building block mapped into a simple one. Consider the case of mapping $\alpha_{\mathbb{HH}}^{n}$ to $\alpha_{\mathbb{R}1}^{\{r,s\}}$. We have the diagram:

$$\begin{tikzcd}
\mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\nu_r"] \arrow[dd] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\nu_c"] \arrow[dd] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[dr, "\nu_h"] \arrow[dd] & \\
& \mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z} \arrow[dd] \\
\mathbb{Z} \arrow[rr] \arrow[dr, "\mu_r"] & & \mathbb{Z} \arrow[rr] \arrow[dr, "\mu_c"] & & \mathbb{Z} \arrow[dr, "\mu_h"] & \\
& \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] & & \mathbb{Z}\oplus\mathbb{Z}
\end{tikzcd}$$

Our conditions imply: $\nu_r(x, y) = kx + ky$, where $2kn = m = r + s$; $\mu_r(x) = (kx, kx)$; $\nu_c(x, y) = tx + ty$, where $k = 2t$; $\mu_c(x) = (tx, tx)$;

$\nu_h(x, y) = jx + jy$, where $8jn = m$, so $t = 2j$; and $\mu_h(x) = (jx, jx)$. We have that the special element is $(k, l) = (2tn, 2tn)$. Since the multiplicity of the embedding from $M_n(\mathbb{H})$ into $M_m(\mathbb{R})$ is a multiple of 8, and the special element is symmetrical, the orthogonal projections required for the existence of an equivariant $*$-homomorphism exist.

If we look at mapping $\alpha^n_{\mathbb{H}\mathbb{H}}$ to $\alpha^m_{\mathbb{C}2}$, we have the diagram:

$$\begin{tikzcd}
\mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\nu_r"] \arrow[dd] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\nu_c"] \arrow[dd] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[dr, "\nu_h"] \arrow[dd] & \\
& \mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z} \arrow[dd] \\
\mathbb{Z} \arrow[rr] \arrow[dr, "\mu_r"] & & \mathbb{Z} \arrow[rr] \arrow[dr, "\mu_c"] & & \mathbb{Z} \arrow[dr, "\mu_h"] & \\
& \mathbb{Z} \arrow[rr] & & \mathbb{Z} \arrow[rr] & & \mathbb{Z},
\end{tikzcd}$$

and our conditions give: $\nu_r(x, y) = rx + ry$, where $2rn = m$, and $\mu_r(x) = 2rx$. Commuting with the flips gives $\nu_c(x, y) = (jx + ky, kx + jy)$, for positive integers j, k with $(j + k)n = m$. Following the element $(n, 0)$ around the first square in the top face shows $(2nj, 2nk) = (rn, rn)$, so $r = 2j = 2k$. We also have $\mu_c(x) = rx$; $\nu_h(x, y) = kx + ky$; and $\mu_h(x) = kx$. Since the multiplicity of the mapping from $M_n(\mathbb{H})$ to $M_m(\mathbb{C})$ is a multiple of 4, the required orthogonal projections exist, and we have a map with this invariant as shown in section three. The other cases in this group can be handled in a similar fashion.

Consider now embeddings of $M_n(\mathbb{C})$ into $M_{nk}(\mathbb{C})$. In the cases where the actions on the centre are the same, straightforward inspections of the diagrams show that the maps described in section three suffice to give all possible morphisms of the invariants.

For $\alpha^{\{r,s\}}_{\mathbb{C}1}$ to $\alpha^n_{\mathbb{C}2}$ we have the diagram:

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr] \arrow[dr, "\nu_r"] \arrow[dd] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\nu_c"] \arrow[dd] & & \mathbb{Z} \arrow[dr, "\nu_h"] \arrow[dd] & \\
& \mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z} \arrow[dd] \\
\mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\mu_r"] & & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\mu_c"] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[dr, "\mu_h"] & \\
& \mathbb{Z} \arrow[rr] & & \mathbb{Z} \arrow[rr] & & \mathbb{Z}
\end{tikzcd}$$

Commutativity of the diagram, and the maps all being unital and intertwining the α_* and $\hat{\alpha}_*$ maps, implies that we have: $\nu_r(x) = kx$ for some positive integer k, $\nu_c(x, y) = (kx, ky)$ $\nu_h(x) = k(x)$ $\mu_r(x, y) = kx + ky$ $\mu_c(a, b, c, d) = sa + sb + sc + sd$, and $\mu_h(x, y) = sx + sy$, where s is a positive integer with $k = 2s$. It follows that k is even, and we have exactly the invariant of the morphism shown to exist in section three.

For $\alpha_{\mathbb{C}1}^{\{r,s\}}$ to $\alpha_{\mathbb{C}3}^{2n}$ we have a similar diagram as for $\alpha_{\mathbb{C}1}^{\{r,s\}}$ to $\alpha_{\mathbb{C}2}^{n}$, but with different maps. The conditions on the diagram imply that the multiplicity of the map from $K_0(A)$ to $K_0(B)$ must be even, and a map giving the required morphism of invariants was shown to exist in section three.

For the mappings from $M_n(\mathbb{R})$ or $M_n(\mathbb{H})$ to $M_{nk}(\mathbb{C})$, straightforward inspection of the diagrams show that the embeddings described in section three give all possible values of the invariant. We leave verifying the details to the reader.

Finally, we consider maps from $M_n(\mathbb{C})$ to $M_{2nk}(\mathbb{R})$ and $M_{nk}(\mathbb{H})$.

For $\alpha_{\mathbb{C}2}^{n}$ to $\alpha_{\mathbb{H}1}^{\{r,s\}}$ we have the diagram:

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr] \arrow[dr, "\nu_r"] \arrow[dd] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\nu_c"] \arrow[dd] & & \mathbb{Z} \arrow[dr, "\nu_h"] \arrow[dd] & \\
& \mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z} \arrow[dd] \\
\mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\mu_r"] & & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\mu_c"] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[dr, "\mu_h"] & \\
& \mathbb{Z} \arrow[rr] & & \mathbb{Z} \arrow[rr] & & \mathbb{Z}
\end{tikzcd}$$

Commutativity of the diagram, and the maps all being unital and intertwining to α_* and $\hat{\alpha}_*$ maps implies that we have: $\nu_r(x) = kx$, where $m = kn$; $\nu_c(x, y) = kx + ky$; $\nu_h(x) = (2kx, 2ky)$; $\mu_r(x, y) = (lx, ly)$, where $2l = k$; $\mu_c(x) = (kx, ky)$, and $(2r, 2s) = (m, m)$, so $(r, s) = (ln, ln)$; and $\mu_h(x) = (2kx, 2kx)$. Equivariant *-homomorphisms with these invariants were constructed in section three.

For $\alpha_{\mathbb{C}2}^{n}$ to $\alpha_{\mathbb{R}2}^{2m}$ we have the diagram:

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr] \arrow[dr, "\nu_r"] \arrow[dd] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] \arrow[dr, "\nu_c"] \arrow[dd] & & \mathbb{Z} \arrow[dr, "\nu_h"] \arrow[dd] & \\
& \mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z} \arrow[rr] \arrow[dd] & & \mathbb{Z} \arrow[dd] \\
\mathbb{Z} \arrow[rr] \arrow[dr, "\mu_r"] & & \mathbb{Z} \arrow[rr] \arrow[dr, "\mu_c"] & & \mathbb{Z} \arrow[dr, "\mu_h"] & \\
& \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[rr] & & \mathbb{Z}\oplus\mathbb{Z}
\end{tikzcd}$$

Commutativity of the diagram, and the maps all being unital and intertwining to α_* and $\hat{\alpha}_*$ maps implies that we have: $\mu_r(x) = kx$, where $m = kn$; $\mu_c(x) = (kx, kx)$; $\mu_h(x) = 2kx$; $\nu_r(x) = 2kx$; $\nu_c(x, y) = kr + ky$; and $\nu_h(x) = kx$. It follows from this alone that the multiplicity of the embedding from $M_n(\mathbb{C})$ into $M_{2m}(\mathbb{R})$ must be even. That this is insufficient to ensure the existence of an equivariant *-homomorphism was discussed in section three. The requirement that $S(\alpha^n_{\mathbb{C}2})$ be mapped into $S(\alpha^{2m}_{\mathbb{R}2})$ ensures that it be a multiple of four, which is sufficient.

For $\alpha^{2n}_{\mathbb{C}3}$ to $\alpha^{2m}_{\mathbb{R}2}$ we have a similar diagram to the one above, but with different order units and maps on the back face. Here the only restrictions on the multiplicity of the embedding we get is that coming from embeddings of matrix algebras over $\mathbb{C}$ into ones over $\mathbb{R}$, and all of these are achieved by the maps found in section three.

The other cases in this group are handled similarly. □

Remark 3. Note that we only require the inclusion of the semi-group S in our invariant for the case $\alpha^n_{\mathbb{C}2}$ mapping into $\alpha^{2m}_{\mathbb{R}2}$.

Next, we prove our uniqueness theorem.

Theorem 3. *Let $(A_\varphi, \mathbb{Z}_2, \alpha)$ and $(B_\varphi, \mathbb{Z}_2, \beta)$ be two finite dimensional real C^*-dynamical systems and let ψ and ζ be two unital equivariant *-homomorphisms from A_φ to B_φ such that* $\mathrm{Inv}(\psi) = \mathrm{Inv}(\zeta)$. *Then there exists a unitary $u \in B^\beta_\varphi$ such that $\zeta = Ad\, u \circ \psi$.*

Proof. Let $(A_\varphi, \mathbb{Z}_2, \alpha)$, $(B_\varphi, \mathbb{Z}_2, \beta)$, ψ, and ζ be as in the statement of the theorem, and let $\tilde{\psi}, \tilde{\zeta} : A_\varphi \rtimes^{\mathbb{R}}_\alpha \mathbb{Z}_2 \to B_\varphi \rtimes^{\mathbb{R}}_\beta \mathbb{Z}_2$ be the canonical extensions of ψ, ζ.

It will suffice to consider the case where B is one of the basic building blocks.

Consider first the case where β is of the form $\beta(x) = Ad\, v(x)$ for some unitary v. By the uniqueness result of [9] applied to the bottom face of the diagram for the morphism of invariants $\mathrm{Inv}(\psi) = \mathrm{Inv}(\zeta)$, there exists a unitary $W \in B_\varphi \rtimes^{\mathbb{R}}_\beta \mathbb{Z}_2$ such that $\tilde{\zeta} = Ad\, W \circ \tilde{\psi}$. Write $W = w_e + w_g g$. Since $\tilde{\zeta}(g) = \tilde{\psi}(g) = g$, we have that $Wg = gW$. Thus $Wg = w_e g + w_g = g w_e + g w_g g = \alpha(w_e) g + \alpha(w_g)$, and we have $w_e = \alpha(w_e)$ and $w_g = \alpha(w_g)$. For all $x \in A$, we have $\zeta(x) = W \psi(x) W^* = (w_e + w_g g)\psi(x)(w_e + w_g g)^* = (w_e \psi(x) w_e^* + w_g(\beta(\psi(x)) w_g^*) + (w_g \beta(\psi(x)) w_e^* + w_e \psi(x) w_g^*) g = (w_e \psi(x) w_e^* + w_g(\beta(\psi(x)) w_g^*)$, and $(w_g \beta(\psi(x)) w_e^* + w_e \psi(x) w_g^*) = 0$. Now consider the element $u = w_e + w_g v$. Then u is in the fixed point sub-algebra of B and, for all $x \in A$, we have $u\psi(x)u^* = (w_e + w_g v)\psi(x)(w_e + w_g v)^* = (w_e \psi(x) w_e^* + w_g(\beta(\psi(x)) w_g^*) + (w_g \beta(\psi(x)) w_e^* + w_e \psi(x) w_g^*) u = (w_e \psi(x) w_e^* + w_g(\beta(\psi(x)) w_g^*) = \zeta(x)$, so u is a unitary meeting our requirements.

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