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Infinite multiplier projections and dichotomy of C^*-algebras

Henning Petzka*

Westfälische Wilhelms-Universität Münster, Mathematisches Institut, Einsteinstraße 62 48149 Münster
e-mail: petzka@uni-muenster.de

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Abstract. We study infiniteness of multiplier projections of a stabilized C^*-algebra and the connection to dichotomy of a C^*-algebra A in the sense of A being either stably finite or purely infinite. The main result is the reduction of the dichotomy problem for real rank zero algebras to a property on multiplier projections, which could possibly hold for general separable C^*-algebras.

1. Introduction

We study possible characterizations of when a C^*-algebra is either stably finite or purely infinite in terms of multiplier projections of the stabilization of the algebra in question. It was asked in [19] whether a characterization could be achieved in terms of a new regularity property, which requires non-full multiplier projections to be stably finite. Another possible characterization could be the absence of infinite, non-properly infinite multiplier projections. Pardo showed in [18] that, if a multiplier projection of a separable simple stable C^*-algebra with stable rank one and real rank zero is infinite, then the projection is properly infinite and full. We generalize this result considerably to separable simple stable regular C^*-algebras and to separable simple stable C^*-algebras with real rank zero and finite stable rank.

In [26], Rørdam proved the existence of a simple C^*-algebra, which is neither stably finite nor purely infinite. He further computed the real rank of his algebra in [28] to be strictly larger than zero. Hence it left open the following important question: Are all simple C^*-algebras of real rank zero

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either stably finite or purely infinite? This question is known as the dichotomy problem of simple real rank zero algebras.

Substantial progress on the dichotomy problem has been achieved by Zhang, who proved dichotomy to hold for simple real rank zero algebras with the corona factorization property, and Ortega, Perera and Rørdam found an independent proof of the same result (see [16]). In this paper we try to answer the question on dichotomy with the help of multiplier projections over stabilized C^*-algebras. While we are not able to answer it in the general case, progress is made by reducing the dichotomy problem to a natural question in terms of multiplier projections and stability: If A is a simple C^*-algebra with real rank zero and Q a properly infinite multiplier projection in $\mathcal{M}(A \otimes \mathcal{K})\backslash(A \otimes \mathcal{K})$, is then Q necessarily full in the multiplier algebra? Alternatively, if Q is as before, is then the hereditary subalgebra $Q(A \otimes \mathcal{K})Q$ of $A \otimes \mathcal{K}$ a stable algebra? Note that the property underlying the first question (i.e., that properly infinite multiplier projections are full) is a natural complement to the corona factorization property, which asks every full projection in the multiplier algebra of the stabilization of A to be properly infinite.

It turns out that the above property on multiplier projections (properly infinite implies full) is connected to a weakening of being regular as introduced by Rørdam in [27]. Regularity for simple stable exact C^*-algebras says that non-unital hereditary subalgebras with no bounded trace are stable. In general, it is not true that a non-unital simple exact C^*-algebras is stable if it has no bounded trace. Firstly, Rørdam constructed in [24] a non-stable simple exact C^*-algebra A, such that $M_2(A)$ is stable. (Since traces extend to matrix algebras, the latter implies that A cannot have a bounded trace.) Secondly, the author constructed in [20] a simple exact C^*-algebra C with no bounded trace and such that no matrix algebra over C is stable. In comparison to the C^*-algebra constructed by Rørdam, the multiplier algebra of the C^*-algebra constructed in [20] has a bounded trace on its multiplier algebra (which is necessarily zero on the canonical ideal).

In respect to this observation we say that a simple stable C^*-algebra is multiplier regular if every non-unital hereditary subalgebra with no normalized 2-quasitrace on its multiplier algebra is stable. Further, we introduce a weakening of asymptotic regularity, a property introduced by Ng in [15]. By definition, a separable simple stable C^*-algebra is asymptotically multiplier regular, if for every non-unital hereditary subalgebra B of A with no normalized 2-quasitrace on its multiplier algebra, some matrix algebra over it is stable. It follows from Brown's Theorem that one can study these properties in terms of multiplier projections and that asymptotic multiplier regularity is connected to the first of the above questions related to the dichotomy problem of real rank zero algebras: A separable simple stable C^*-algebra is asymptotically multiplier regular if and only if every properly infinite multiplier projection that is not in the canonical ideal is full in the multiplier algebra. To the knowledge of the author, there is no C^*-algebra, which is known not to be asymptotically multiplier regular, and it follows from the results of [19] that the most natural approach to the construction of a possible (non-simple, infinite real rank) counterexample fails.

The content of the paper is organized as follows. In Section 2 and 3 we recall notation, concepts and important results which are central to the proofs of the results of this paper. Section 4 contains a study of infiniteness of multiplier projections of stabilized algebras. In Section 5 we introduce our new regularity properties, which are weak versions of the regularity properties of Section 3 in a natural sense, and we characterize these properties in terms of multiplier projections. Section 6, which may be read independently of the rest of the paper, contains the proof that a simple real rank zero algebra A is either stably finite or purely infinite, provided that properly infinite multiplier projections in $\mathcal{M}(A\otimes\mathcal{K})\backslash(A\otimes\mathcal{K})$ are full. Finally, in the last section we study general characterizations of when a C^*-algebra is necessarily stably finite or purely infinite in terms of multiplier projections.

2. Preliminaries and notation

In this section we will fix notation and recall some key results, which serve as important tools in the arguments of some proofs.

By $M_n(A)$ we will denote the matrix algebra over A of size $(n \times n)$. By $A \otimes \mathcal{K}$ we denote the stabilization of A. We will denote the multiplier algebra of a C^*-algebra by $\mathcal{M}(A)$. We refer the reader to [2] for an introduction to multiplier algebras of C^*-algebras.

As customary, we denote Murray-von Neumann comnparison of projections by '$\sim$', i.e., for two projections p, q in a C^*-algebra A we write $p \sim q$ and say that p and q are equivalent, whenever there is some partial isometry $v \in A$, such that $p = v^*v$ and $q = vv^*$. We write $p \precsim q$, if there is some projection $q_0 \leq q$ such that $p \sim q_0$. A projection is called infinite, if it is equivalent to a proper subprojection of itself. A projection is called properly infinite, if $p \oplus p \precsim p$ in $A \otimes \mathcal{K}$.

A quasitrace on a C^*-algebra A is a function $\tau : A \to \mathbb{C}$, such that

$$0 \leq \tau(x^*x) = \tau(xx^*),\ \tau(a + ib) = \tau(a) + i\tau(b) \text{ for all } a, b \in A_+,$$

and such that τ is linear on commuting subalgebras. If τ extends to a quasitrace on $M_2(A)$, then τ is called a 2-quasitrace. It is further called normalized, if A is unital and $\tau(1_A) = 1$. A trace is a linear 2-quasitrace, and a normalized trace is called a tracial state. It follows from results by Haagerup that a quasitrace on an exact C^*-algebra is a trace. By a trace function we will mean a linear function $\tau : A_+ \to [0, \infty]$ such that $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$. It is called semifinite, if $\tau(x) = \sup\{\tau(y) \mid y \in A,\ \tau(y) < \infty\}$.

Brown's Theorem ([4, Corollary 2.6]) is an important ingredient in the proofs of this paper. We recall two consequences we will need.

Theorem 2.1 ([4,10]). *Suppose that C is a separable stable C^*-algebra. Then the following two statements hold.*

- *Every hereditary subalgebra of C is of the form QCQ for some multiplier projection Q.*
- *A full hereditary subalgebra $D = QCQ$ is stable if and only if Q is equivalent to the multiplier unit.*

We recall two more results that we will use several times. The first one is a result by Rørdam from [25].

Theorem 2.2 ([25, Theorem 2.5]). *Let A be a unital C^*-algebra. Then A admits no normalized 2-quasitrace if and only if $M_n(A)$ is properly infinite for some $n \in \mathbb{N}$.*

The second important tool for the results of this paper is a result by Kirchberg and Rørdam from [8], generalizing the fact that infinite projections in simple C^*-algebras are properly infinite.

Theorem 2.3 ([8, Corollary 3.15]). *A non-zero projection p in a C^*-algebra is properly infinite if and only if p is either zero or infinite in each quotient of A.*

3. Regularity properties

We review the definitions of regularity and the corona factorization property. Recall that a projection p in a C^*-algebra A is called full (in A), if it is not contained in any proper closed ideal of A.

Definition 3.1. *A C^*-algebra A has the corona factorization property if every full multiplier projection $Q \in \mathcal{M}(A \otimes \mathcal{K})$ is properly infinite.*

There are several equivalent ways of characterizing when a separable C^*-algebra has the corona factorization property. The following proposition is a small selection of some of these characterizations. The third one of these characterizations is irrelevant for this paper, but was included for the reason that the corona factorization property originated from extension theory ([6]).

Proposition 3.2 (Corona factorization property [11,10,6]). *Let A be a separable C^*-algebra. Then the following properties are equivalent.*

(i) A has the corona factorization property.
(ii) A full hereditary subalgebra D of $A \otimes \mathcal{K}$ is stable if and only if $M_n(D)$ is stable for some positive integer n.
(iii) Every full extension of $A \otimes \mathcal{K}$ is nuclearly absorbing.

There is a second regularity property that usually goes along with the study of the corona factorization property. This property, introduced by Rørdam in [27], is usually simply called 'regularity'.

Definition 3.3 ([17]). *A C^*-algebra has property (S) if it has no non-zero unital quotient and no non-zero bounded 2-quasitrace.*

Definition 3.4 ([27]). *A C^*-algebra is called regular if, whenever $D \subseteq A \otimes \mathcal{K}$ is a full hereditary subalgebra with property (S), then D is stable.*

We remark that regularity was defined in [27] by assuming the hereditary subalgebra to have no bounded trace, instead of not having any bounded 2-quasitraces. Property (S), however, was defined in [17] using 2-quasitraces, with the goal to characterize a suitable form of regularity in terms of the Cuntz semigroup. By Haagerup's result that quasitraces on exact C^*-algebras are traces, it makes no difference for exact C^*-algebras.

Ng introduced another regularity property in [15], which he called asymptotic regularity. This property naturally fits into the study of regularity and the corona factorization property in the sense of Proposition 3.6. (Again, asymptotic regularity, as defined by Ng, considers traces instead of quasitraces.)

Definition 3.5 ([15]). *A C^*-algebra is called asymptotically regular if, whenever $D \subseteq A \otimes \mathcal{K}$ is a full hereditary subalgebra with property (S), then $M_n(D)$ is stable for some $n \in \mathbb{N}$.*

Using item (ii) of Proposition 3.2 one verifies the following relation between corona factorization property, regularity and asymptotic regularity.

Proposition 3.6 ([15]). *A C^*-algebra A is regular if and only if A has the corona factorization and is asymptotically regular.*

4. Characterization of infiniteness

In this section we will give criteria for finiteness and infiniteness of multiplier projections over a stable C^*-algebra.

We start by recalling notation from [26]. If A is a stable C^*-algebra, then we can find a unital copy of $\mathcal{O}_\infty$ in its multiplier algebra $\mathcal{M}(A)$, such that the infinite sum $\sum_{n=1}^{\infty} S_n S_n^*$ converges strictly to the multiplier unit. Then, for every norm-bounded sequence $(a_n)_n$ in $\mathcal{M}(A)$, the infinite sum $\sum_{n=1}^{\infty} S_n a_n S_n^*$ converges strictly to an element $a \in \mathcal{M}(A)$. We will denote the strict limit by $a = \bigoplus_{n=1}^{\infty} a_n$, and we will denote the m-fold direct sum $b \oplus b \oplus \cdots \oplus b$ of an element $b \in \mathcal{M}(A)$ by $n \cdot b$. (We allow m to be any natural number or $m = \infty$.) Up to unitary equivalence, the elements a and $m \cdot b$ are independent of the choice of isometries $\{S_n\}_{n=1}^{\infty}$.

Proposition 4.1. *Let A be a separable simple stable C^*-algebra containing a non-zero projection (for example $A = A_0 \otimes \mathcal{K}$ for a unital separable simple C^*-algebra A_0), and let $Q \in \mathcal{M}(A)$ be a projection. Then the following statements hold.*

(a) If there is some projection p in A such that $p \not\precsim Q$, then Q is finite.
(b) If Q is infinite, then $p \precsim Q$ for all projections $p \in A$.
(c) If Q is infinite, then $n \cdot p \precsim Q$ for all projections $p \in A$ and any $n \in \mathbb{N}$.

Proof. Since (a) follows immediately from (b), and (b') is just a rephrasing of (b) (for simple stable C^*-algebras), it suffices to show (b).

By assumption there is some non-zero multiplier projection P such that $Q \sim P \oplus Q$. By a standard argument we get that $n \cdot P \precsim Q$ for all $n \in \mathbb{N}$. Consider the ideal in $\mathcal{M}(A)$ generated by P. Since A is an essential ideal in $\mathcal{M}(A)$, the ideal generated by P has non-zero intersection with A. By simplicity of A, it contains all of A. Therefore, for any projection $p \in A$, p is in the ideal generated by P and there is some $m \in \mathbb{N}$ such that $p \precsim m \cdot P$. Now we get that $p \precsim m \cdot P \precsim Q$. □

Using the previous proposition we are able to answer a question left open in [20]. In the proof of [20, Theorem 6.3] it is shown that there is a non-unital C^*-algebra A and a projection $E \in A$ such that every multiple of the multiplier unit of A majorizes only a finite number of mutually orthogonal projections equivalent to E. The simple argument from the previous proposition shows that $\mathcal{M}(A)$ is stably finite. Therefore [20, Theorem 6.3] can be strengthened to the following result.

Corollary 4.2. *There exists a separable nuclear (non-unital) simple C^*-algebra A with no bounded trace, such that the multiplier algebra $\mathcal{M}(A)$ is stably finite.*

It does not seem easy to find a characterization distinguishing proper infiniteness of multiplier projections from non-proper infiniteness in terms of which (multiplier) projections it majorizes. A natural candidate for a characterization of properly infinite multiplier projections would be to ask a properly infinite multiplier projection to majorize infinitely many copies of any projection q in the canonical ideal A (and not only arbitrarily finite many).

Proper infiniteness of Q provides us with a sequence of mutually orthogonal, mutually equivalent (properly infinite) multiplier projections Q_j, such that for each j there exist projections $q_j \in A$ with $q \sim q_j < Q_j$. While $\infty \cdot q$ is well-defined in $\mathcal{M}(A)$ (using the notation from the beginning of this section), the strict convergence of the sequence of $\left(\sum_{j=1}^{n} q_j\right)_n$ is not guaranteed.

Also it is not known to the author, whether the converse of Proposition 4.1(a) holds, i.e., whether a multiplier projection $Q \in \mathcal{M}(A)$ over a given stable C^*-algebra A, which majorizes every projection p in A, is necessarily infinite.

Nevertheless, we do have a criteria implying proper infiniteness for certain multiplier projections. Before we are ready to state this criteria, we need to recall a result on properly infinite full multiplier projections.

Lemma 4.3 ([26, Lemma 4.3]). *Let A be a separable stable C^*-algebra, and let $Q \in \mathcal{M}(A)$ be a multiplier projection. Then Q is equivalent to the multiplier unit if and only if Q is both properly infinite and full.*

Lemma 4.4. *Let A be a separable stable C^*-algebra and suppose that Q is a multiplier projection in $\mathcal{M}(A)\backslash A$, such that QAQ has an approximate unit of projections. If for all projections $q \in A$ there exists a projection $q_0 \in A$ such that $q \sim q_0 < Q$ and $q \preceq Q - q_0$, then Q is properly infinite and full in $\mathcal{M}(A)$.*

Proof. The condition on the multiplier projection Q implies stability of QAQ by the Hjelmborg-Rørdam criteria for stability of C^*-algebras with an approximate unit of projections [7, Theorem 3.3]. By Brown's theorem (see Theorem 2.1), Q is equivalent to the multiplier unit, hence properly infinite and full by Lemma 4.3. □

From the characterization in Proposition 4.1 one might hope at first that every infinite multiplier projection is equivalent to the multiplier unit. That this is not the case follows from [19, Proposition 4.9].

Proposition 4.5. *Let A be a separable simple stable C^*-algebra which is neither stably finite nor purely infinite. Assume that some proper ideal of the multiplier algebra contains a projection not in the canonical ideal. (Since A is neither stably finite nor purely infinite, $\mathcal{M}(A)/A$ is not simple ([23]).) Then there is an infinite multiplier projection $Q \in \mathcal{M}(A)\backslash A$, which is not equivalent to the multiplier unit.*

Proof. Find an infinite projection p in A. Find a non-full multiplier projection $Q \in \mathcal{M}(A)\backslash A$. Using stability of A, choose two isometries S_1, S_2 in $\mathcal{M}(A)$ with orthogonal range projections.

Consider now the projection $R = S_1 p S_1^* + S_2 Q S_2^*$. Q is not full, so neither is R, and R is infinite, majorizing an infinite projection. □

Corollary 4.6. *There exists a separable simple stable C^*-algebra A and a multiplier projection $Q \in \mathcal{M}(A)\backslash A$ such that Q is infinite, but not equivalent to the multiplier unit.*

Proof. Considering A to be the stabilization of the simple C^*-algebra with both a non-zero finite and an infinite projection constructed by Rørdam in [26], we can find an infinite projection $p_{inf} \in A$ and a finite non-full multiplier projection Q_{fin} (see [19, Proposition 4.9] for the construction of Q_{fin}). Now apply Proposition 4.5. □

In [24], Rørdam proved the existence of a finite multiplier projection Q such that $2 \cdot Q$ is equivalent to the multiplier unit. We can find a somewhat related example.

Proposition 4.7. *There exists a separable C^*-algebra A and a multiplier projection $R \in \mathcal{M}(A \otimes \mathcal{K})$ such that R is finite, and $2 \cdot R$ is infinite, but not equivalent to the multiplier unit.*

Proof. In the example of Corollary 4.6, let $R_0 = Q_{fin}$. Then, for suitable $p \in A$, $R := R_0 \oplus p_{inf}$ satisfies that $2 \cdot R$ majorizes p_{inf}. Indeed, $Q > p_{fin}$ for some finite projection p_{fin} in A. (This follows by construction of Q_{fin} in [19]). Since A is simple, some multiple of p_{fin}, say $m \cdot p_{fin}$, will majorize p_{inf}. If $2 \cdot R_0$ does not majorize p_{inf}, exchange R_0 with the direct sum of R_0 with a suitable multiple of p_{fin}. Then new projection R is still finite and non-full, and $2 \cdot R$ is infinite, since it majorizes p_{inf}. □

In a positive direction we have the following result, which generalizes results by Pardo in [18], where the case of a σ-unital simple C^*-algebra of real rank zero and stable rank one was considered.

Proposition 4.8. *Let A be a separable simple unital C^*-algebra of real rank zero. If $sr(A) < \infty$, then every infinite multiplier projection $Q \in \mathcal{M}(A \otimes \mathcal{K})$ is equivalent to the multiplier unit.*

Proof. We show that for any given projection $q \in A \otimes \mathcal{K}$ there is a projection q_0 in A such that $q \sim q_0 < Q$ and such that $q \preceq Q - q_0$. Then, by Lemma 4.4, Q is equivalent to the multiplier unit.

Since Q is assumed to be infinite, we can apply Proposition 4.1 to see that for all $N \in \mathbb{N}$ we have $N \cdot q_0 \preceq Q$. Hence for all $N \in \mathbb{N}$,

$$N \cdot q_0 \preceq q_0 \oplus (Q - q_0). \qquad (*)$$

Since $sr(A)$ is finite, also $sr(q_0 A q_0)) =: n$ is a finite integer ([21]). It follows from [2, V.3.1.25] that whenever $(n+1) \cdot q_0 \oplus p_1 \sim q_0 \oplus p_2$ for some projections p_1, p_2 in $A \otimes \mathcal{K}$, then $n \cdot q_0 \oplus p_1 \sim p_2$.

We now choose $N = (n+1)$ in (∗), which implies that

$$(n+1) \cdot q_0 \preceq q_0 \oplus (Q - q_0).$$

Since the left hand side is in $A \otimes \mathcal{K}$ the subequivalence must take place in $A \otimes \mathcal{K}$, i.e., there is some projection $p_2 \in A \otimes \mathcal{K}$, $p_2 \leq Q - q_0$ such that

$$(n+1) \cdot q_0 \preceq q_0 \oplus p_2.$$

Hence there is a projection p_1 in $A \otimes \mathcal{K}$ such that

$$(n+1) \cdot q_0 \oplus p_1 \sim q_0 \oplus p_2.$$

The cancellation result for C^*-algebras with finite stable rank from above implies that

$$n \cdot q_0 \oplus p_1 \sim p_2.$$

In particular,

$$q_0 \preceq p_2 \leq Q - q_0.$$

□

Remark 4.9. To the knowledge of the author, there is no simple stably finite C^*-algebra with real rank zero known, which has stable rank greater than one. For C^*-algebras with stable rank one, the result of the last proposition was shown by Pardo in [18].

As a second positive result we have the following, quite general, result.

Proposition 4.10. *Let A be a separable exact simple unital C^*-algebra, If A is regular, then every infinite multiplier projection $Q \in \mathcal{M}(A \otimes \mathcal{K}) \backslash (A \otimes \mathcal{K})$ is equivalent to the multiplier unit.*

Proof. We will denote the stabilization of A by C. Let Q be an infinite multiplier projection and p a non-zero projection in C. Then $N \cdot p \preceq Q$ for all $N \in \mathbb{N}$ by Proposition 4.1. Consider the full hereditary subalgebra QCQ of C. Suppose τ is a bounded trace on QCQ. Then τ extends to a semifinite trace on $QCQ \otimes \mathcal{K} \cong C$, which is faithful by simplicity of C. The semifinite trace τ extends to a trace function $\bar{\tau}$ onto the positive part of $\mathcal{M}(C)$, such that $\bar{\tau}(Q) < \infty$. Since $N \cdot p \preceq Q$ for all N, we have that $\bar{\tau}(Q) \geq N \cdot \tau(p) > 0$ for all $N \in \mathbb{N}$, a contradiction. Hence QCQ is a full hereditary subalgebra without unital quotients and without a bounded trace. By regularity of C, QCQ must be stable and Q is equivalent to the multiplier unit by Theorem 2.1.

□

5. Weakened regularity properties

In this section we will consider new regularity properties for C^*-algebras, which are, in a natural sense, weak versions of regularity and asymptotic regularity.

Recall from Section 3 that a stable C^*-algebra is regular (resp. asymptotically regular), whenever every full hereditary subalgebra with property (S) is stable (resp. stable after tensoring with some large enough matrix algebra). In the simple case, the hypothesis of having property (S) simply requires the hereditary subalgebra in question to be non-unital and not to have any bounded 2-quasitrace. In this section we will strengthen this hypothesis of having property (S) to get weaker versions of both regularity and asymptotic regularity.

It is well known that a bounded trace on a C^*-algebra C extends to a bounded trace on its multiplier algebra. Conversely though, if there exists no bounded trace on a (non-unital) C^*-algebra C, then there may still exist a trace or quasitrace on $\mathcal{M}(C)$, which is zero on the canonical ideal C. An example of such a C^*-algebra was constructed by the author in [20] (see Corollary 4.2).

Definition 5.1. *A C^*-algebra D is said to have property $(\tilde{S})$ whenever D has no non-unital quotients and the multiplier algebra $\mathcal{M}(D)$ has no normalized quasitrace.*

Definition 5.2. *Let A be a C^*-algebra. Then A is called multiplier regular if, whenever $D \subseteq A \otimes \mathcal{K}$ is a full hereditary subalgebra with property $(\tilde{S})$, then D is stable.*

Definition 5.3. *Let A be a C^*-algebra. Then A is called asymptotically multiplier regular if, whenever $D \subseteq A \otimes \mathcal{K}$ is a full hereditary subalgebra with property $(\tilde{S})$, then $M_n(D)$ is stable for some $n \in \mathbb{N}$.*

We will see in Proposition 5.8 and in Theorem 5.9 equivalent ways of defining asymptotic multiplier regularity for separable and for separable simple C^*-algebras respectively.

The following proposition shows the new regularity properties defined above to be weak versions of the existing notions.

Proposition 5.4. *Let A be an exact C^*-algebra. Then the following three statements hold.*

(i) If A is regular, then A is multiplier regular.
(ii) If A is asymptotically regular, then A is asymptotically multiplier regular.
(iii) A is multiplier regular if and only if A has the corona factorization property and is asymptotically multiplier regular.

Proof. Statements (i) and (ii) follow from the fact that bounded traces on non-unital C^*-algebras extend to bounded traces on its multiplier algebras. Statement (iii) can be proved along the same lines as Proposition 3.6. □

Besides being a natural weakening of asymptotic regularity, asymptotic multiplier regularity can be rephrased nicely to fit into the framework of studying full and properly infinite multiplier projections. Before discussing a characterization of asymptotic multiplier regularity in terms of multiplier projections, we characterize property $(\tilde{S})$ in terms of multiplier projections.

Lemma 5.5 ([2, II.5.3.5]). *Let A be a C^*-algebra and B a full hereditary subalgebra of A. Then there is a one-to-one correspondence of closed ideals of A and closed ideals of B, where an ideal I of A corresponds to the ideal $I \cap B$ of B.*

For a C^*-algebra A and a closed ideal I of A, let $\pi_I : A \to A/I$ denote the natural projection map. Then π_I extends to a map $\bar{\pi}_I : \mathcal{M}(A) \to \mathcal{M}(A/I)$ (see e.g. [2, II.7.3.9]).

Definition 5.6. *Let A be a C^*-algebra and $Q \in \mathcal{M}(A)$ be a multiplier projection. We say that Q is nowhere in the canonical ideal if for all proper ideals I of A we have that $\bar{\pi}_I(Q) \in \mathcal{M}(A/I)\backslash(A/I)$.*

Note that, if Q is a multiplier projection in $\mathcal{M}(A)$ which is nowhere in the canonical ideal, then in particular the hereditary subalgebra QAQ of A generated by Q is full in A. Indeed, suppose $AQA = I$ is a proper ideal of A, then $\bar{\pi}_I(Q)$ is a multiplier projection of A/I such that $(A/I)\bar{\pi}_I(Q)(A/I) = \pi_I(AQA) = \{0\}$, so the ideal of $\mathcal{M}(A/I)$ generated by $\bar{\pi}_I(Q)$ has zero intersection with A/I. Since A/I is an essential ideal in its multiplier algebra, $\bar{\pi}_I(Q) = 0$, in particular $\bar{\pi}_I(Q) \in A/I$.

Lemma 5.7. *Let A be a stable C^*-algebra and $B = QAQ$ a full hereditary subalgebra, with Q denoting some multiplier projection in $\mathcal{M}(A)$ (see Theorem 2.1). Then B has property $(\tilde{S})$ if and only if there is some $n \in \mathbb{N}$ such that $n \cdot Q$ is properly infinite and Q is nowhere in the canonical ideal.*

Proof. We first note that for the multiplier algebra of $B = QAQ$ we have $\mathcal{M}(B) \cong Q\mathcal{M}(A)Q$ (see for example [9, Lemma 11]). By Theorem 2.2, the non-existence of normalized quasitraces on $\mathcal{M}(B)$ is equivalent to Q being stably properly infinite, i.e., there is some $n \in \mathbb{N}$ such that $n \cdot Q$ is properly infinite.

By Lemma 5.5, any quotient B/J of B comes from a quotient A/I of A by mapping A/I to $(A \cap B)/(I \cap B)$. In our case, this is the same as mapping A/I to $QAQ/QIQ = \pi(QAQ) = \bar{\pi}(Q)(A/I)\bar{\pi}(Q)$. Hence, $B = QAQ$ has a unital quotient if and only if $\bar{\pi}(Q) \in A/I$ for some ideal I. □

Proposition 5.8. *A separable C^*-algebra A is asymptotically multiplier regular if and only if every properly infinite multiplier projection $Q \in \mathcal{M}(A \otimes \mathcal{K})$, which is nowhere in the canonical ideal, is full in the multiplier algebra.*

Proof. Assume that a given C^*-algebra A has the asymptotic multiplier regularity and let Q be a properly infinite multiplier projection nowhere in the canonical ideal. Then, by Lemma 5.7, $D := Q(A \otimes \mathcal{K})Q$ is a full hereditary subalgebra with property $(\tilde{S})$. It follows by the asymptotic multiplier regularity that there is $n \in |NN$ such that $M_n(D) \cong M_n(D) \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. We get $n \cdot Q \sim \mathbb{1}$ by Brown's Theorem (see Theorem 2.1). In particular, Q is full.

Conversely, suppose that every properly infinite multiplier projection, which is nowhere in the canonical ideal, is full in the multiplier algebra. Let D be a full hereditary non-unital subalgebra of $A \otimes \mathcal{K}$ with property $(\tilde{S})$. By Theorem 2.1, $\mathcal{M}(D) \cong \mathcal{M}(Q(A \otimes \mathcal{K})Q) \cong Q\mathcal{M}(A \otimes \mathcal{K})Q$ for some multiplier projection Q. Moreover, by Lemma 5.7, there is $n \in \mathbb{N}$ such that $n \cdot Q$ is properly infinite and Q is nowhere in the canonical ideal. By our assumption that proper infiniteness implies fullness for projections nowhere in the canonical ideal, $n \cdot Q$ is full, and therefore $n \cdot Q \sim \mathbb{1}$ by Lemma 4.3. By Theorem 2.1, $M_n(D) \cong M_n(Q(A \otimes \mathcal{K})Q) \cong M_n(A \otimes \mathcal{K})$ is stable. Hence, A is asymptotically multiplier regular. □

For a simple C^*-algebra A, the condition on the multiplier projection Q in Proposition 5.8 reduces to the property that Q is a properly infinite projection in $\mathcal{M}(A \otimes \mathcal{K}) \setminus A \otimes \mathcal{K}$. This together with Proposition 4.8 implies that separable simple C^*-algebras with real rank zero and finite stable rank have asymptotic multiplier regularity.

It turns out that in the simple case there is even yet another natural way to characterize asymptotic multiplier regularity.

Theorem 5.9. *Let A be a separable simple C^*-algebra. Then the following properties are equivalent.*

(i) A is asymptotically multiplier regular.
(ii) Every properly infinite multiplier projection $Q \in \mathcal{M}(A \otimes \mathcal{K}) \setminus A \otimes \mathcal{K}$ is full in $\mathcal{M}(A)$.
(iii) If Q is a properly infinite multiplier projection in $\mathcal{M}(A \otimes \mathcal{K}) \setminus A \otimes \mathcal{K}$, then Q is equivalent to the multiplier unit of $A \otimes \mathcal{K}$.
(iv) If B is a non-unital C^*-algebra, stably equivalent to A, such that $\mathcal{M}(B)$ is properly infinite, then B is stable.

Proof. It follows from Proposition 5.8 and Lemma 4.3 that (i)–(iii) are equivalent (after noticing that asking the properly infinite multiplier projection Q in Proposition 5.8 to be nowhere in the canonical ideal, reduces in the simple case to the condition that Q does not lie in the canonical ideal).

Since B is a hereditary subalgebra of its stabilization, if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, then there is a multiplier projection $P \in \mathcal{M}(A \otimes \mathcal{K})$ such that $P(A \otimes \mathcal{K})P \cong B$. Now $\mathcal{M}(B)$ is properly infinite if and only if P is properly infinite. It follows that the hypothesis of (iii) is equivalent to the one of (iv) with $B = P(A \otimes \mathcal{K})P$. By Theorem 2.1, also the conclusions of (iii) and (iv) are equivalent. □

In respect of the results of this paper, in particular with respect to Theorem 6.1, it would be very interesting to answer the following question, or at least to answer it for C^*-algebras of real rank zero.

Question 5.10. *Is every separable simple C^*-algebra asymptotically multiplier regular?*

By Proposition 5.9 one can equivalently ask the following question.

Question 5.11. *If A is a non-unital simple C^*-algebra, such that $\mathcal{M}(A)$ is properly infinite, is then A necessarily stable?*

The last question was shown to have a positive answer in the stable rank one case in [25]. The results above give with asymptotic multiplier regularity another sufficient condition for when a non-unital C^*-algebra A with properly infinite multiplier algebra $\mathcal{M}(A)$ must be stable. Hence this holds in particular for all regular C^*-algebras, for example any C^*-algebra with finite radius of comparison [3].

Remark 5.12. The typical example for a non-simple separable C^*-algebra with neither the corona factorization property nor asymptotic regularity is the algebra $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ of continuous functions from the infinite product of 2-spheres into the compacts by constructing multiplier projections of a certain form. It was shown in [19, Corollary 4.3] that all infinite projections of this special form are properly infinite and full. Hence, the same type of multiplier projections are of no help to decide whether the algebra $C(\prod_{j=1}^{\infty} S^2, \mathcal{K})$ is asymptotically multiplier regular or not.

We now turn our attention to a characterization of multiplier regularity. For C^*-algebras of real rank zero, we are also able to characterize multiplier regularity in terms of multiplier projections.

Proposition 5.13. *A separable C^*-algebra of real rank zero is multiplier regular if and only if, whenever Q is a multiplier projection in $\mathcal{M}(A \otimes \mathcal{K})$, which is nowhere in the canonical ideal and such that $m \cdot Q$ is properly infinite for some $m \in \mathbb{N}$, then Q is itself properly infinite.*

Proof. Suppose A is multiplier regular and $Q \in \mathcal{M}(A \otimes \mathcal{K})$ is nowhere in the canonical ideal, such that $m \cdot Q$ is properly infinite for some $m \in \mathbb{N}$. By Lemma 5.7, the subalgebra $Q(A \otimes \mathcal{K})Q$ is a full hereditary subalgebra with property $(\tilde{S})$. Multiplier regularity implies that $Q(A \otimes \mathcal{K})Q$ is stable, hence Q is equivalent to the multiplier unit by Theorem 2.1. In particular, Q is properly infinite.

Conversely assume that for any multiplier projection Q with $m \cdot Q$ properly infinite for some $m \in \mathbb{N}$ and such that Q nowhere in the canonical ideal, we have that Q is properly infinite itself. This clearly implies the corona factorization property of A. It therefore suffices to prove that A is asymptotically multiplier regular.

Using Proposition 5.8 it suffices to show that every properly infinite projection $Q \in \mathcal{M}(A \otimes \mathcal{K}) \setminus A \otimes \mathcal{K}$, which is nowhere in the canonical ideal, is full in the multiplier algebra. Since Q is properly infinite, it follows that for every projection q in $A \otimes \mathcal{K}$ we have $q \sim q_0 < Q$. Now $2 \cdot (Q - q_0) > Q$ is an infinite multiplier projection, since it majorizes a projection equivalent to Q. Since $2 \cdot (Q - q_0)$ agrees with $2 \cdot Q$ in all non-trivial quotients and $2 \cdot Q$ is properly infinite, we apply Theorem 2.3 to see that also $2 \cdot (Q - q_0)$ is properly infinite. By assumption, this implies that $(Q - q_0)$ is properly infinite itself, and in particular it follows that $q \preceq Q - q_0$. By Lemma 4.4, QAQ is stable and hence Q is equivalent to the multiplier unit by Theorem 2.1. In particular, Q is a full multiplier projection. □

Corollary 5.14. *A separable simple C^*-algebra of real rank zero is multiplier regular if and only if every multiplier projection $Q \in \mathcal{M}(A \otimes \mathcal{K}) \setminus A \otimes \mathcal{K}$, such that $m \cdot Q$ is properly infinite for some $m \in \mathbb{N}$ is itself properly infinite.*

Question 5.15. *Does this characterization of multiplier regularity hold for general (simple) C^*-algebras?*

6. Dichotomy of simple real rank zero algebras

In this section we reduce an affirmative solution to the dichotomy problem for real rank zero algebras, i.e., the question on whether all separable simple C^*-algebras of real rank zero are either stably finite or purely infinite, to answering Question 5.10 or Question 5.11 affirmatively. Note that by Theorem 5.9 the hypothesis of asymptotic multiplier regularity on A is the same as assuming every properly infinite multiplier projection in $\mathcal{M}(A \otimes \mathcal{K}) \setminus (A \otimes \mathcal{K})$ to be full. In fact, it is the latter version which is used in the proof.

Theorem 6.1. *Let A be a separable simple C^*-algebra of real rank zero, such that A has the asymptotic multiplier regularity. Then A is either stably finite or purely infinite.*

Proof. Since stable finiteness, pure infiniteness and the property of having real rank zero are preserved under passing to Morita equivalent algebras, we may assume without loss of generality that the given C^*-algebra A is stable.

We assume that A is neither stably finite nor purely infinite. Since stably projectionless algebras are stably finite, A contains an infinite projection p_{inf}.

Let us consider Lin's ideal L in the multiplier algebra of A. L is the minimal ideal of the multiplier algebra properly containing A, i.e., every other ideal of the multiplier algebra properly containing A contains L ([13]). Since A is neither elementary nor purely infinite, $L \neq \mathcal{M}(A)$ ([23]). Using that A is of real rank zero, we can find a multiplier projection Q in $L \backslash A$ ([30]). Let $C^*(S_1, S_2)$ be a unital copy of the Cuntz algebra $\mathcal{O}_2$ in $\mathcal{M}(A)$. The projection $P := S_1 Q S_1^* + S_2 p_{inf} S_2^*$ lies in $L \backslash A$, and it majorizes the infinite projection p_{inf}, so P must itself be infinite. We note that, due to asymptotic multiplier regularity and Theorem 5.9, P is not properly infinite (since P is in the proper ideal L.) It follows from Theorem 2.3 that there is some ideal I of $\mathcal{M}(\mathcal{A})$ such that P is a non-zero finite projection in the corresponding quotient algebra $\mathcal{M}(A)/I$. But P is infinite modulo the zero ideal, and $\pi_J(P) = 0$ for all multiplier ideals containing the (minimal) ideal L. Hence the only possible ideal to satisfy this condition is $I = A$. Hence, P is finite in $\mathcal{M}(A)/A$. By a theorem of Zhang ([30, Theorem 1.3(a)]) this is in contradiction to A having real rank zero. □

It follows that if Question 5.10 (or Question 5.11) has a positive answer, then all simple real rank zero C^*-algebras are either stably finite or purely infinite. It seems reasonable to expect an affirmative answer to Question 5.10. This is why Theorem 6.1 was phrased in the way above. Nevertheless, one should note the natural alternative way to phrase the theorem: If there is a real rank zero algebra which is neither stably finite nor properly infinite, then the same arguments would lead to an example of a properly infinite non-full multiplier projection, which is not in the canonical ideal, and hence to a negative answer to Question 5.10.

7. Multiplier projections and dichotomy

In this section we study properties on multiplier projections of a stabilized C^*-algebra implying that the given algebra is either stably finite or purely infinite. The first result is indeed a characterization of this dichotomy, but it doesn't seem very useful for applications. In Proposition 7.3, two more candidates for properties characterizing dichotomy are given, but we only have a proof for one direction: both properties imply the C^*-algebra in question to be either stably finite or purely infinite.

Lemma 7.1. *Let A_0 be a simple unital C^*-algebra, $A = A_0 \otimes \mathcal{K}$, and suppose that Q is a properly infinite multiplier projection in $\mathcal{M}(A)$. Then QAQ contains a (non-zero) stable subalgebra.*

Proof. Since Q is properly infinite, there exists a sequence $Q_1 \sim Q_2 \sim Q_3 \sim \cdots \sim Q$ of mutually orthogonal multiplier projections. By simplicity of A and by proper infiniteness of each Q_j, we have $1_{A_0} \sim q_j < Q_j$ for all j. Hence there is a sequence of mutually orthogonal and mutually equivalent projections $(q_j)_{j \in \mathbb{N}}$ in QAQ. It follows that QAQ contains a (non-zero) stable subalgebra. □

Theorem 7.2. *Let A_0 be a simple unital exact C^*-algebra and $A = A_0 \otimes \mathcal{K}$. Then the following conditions are equivalent.*

(i) A_0 is either stably finite or purely infinite.
(ii) A is either stably finite or purely infinite.
(iii) Either there exists a multiplier projection in $\mathcal{M}(A)$ which is stably finite, or all multiplier projections are properly infinite.

Proof. The equivalence between (i) and (ii) is trivial.

Suppose that all multiplier projections are properly infinite. Then every hereditary subalgebra of A (which is necessarily of the form QAQ for some multiplier projection Q, see Theorem 2.1) contains a (non-zero) stable subalgebra by Lemma 7.1. It follows that A is purely infinite (see e.g. [22, Proposition 4.1.1]). Hence, if A is not purely infinite, then there exists a multiplier projection, which is not properly infinite. Further, if A is not stably finite, then A contains an infinite projection p_{inf}. Using simplicity and the fact that A is an essential ideal in its multiplier algebra, one gets that for each multiplier projection Q there is some $m \in \mathbb{N}$ such that $m \cdot Q \succeq p_{inf}$. Majorizing an infinite projection, $m \cdot Q$ is itself infinite. It follows that there doesn't exist any stably finite multiplier projection.

We have therefore shown that, if there is a stably finite multiplier projection, or if all multiplier projections are properly infinite, then A must be either stably finite or purely infinite, which proves that (iii) implies (ii).

Conversely, suppose that A is either stably finite or purely infinite. In the latter case, every multiplier projection, that is not in the canonical ideal, is full in $\mathcal{M}(A)$ and A has the corona factorization property, hence all multiplier projections are properly infinite. If A is stably finite, then there is a tracial state τ_0 on A_0, extending to a semifinite trace function on A. Consider a non-unital hereditary subalgebra B_0 of A_0, and find a multiplier projection Q in $\mathcal{M}(A)$, not in the canonical ideal, such that $B = QAQ$. The restriction of τ_0 onto B_0 gives a bounded trace on B_0.

The natural extension $\overline{\tau}$ of τ to matrix algebras over $Q\mathcal{M}(A)Q \cong \mathcal{M}(B)$ satisfies that $\overline{\tau}(m \cdot Q) < \infty$. If some multiple of Q was infinite, then $N \cdot 1_{A_0} \preceq m \cdot Q$ for all $N \in \mathbb{N}$ by Proposition 4.1. It follows that τ must be zero on A, a contradiction. Hence, Q is stably finite. □

Assume that Question 5.10 has a positive answer. Then, by Theorem 5.9, every non-full multiplier projection of a separable simple stable C^*-algebra is either finite or infinite, but cannot be properly infinite. That means, that Proposition 4.5 can be rephrased as follows:

> Let A be a separable simple stable C^*-algebra which is neither stably finite nor purely infinite. Assume that some proper ideal of the multiplier algebra contains a projection not in the canonical ideal. Then the multiplier algebra of A contains an infinite projection, which is not properly infinite.

In this light, and considering our results from Section 4, in particular Proposition 4.8 and Proposition 4.10, it seems natural to try to relate the conditions $(P1) - (P3)$ of the following proposition.

Proposition 7.3. *Let A be a separable simple stable C^*-algebra with asymptotic multiplier regularity, and such that each multiplier ideal contains a projection not in the canonical ideal. Consider the following three properties.*

(P1) There is no infinite multiplier projection in $\mathcal{M}(A)$, which is not properly infinite.

(P2) A multiplier projection in $\mathcal{M}(A)$ is stably finite if and only if it is non-full.

(P3) A is either stably finite or purely infinite.

Then $(P1) \Rightarrow (P2) \Rightarrow (P3)$

Proof. We show first, that (P1) implies (P2), so let us assume (P1) holds. It is clear that stably finite multiplier projections are non-full. Conversely, if Q is a non-full multiplier projection in $\mathcal{M}(A)\setminus A$, then $n \cdot Q$ is not equivalent to the multiplier unit for any $n \in \mathbb{N}$. By the assumption of asymptotic multiplier regularity and Theorem 5.9, $n \cdot Q$ cannot even be properly infinite. By assumption (P1), $n \cdot Q$ is finite for all $n \in \mathbb{N}$.

To see that (P2) implies (P3), assume A is neither stably finite nor purely infinite. Then, by Theorem 7.2, no multiplier projection is stably finite. By (P2) all multiplier projections are full. Since by assumption every ideal of $\mathcal{M}(A)$ contains a projection, it follows that $\mathcal{M}(A)/A$ is simple. But by the results in [23], the corona algebra is simple only for $A = \mathcal{K}$ or A purely infinite, so we ended up with a contradiction. □

Question 7.4. *Are the three conditions (P1)–(P3) equivalent for all simple stable separable C^*-algebras? Are they equivalent if we assume in addition real rank zero?*

Suppose A is a stable simple C^*-algebra which is asymptotically multiplier regular, and which is neither stably finite, nor purely infinite (if such an algebra exists). Then A contains an infinite projection p_{inf} and a non-full multiplier projection Q. Choosing any two partial isometries S_1, S_2 in $\mathcal{M}(A)$ with orthogonal range projections, the projection $p_{inf} \oplus Q = S_1 p_{inf} S_1^* + S_2 Q S_2^*$ is an infinite multiplier projection (majorizing an infinite projection), while it is not properly infinite (because it generates the same ideal as Q and full projections are assumed to be properly infinite). If one tries to show that (P3) implies (P1), then one is actually trying to show that any infinite multiplier projection, which is not properly infinite, arises as in the example above, i.e., it is of the form $p_{inf} \oplus Q$ with p_{inf} an infinite projection in A, and Q a non-full multiplier projection.

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