J. Ramanujan Math. Soc. **30**, No.3 (2015) 309–330

Sublagrangian constructions

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Communicated by: Prof. Purna Bangere

Received: October 29, 2014

Abstract. In this article we show multiple ways of constructing sublagrangians of symmetric forms $\varphi : \mathcal{E}_\bullet \xrightarrow{\sim} \mathcal{E}_\bullet^\#$ in the bounded derived category $\mathcal{D}^b(\mathscr{V}(X))$ of complexes of locally free sheaves over quasi-projective regular (sometimes without regularity) schemes X, over noetherian affine schemes $Spec(A)$. By application of the sublagrangian theorem of Balmer, for Witt theory of triangulated categories, we prove some results regarding structure of forms in the Witt groups $W^n(\mathscr{D}^k(X))$ of the filtered subcategories.

Mathematics Subject Classification. 19G12, 18E30, 13C09, 13D10, 11E12, 11E81.

1. Introduction

Suppose $(K, \#, \varpi)$ is a triangulated category with δ-duality #, containing $1/2$ and satisfying (TR4+) (see [B2]). Suppose $(\mathcal{E}, \varphi)$ is a symmetric form in K and $(\mathcal{L}, \nu)$ is a sublagrangian of $(\mathcal{E}, \varphi)$, *meaning* $\nu : \mathcal{L} \longrightarrow \mathcal{E}$ *is a morphism with* $\nu^\# \varphi \nu = 0$. Given this data, Balmer's Sub-Lagrangian theorem ([B2, 4.20]), gives a method of constructing another symmetric form $(\mathcal{R}, \psi)$, which is Witt equivalent to $(\mathcal{E}, \varphi)$. First, we recall the Sub-Lagrangian theorem ([B2, 4.20]) and the method. Embed ν in an exact triangle and consider the following diagram of exact triangles

$$\begin{array}{ccccccc} T^{-1}\mathcal{N} & \xrightarrow{\nu_0} & \mathcal{L} & \xrightarrow{\nu} & \mathcal{E} & \xrightarrow{\nu_1} & \mathcal{N} \\ \downarrow{\scriptstyle T^{-1}\mu_0^\#} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mu_0^\#} \\ T^{-1}\mathcal{L}^\# & \longrightarrow & \mathcal{N}^\# & \longrightarrow & \mathcal{E}^\# & \longrightarrow & \mathcal{L}^\# \end{array} \tag{1}$$

*Partially supported by a General Research Grant from KU where $\mathcal{N}$ is the cone of ν, the second line is the dual of the first line and μ_0 is assumed to be a very good morphism (see [B2]). Let $\mathcal{R}$ be the cone of μ_0. Then the Sub-Lagrangian theorem ([B2, 4.20]) asserts that there is a symmetric form $\psi : \mathcal{R} \xrightarrow{\sim} \mathcal{R}^{\#}$ such that $[(\mathcal{E}, \varphi)] = [(\mathcal{R}, \psi)]$ in the Witt group $W(K, \#, \omega)$.

The Sub-Lagrangian theorem has numerous applications that led to many interesting results (e.g. ([B1,B3,BW] and others). This tool was developed ([B2]), mainly with the intent, to be applied to Witt theory of Derived categories of various exact subcategories of the category $Coh(X)$ of coherent sheaves over noetherian schemes. (*Apparently, the examples of triangulated categories that are not derived categories are not there in abundance*). Obviously, replacing one symmetric space $(\mathcal{E}, \varphi)$ by another one $(\mathcal{R}, \psi)$ would not bring any extra milage. What kind of dividend the Sub-Lagrangian theorem will bring depends entirely on smart construction of sublagrangians $(\mathcal{L}, \nu)$ of the form $(\mathcal{E}, \varphi)$ that yields Witt equivalent forms $(\mathcal{R}, \psi)$ with further desirable properties. The examples of sublagrangians in the literature could be classified in to two group. The first group of constructions are fully formal, within the realm of triangulated categories with duality. Such a construction was used to establish the twelve term exact sequence ([B2]), which is omnipresent in the literature of Derived Witt theory and eventually led to the proof of exactness of the Gersten-Witt complex ([BW,B4,Bet]), for regular local rings containing a field.

The second set of examples belong to, down to earth, paradigm of derived categories. However, there were only limited amount of efforts to construct sublagrangians that are beyond routine. The existing examples are given by complexes $\mathcal{L}$ concentrated at a single degree, in spite of their important consequences. In this article, we show multiple ways of constructing sublagrangians, which also lead to other consequences. Recall that other such constructions were given in [M1,MS]. (*For the rest of this introduction, the readers are referred to (2.3) for notations, as needed.*)

We consider quasi-projective schemes X over affine schemes $Spec(A)$. Let $\mathscr{D}^k(X) \subseteq \mathcal{D}^b(\mathscr{V}(X))$ denote the filtration, by grade of the homologies, of the bounded derived category $\mathcal{D}^b(\mathscr{V}(X))$ of complexes $\mathcal{E}_\bullet$ of locally free sheaves over X. For elements $x \in W^n(\mathscr{D}^k(X))$, we obtain symmetric forms $(\mathcal{E}_\bullet, \varphi)$ with interesting structures, representing x, i.e. $x = [(\mathcal{E}_\bullet, \varphi)]$. Two sets of results are obtained. Either set provide information on the range $\mathscr{R}(\mathcal{E}_\bullet)$ of degrees, where $\mathcal{E}_\bullet$ resides (see 2.3(7)). The first set of results provide further structure regarding the vanishing of the homologies $\mathcal{H}_i(\mathcal{E}_\bullet)$, at certain degrees i, when X is regular. We obtain representations with $length(\mathscr{R}(\mathcal{E}_\bullet)) \leq \min\{\dim X, 2k\}$. We have exactly similar results with skew duality, as well. These groups $W^n(\mathscr{D}^k(X))$ were studied extensively in ([B2,B3,BW]). Recall, for $k = d$ these groups were computed in ([BW]) when X is regular and subsequently in ([M1,MS]) in the non-regular case.

Now we will introduce the results in this article. Due to 4-periodicity, we need to state the results on four groups $W^r(\mathscr{D}^k(X))$, for $r = k-1, k, k+1, k+2$ only. However, in this introduction, we state the results only on $W^k(\mathscr{D}^k(X))$, and the other three will be stated later (§3.1).

Theorem 1.1. *Let X be a regular quasi-projective scheme over a noetherian affine scheme $Spec(A)$, with $\dim X = d$, as in (2.3). Let $x = [(\mathcal{E}_\bullet, \varphi)] \in W^k(\mathscr{D}^k(X))$ or $x = [(\mathcal{E}_\bullet, \varphi)] \in W^k_-(\mathscr{D}^k(X))$. Then $(\mathcal{E}_\bullet, \varphi)$ is Witt equivalent to a form $(\mathcal{R}_\bullet, \psi)$ with the following properties:*

1. *$\mathcal{H}_i(\mathcal{R}_\bullet) = 0 \ \forall \ i > 0$ and $grade(\mathcal{H}_0(\mathcal{R}_\bullet)) = k$, unless $\mathcal{H}_0(\mathcal{R}_\bullet) = 0$.*
2. *Further,*

$$\mathscr{R}(\mathcal{R}_\bullet, \psi) \subseteq \begin{cases} \left[k + \frac{d-k}{2}, -\frac{d-k}{2}\right] & \textit{if } d-k \textit{ is even} \\ \left[k + \frac{d-k-1}{2}, -\frac{d-k-1}{2}\right] & \textit{if } d-k \textit{ is odd} \end{cases}$$

In particular, the total length of the range $\leq d$.

In Section 4, we use an extension ([M2]) of a method of constructing certain chain complex maps, originally due to Foxby ([F]), to construct another sublagrangian, for quasi-projective schemes X over noetherian affine schemes $Spec(A)$, without any regularity condition. This gives another bound of the range of the symmetric forms. For example, for a quasi-projective scheme X, any element $x \in W^k(\mathscr{D}^k_g(X))$ or $x \in W^k_-(\mathscr{D}^k_g(X))$ can be represented by a symmetric form $(\mathcal{E}_\bullet, \varphi)$ such that

$$\mathscr{R}(\mathcal{E}_\bullet, \psi) \subseteq \begin{cases} \left[k + \frac{k}{2}, -\frac{k}{2}\right] & \text{if } k \text{ is even} \\ \left[k + \frac{k-1}{2}, -\frac{k-1}{2}\right] & \text{if } k \text{ is odd.} \end{cases}$$

where $\mathscr{D}^k_g(X)$ denotes the filtration by grade of the homologies (see §4).

In §5, we comment on the implications of the main theorems, for the Witt groups $W^r(\mathscr{D}^{d-1}(X))$ and $W^r_-(\mathscr{D}^{d-1}(X))$. We prove (*see (2.3) of notations*) there are surjective homomorphisms $W(\mathscr{A}(X, d-1)) \twoheadrightarrow W^{d-1}(\mathscr{D}^{d-1}(X)) \cong W^{d-3}_-(\mathscr{D}^{d-1}(X))$ of Witt groups, and $W^{d-2}(\mathscr{D}^{d-1}(X)) = 0$. However, $W^d(\mathscr{D}^{d-1}(X)) = 0 \iff H^d(X, W) = 0$, which is standard.

The article is organized as follows. In §2, we prove some preliminary lemmas. In §3, we prove Theorem 1.1. The results on the other three groups $W^{k-1}(\mathscr{D}^k(X))$, $W^{k+1}(\mathscr{D}^k(X))$, $W^{k+2}(\mathscr{D}^k(X))$ are stated in §3.1. In §4, we state and prove second set of results, using the extension of Foxby's construction. In §5 we discuss some of the consequences.

I would like to thank Jean Fasel for reading parts of an earlier version of this article, pointing to some errors and to the Remark 5.4. I would also like to thank Charles A. Weibel for his suggestions regarding the introduction of this article and for alerting me about standard notations.

2. Preliminaries

In this section, we develop some preliminary lemmas and introduce some notations. First, we recall the definition of grade.

Definition 2.1. *For a coherent sheaf $\mathcal{F}$ over noetherian schemes $(X, \mathcal{O}_X)$, define $grade(\mathcal{F}) = \min\{r : \mathcal{E}xt^r(\mathcal{F}, \mathcal{O}_X) \neq 0\}$. For facts about grade of a module, readers are referred to any standard textbook (e.g. [Mh]). For a module M over a Cohen-Macaulay local ring A, the $grade(M) = height(ann(M))$.*

We also record the following easy lemma.

Lemma 2.2. *Suppose A is a commutative noetherian ring, with* $\dim A = d$. *Let C, N are two A-modules, with $proj \dim(C) < \infty$. Then, for $n \geq 0$, $height(ann(Ext^n(C, M)) \geq n$. More precisely, for $\wp \in Spec(A)$,* $\text{depth}(A_\wp) < n \Longrightarrow Ext^n(C, M)_\wp = 0$.

Now we set up some notations.

Notations 2.3. *The readers may be better advised to refer to these notations, as needed. For unexplained notations, readers are referred to ([M1,MS,W]).*

1. Unless stated otherwise, $(X, \mathcal{O}_X)$ will denote a noetherian scheme with $\dim X = d$ and $1/2 \in \mathcal{O}_X$. Without exception, we assume that all coherent sheaves over X are quotient of a locally free sheaf. In fact, for our final results, we assume X is a quasi-projective scheme over a noetherian affine scheme $Spec(A)$. The category of coherent $\mathcal{O}_X$-modules will be denoted by $Coh(X)$. The category of locally free $\mathcal{O}_X$-modules will be denoted by $\mathcal{V} = \mathcal{V}(X)$. For $\mathcal{F} \in Coh(X)$, $\dim_{\mathcal{V}}(\mathcal{F})$ will denote the $\mathcal{V}$-dimention of $\mathcal{F}$. Denote $\mathbb{H}(X) = \{\mathcal{F} \in Coh(X) : \dim_{\mathcal{V}}(\mathcal{F}) < \infty\}$. As usual, $\mathcal{D}^b(\mathcal{V}(X))$ will denote the bounded derived category of the complexes of locally free $\mathcal{O}_X$-modules and T will denote the translation functor.
2. For complexes $\mathcal{E}_\bullet \in \mathcal{D}^b(\mathcal{V}(X))$, the homologies will be denoted by $\mathcal{H}_i(\mathcal{E}_\bullet)$, as usual.
3. Also, recall (see [M1]) the resolution functor $\zeta_k : \mathbb{H}(X) \longrightarrow \mathcal{D}^b(\mathcal{V}(X))$ sending $\mathcal{F}$ to a finite $\mathcal{V}$-resolution, placing the only nonzero homology at degree $-k$.
4. For integers $k \geq 0$, let $Coh(X, k) \subseteq Coh(X)$ denote the full subcategory of objects $\mathcal{F} \in Coh(X)$ such that $grade((Supp(\mathcal{F})) \geq k$ (see 2.1). Also, let $\mathbb{H}_k(X) = \{\mathcal{F} \in Coh(X, k) : \dim_{\mathcal{V}}(\mathcal{F}) = k\}$. This is the category of locally Cohen-Macaulay $\mathcal{O}_X$-modules $\mathcal{F}$, with finite locally free dimension and $grade(Supp(\mathcal{F})) = k$. Note that $\mathbb{H}_k(X)$ is

an exact category. For $\mathcal{F} \in \mathbb{H}_k(X)$, denote $\mathcal{F}^\vee = \mathcal{E}xt^k(\mathcal{F}, \mathcal{O}_X)$. The association $\mathcal{F} \mapsto \mathcal{F}^\vee$ defines a duality on $\mathbb{H}_k(X)$. Denote $\mathcal{A}(X,k) = \left(\mathbb{H}_k(X),^\vee, (-1)^{\frac{k(k-1)}{2}} \varpi_k\right)$, where $\varpi =^{\vee\vee}$. (*Consult* ([BW]) *for standard sign conventions.*)

5. For integers $k \geq 0$, denote $\mathscr{D}^k = \mathscr{D}^k(X) = \{\mathcal{E}_\bullet \in \mathcal{D}^b(\mathscr{V}(X)) : \forall\, i\ \mathcal{H}_i(\mathcal{E}_\bullet) \in Coh(X,k)\}$. Accordingly, we obtain the usual filtration of $\mathcal{D}^b(\mathscr{V}(X))$, by derived subcategories with dualities $\mathcal{D}^b(\mathscr{V}(X)) = \mathscr{D}^0 \supseteq \mathscr{D}^1 \supseteq \cdots \supseteq \mathscr{D}^r \supseteq \cdots \supseteq \mathscr{D}^d \supseteq 0$.
6. Given an exact category $\mathscr{A}$ with duality $W(\mathscr{A})$ or $W_+(\mathscr{A})$ (*resp.* $W_-(\mathscr{A})$) will denote the Witt groups, with respect to plus (*resp.* skew) duality. Likewise, for triangulated categories $\mathscr{T}$ with duality, $W^n(\mathscr{T})$ or $W^n_+(\mathscr{T})$ (*resp.* $W^n_-(\mathscr{T})$) denote the n-shifted Witt groups with plus duality (*resp.* with skew duality).
7. For a complex $\mathcal{E}_\bullet \in \mathcal{D}^b(\mathscr{V}(X))$, denote $\mathscr{R}(\mathcal{E}_\bullet) = [m_0, n_0]$ if $\mathcal{E}_i = 0$ unless $m_0 \geq i \geq n_0$ and $\mathcal{E}_{m_0} \neq 0$ and $\mathcal{E}_{n_0} \neq 0$. For integers $m \geq n$, we also write $\mathscr{R}(\mathcal{E}_\bullet) \subseteq [m,n]$, $m \geq m_0 \geq n_0 \geq n$. (*Note that we write* m, n *in decreasing order.*) $\mathscr{R}(\mathcal{E}_\bullet)$ will be called the "**range**" of $\mathcal{E}_\bullet$, which was referred to as "support" in [B3].

The following is a key lemma that will be used numerous times, subsequently.

Lemma 2.4. *Let A be a noetherian commutative ring with* $\dim A = d$. *Let*

$$0 \longrightarrow Q_k \xrightarrow{d_k} Q_{k-1} \longrightarrow \cdots \longrightarrow Q_1 \xrightarrow{d_1} Q_0 \longrightarrow Q_{-1} \longrightarrow \cdots$$

be a complex of projective A-modules such that (1) $\mathcal{H}_i(Q_\bullet) = 0$ for $i = 1, \ldots k$, and (2) $H_0(Q_\bullet) \neq 0$ with $grade(H_0(Q_\bullet)) = r_0$. Then, $k \geq r_0$.

Proof. We can assume that A is local. Write $I = Ann(H_0(Q_\bullet))$ and $M = \frac{Q_0}{d_1(Q_1)}$. Let $\wp \in Supp(H_0(Q_\bullet)) = V(I)$ be minimal. Then $H_0(Q_\bullet)_\wp$ has finite length. Since $H_0(Q_\bullet) \subseteq M$, we have $depth(M_\wp) = 0$. Therefore, $\dim_{\mathscr{V}}(M_\wp) = depth A_\wp \geq depth_{I_\wp} A_\wp \geq r_0$. Since $(Q_\bullet)_\wp$ provides a projective resolution of $M_\wp$, we have $k \geq \dim_{\mathscr{V}}(M_\wp) \geq r_0$. The proof is complete. □

The following is an immediate extension of ([M1, 5.3]).

Lemma 2.5. *Suppose X is a noetherian scheme, as in (2.3). Assume any object $\mathcal{F} \in Coh(X)$ is quotient of a locally free sheaf on X (as in [M1, 2.1]). Let $\mathcal{L}_\bullet, \mathcal{G}_\bullet \in Ch^b(\mathscr{V}(X))$ be complexes and $\eta_\bullet : \mathcal{L}_\bullet \longrightarrow \mathcal{G}_\bullet$ be a morphism such that $\mathcal{H}_r(\mathcal{G}_\bullet) = 0\ \forall r \geq 0$ and $\mathcal{H}_r(\mathcal{L}_\bullet) = 0\ \forall r < 0$. Then, $\eta_\bullet = 0$ in $\mathcal{D}^b(\mathscr{V}(X))$.*

Proof. First, we can assume $\eta_\bullet$ is denominator free. By replacing $\mathcal{L}_\bullet$ by a quasi-isomorphic complex, we assume, $\mathcal{L}_i = 0\ \forall i < 0$. The functor $\mathcal{D}^b(\mathscr{V}(X)) \xrightarrow{\sim} \mathcal{D}^b(\mathbb{H}(X))$ is an equivalence of categories. We will prove $\eta_\bullet = 0$ in $\mathcal{D}^b(\mathbb{H}(X))$. Now define $\mathcal{F}_i = \mathcal{G}_i\ \forall\, i \geq 1$, $\mathcal{F}_0 = \ker(\mathcal{G}_0 \longrightarrow \mathcal{G}_{-1})$ and $\mathcal{F}_i = 0\ \forall\, i \leq 0$. Then, $\eta_\bullet$ factors through a morphism $\eta'_\bullet : \mathcal{L}_\bullet \longrightarrow \mathcal{F}_\bullet$. Since $\mathcal{F}_\bullet$ is acyclic, $\eta'_\bullet = 0$ and hence $\eta_\bullet = 0$. The proof is complete. $\square$

The Following are extensions of ([M1, 3.3, 3.4]). To provide a flavor, unlike in ([M1]), we state and prove the formal versions.

Lemma 2.6. *Suppose $\mathcal{C}$ is an abelian category. Let $g : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism in $\mathcal{C}$. Let $\mathcal{F}_\bullet, \mathcal{G}_\bullet$ be bounded complexes in $Ch^{\geq 0}(\mathcal{C})$ such that $\mathcal{G}_\bullet$ is a resolution of $\mathcal{G}$ and $\mathcal{H}_0(\mathcal{F}_\bullet) = \mathcal{F}$. Then, there is a bounded complex $\Gamma_\bullet \in Ch^b(\mathcal{C})$ and a diagram $\mathcal{F}_\bullet \xleftarrow[t_\bullet]{} \Gamma_\bullet \xrightarrow[g_\bullet]{} \mathcal{G}_\bullet$ of morphisms, where $t_\bullet$ is a quasi-isomorphism, and $\mathcal{H}_0(g_\bullet) = g$. In particular, g lifts to a morphism $\mathcal{F}_\bullet \longrightarrow \mathcal{G}_\bullet$ of complexes in the bounded derived category $\mathcal{D}^b(\mathcal{C})$.*

Proof. We represent the complexes and g as follows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{F}_r & \xrightarrow{d_n} & \mathcal{F}_{n-1} & \longrightarrow \cdots \longrightarrow & \mathcal{F}_0 & \twoheadrightarrow & \mathcal{F} \\ & & & & & & & & \downarrow g \\ 0 & \longrightarrow & \mathcal{G}_r & \xrightarrow[\partial_n]{} & \mathcal{G}_{n-1} & \longrightarrow \cdots \longrightarrow & \mathcal{G}_0 & \twoheadrightarrow & \mathcal{G} \end{array} \quad \text{and let} \quad \begin{array}{ccc} \Gamma_0 & \overset{t_0}{\dashrightarrow} & \mathcal{F}_0 \\ g_0 \downarrow & & \downarrow g d_0 \\ \mathcal{G}_0 & \xrightarrow[\partial_0]{} & \mathcal{G} \end{array}$$

be the pullback. Since, ∂_0 is surjective, so is t_0. So, $\delta_0 := d_0 t_0 : \Gamma_0 \twoheadrightarrow \mathcal{F}$ is surjective. We construct Γ_n by induction. We denote $Z'_n = \ker(d_n)$, $B'_n = image(d_{n+1})$ and $B_n = \ker(\partial_n) = image(\partial_{n+1})$. Also, as we construct Γ_n, by induction, the differentials will be denoted by $\delta_n : \Gamma_n \longrightarrow \Gamma_{n-1}$.

Write $\mathscr{B}_0 = \ker(\delta_0)$, and g'_0, t'_0 be the restriction maps. Define Γ_1 by combining following two pullbacks:

$$\begin{array}{ccccccccc} \mathcal{F}_1 & & \xrightarrow{\quad d_1 \quad} & & B'_0 & \longrightarrow & \mathcal{F}_0 & \xrightarrow{d_0} & \mathcal{F} \\ t_1 \uparrow & & & \nearrow & \uparrow t'_0 & & \uparrow t_0 & & \| \\ \Gamma_1 & \xrightarrow{p} & \Gamma'_1 & \xrightarrow{p'} & \mathscr{B}_0 & \longrightarrow & \Gamma_0 & \xrightarrow[\delta_0]{} & \mathcal{F} \\ & \searrow^{g_1} & q \downarrow & & \downarrow g'_0 & & \downarrow g_0 & & \downarrow \varphi \\ & & \mathcal{G}_1 & \xrightarrow[\partial_1]{} & B_0 & \longrightarrow & \mathcal{G}_0 & \xrightarrow[\partial_0]{} & \mathcal{G} \end{array}$$

In this diagram Γ'_1 is the pullback of ∂_1 and g'_0 and Γ_1 is the pullback of $d_1, t'_0 p'$. The maps t_1, g_1 are defined as in the diagram. It follows, t_1 is surjective. By the properties of pullback diagram, the restriction $t_1 : \ker(p) \xrightarrow{\sim}$

Z'_1 is an isomorphism. So, t_1 induces a surjective morphism $\mathscr{Z}_1 := \ker(\delta_1) \twoheadrightarrow Z'_1$. It follows $\mathcal{H}_0(\Gamma_\bullet) = \mathcal{F}$.

Now suppose $\Gamma_n, t_n, g_n, \delta_n$ has been defined such that the restriction $t_n : \mathscr{Z}_n := \ker \delta_n \twoheadrightarrow Z'_n$ is surjective. Define $\Gamma_{n+1}, t_{n+1}, g_{n+1}$, as in the diagram:

$$\begin{array}{ccccccccccc}
Z'_{n+1} & \longrightarrow & \mathcal{F}_{n+1} & & \xrightarrow{\quad d_{n+1} \quad} & & B'_n & \hookrightarrow & Z'_n & \hookrightarrow & \mathcal{F}_n \\
\| & & \uparrow{\scriptstyle t_{n+1}} & & \nearrow & & \uparrow{\scriptstyle t'_n} & & \uparrow{\scriptstyle t_n} & & \uparrow{\scriptstyle t_n} \\
ker(p) & \longrightarrow & \Gamma_{n+1} & \xrightarrow[p]{} & \Gamma'_{n+1} & \xrightarrow[p']{} & t_n^{-1}(B'_n) & \hookrightarrow & \mathcal{Z}_n & \hookrightarrow & \mathcal{E}_n \\
& & & {\scriptstyle g_{n+1}} \searrow & \downarrow & & \downarrow{\scriptstyle g'_n} & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle g_n} \\
& & & & \mathcal{G}_{n+1} & \xrightarrow[\partial_{n+1}]{} & B_n & = & B_n & \hookrightarrow & \mathcal{G}_n
\end{array}$$

here Γ'_{n+1} is the pullback of ∂_{n+1} and g'_n and Γ_{n+1} is the pullback of $d_{n+1}, t'_n p'$. The maps t_{n+1}, g_{n+1} are defined as in the diagram. It follows, t_{n+1} is surjective. By the properties of pullback, the restriction $t_{n+1} : \ker(p) \xrightarrow{\sim} Z'_{n+1}$ is an isomorphism. So, t_{n+1} induces a surjective morphism $\mathscr{Z}_{n+1} := \ker(\delta_{n+1}) \twoheadrightarrow Z'_{n+1}$. By construction, $\mathcal{H}_n(\Gamma_\bullet) = \mathcal{H}_n(\mathcal{F}_\bullet)$. The proof is complete. □

A version of (2.6) for derived categories of resolving subcategories follows similarly. Consult ([M1]), for a definition of a resolving subcategory of an abelian category.

Lemma 2.7. *Suppose $\mathscr{V}$ is a resolving subcategory of an abelian category $\mathcal{C}$. Further assume* (as in [M1]), *if $(\mathcal{R}_\bullet, d_\bullet)$ is a $\mathscr{V}$-resolution of $\mathcal{H}_0(\mathcal{R}_\bullet) \in \mathcal{C}$ and if $\mathcal{H}_0(\mathcal{R}_\bullet)$ has finite $\mathscr{V}$-dimension,* $\ker(d_n) \in \mathscr{V}$ *for all $n \gg 0$.*

Let $g : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism in $\mathcal{C}$. Let $\mathcal{E}_\bullet$, $\mathcal{Q}_\bullet$ be two bounded complexes in $Ch^{\geq 0}(\mathcal{C})$ such that $\mathcal{Q}_\bullet$ is a resolution of $\mathcal{G}$ and $\mathcal{H}_0(\mathcal{E}_\bullet) = \mathcal{F}$. Then, there is a bounded complex $\mathcal{L}_\bullet \in Ch^b(\mathscr{V})$ and a diagram $\mathcal{E}_\bullet \xleftarrow[t_\bullet]{} \mathcal{L}_\bullet \xrightarrow[g_\bullet]{} \mathcal{Q}_\bullet$ of morphisms, where $t_\bullet$ is a quasi-isomorphism, and $\mathcal{H}_0(g_\bullet) = g$. In particular, g lifts to a morphism $\mathcal{E}_\bullet \longrightarrow \mathcal{Q}_\bullet$ of complexes in the bounded derived category $\mathcal{D}^b(\mathscr{V})$.

Proof. By (2.6), there is a digram $\mathcal{E}_\bullet \xleftarrow[\tau_\bullet]{} \Gamma_\bullet \xrightarrow[\gamma_\bullet]{} \mathcal{Q}_\bullet$ where $\Gamma_\bullet$ is a bounded complex in $Ch^{\geq 0}(\mathcal{C})$ and $\tau_\bullet$ is a quasi-isomorphism. By resolving category version of ([M1, 3.2]), there is a quasi-isomorphism $\eta_\bullet : \mathcal{L}_\bullet \longrightarrow \Gamma_\bullet$, where $\mathcal{L}_\bullet$ is in $Ch^b(\mathscr{V})$. Now, the proof is complete with $t_\bullet = \tau_\bullet \eta_\bullet$ and $g_\bullet = \gamma_\bullet \eta_\bullet$. □

We underscore that (2.7) applies to the subcategory $\mathscr{V}(X) \subseteq Coh(X)$, when X is a noetherian scheme, as in (2.3).

Lemma 2.8. *Suppose X is a noetherian scheme with* $\dim X = d$*, as in (2.3). Suppose $(\mathcal{E}_\bullet, \varphi)$ is a symmetric form in $T^k\mathcal{D}^b(X)$, and $\mathcal{H}_i(\mathcal{E}_\bullet) = 0$ $\forall i < -n$ for some integer $n \geq 0$. Then, $(\mathcal{E}_\bullet, \varphi)$ is isometric, to a symmetric form $(\mathcal{E}'_\bullet, \varphi')$ such that the range $\mathscr{R}(\mathcal{E}'_\bullet, \varphi') \subseteq [n+k, -n]$.*

Proof. The proof is same as that of [M1, Lemma 5.3]. □

Remark. A version of Lemma 2.8, for resolving subcategories $\mathscr{V}$, with ω-duality structure as in the set up ([M1, 7.1]), of an abelian category $\mathcal{C}$ can be formulated and would be valid. Exactly the same proof of (2.8) would go through. The following elementary lemma will be of some use for us.

Lemma 2.9. *Suppose X is quasi-projective scheme over a noetherian affine scheme $Spec A$. Let Z be a closed subscheme with $grade(\mathcal{O}_Z) \geq r$. Then, there is a complete intersection closed subscheme Y, containing Z, with $co\dim(Y) = r$.*

Remark. Barring this Lemma (2.9), most of our arguments in the paper would go through for regular noetherian schemes X, as in (2.3).

3. The proof of Theorem 1.1

In this section, we prove Theorem 1.1. We give the proof for plus dulity only. We restate and prove the (1) of (1.1), as follows.

Theorem 3.1. *Let X be a regular noetherian scheme, as in (2.3). Let $x = [(\mathcal{E}_\bullet, \varphi)] \in W^k(\mathscr{D}^k(X))$. Then $(\mathcal{E}_\bullet, \varphi)$ is Witt equivalent to a form $(\mathcal{R}_\bullet, \psi)$ such that*

1. $\mathcal{H}_i(\mathcal{R}_\bullet) = 0 \ \forall \ i > 0$ *and*
2. $grade(\mathcal{H}_0(\mathcal{R}_\bullet)) = k$*, unless* $\mathcal{H}_0(R_\bullet) = 0$.

Proof. Suppose $(\mathcal{E}_\bullet, \varphi)$ is a symmetric form in the shifted category $T^k\mathscr{D}^k(X)$. Let the range $\mathscr{R}(\mathcal{E}_\bullet, \varphi) \subseteq [n+k, -n]$. We can assume $2n+k \geq d+1, n \geq 1$. Assume, for some $\upsilon \geq 1$, $\mathcal{H}_i(\mathcal{E}_\bullet) = 0 \ \forall \ i > \upsilon$ and $\mathcal{H}_\upsilon(\mathcal{E}_\bullet) \neq 0$. Our goal is to knock off $\mathcal{H}_\upsilon(\mathcal{E}_\bullet)$. Write the form $(\mathcal{E}_\bullet, \varphi)$ as follows: $\mathcal{E}_{n+k} \longrightarrow \cdots \longrightarrow \mathcal{E}_n \xrightarrow{d_n} \cdots \longrightarrow \mathcal{E}_\upsilon \xrightarrow{d_\upsilon} \cdots \longrightarrow \mathcal{E}_{-n}$, ignoring zeros on two sides. We prove (1) first. Write $t = k + n - \upsilon$. By (2.4), there are two cases,

1. $grade\,(\mathcal{H}_\upsilon(\mathcal{E}_\bullet)) = t$.
2. $grade\,(\mathcal{H}_\upsilon(\mathcal{E}_\bullet)) < t$.

In either case, we want to construct a symmetric form $(\mathcal{R}_\bullet, \psi)$ with the properties that (1) $(\mathcal{R}_\bullet, \psi)$ is Witt equivalent to $(\mathcal{E}_\bullet, \varphi)$, (2) $\mathcal{H}_i(\mathcal{R}_\bullet) = 0\ \forall\ i \geq v$, and (3) the range $\mathscr{R}(\mathcal{R}_\bullet, \psi) \subseteq [n+k, -n]$. The rest of the proof of (3.1) follows from Proposition 3.4 below. □

The following proposition establishes the point (1) for (3.1).

Proposition 3.2. *Let X be a regular noetherian scheme, as in (2.3). Suppose $(\mathcal{E}_\bullet, \varphi)$ is a symmetric form in $T^k\mathscr{D}^k$, such that (a) $\mathscr{R}(\mathcal{E}_\bullet) \subseteq [n+k, -n]$ for some integer $n \gg 0$, (b) there are u, v with $n+k \geq u \geq v \geq 1$ such that $\mathcal{H}_i(\mathcal{E}_\bullet) = 0\ \forall i \geq v, i \neq u$, and (c) $grade(\mathcal{H}_u(\mathcal{E}_\bullet)) = n+k-u =: \tau$.*

Then, there is a symmetric form $(\mathcal{R}_\bullet, \psi)$ such that (1) $(\mathcal{R}_\bullet, \varphi)$ is Witt equivalent to $(\mathcal{E}_\bullet, \varphi)$, (2) the range $\mathscr{R}(\mathcal{R}_\bullet) \subseteq [n+k, -n]$, and (3) $\mathcal{H}_i(\mathcal{R}_\bullet) = 0\ \forall i \geq v$. (4) Further, $\mathcal{H}_0(\mathcal{R}_\bullet) = \mathcal{H}_0(\mathcal{E}_\bullet)$. In fact, $\mathcal{H}_i(\mathcal{R}_\bullet) = \mathcal{H}_i(\mathcal{E}_\bullet)$ for all $(n+k) - d > i \geq -(n-1)$.

Proof. Write the form $(\mathcal{E}_\bullet, \varphi)$ as:

$$\mathcal{E}_{n+k} \longrightarrow \cdots \longrightarrow \mathcal{E}_n \xrightarrow{d_n} \cdots \longrightarrow \mathcal{E}_v \xrightarrow{d_v} \cdots \longrightarrow \mathcal{E}_{-n},$$

ignoring zeros on two sides. Denote $Z_n := Z_n(\mathcal{E}_\bullet) := \ker(d_n)$, $B_n =: B_n(\mathcal{E}_\bullet) := Image(d_{n+1})$. Consider the exact sequence

$$0 \longrightarrow \mathcal{H}_u(\mathcal{E}_\bullet) \longrightarrow \frac{\mathcal{E}_u}{B_u} \longrightarrow B_{u-1} \longrightarrow 0 \tag{2}$$

Denote $\mathcal{H}_i := \mathcal{H}_i(\mathcal{E}_\bullet)$. Then, $grade(\mathcal{H}_u)) = \tau$ and $\dim_{\mathscr{V}}\left(\frac{\mathcal{E}_u}{B_u}\right) \leq \tau$. Therefore, from the ext-sequence of (2), we obtain the exact sequence:

$$\mathcal{E}xt^\tau\left(\frac{\mathcal{E}_u}{B_u}, \mathcal{O}_X\right) \xrightarrow{\beta} \mathcal{E}xt^\tau(\mathcal{H}_u, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^{\tau+1}(B_{u-1}, \mathcal{O}_X) \longrightarrow 0 \tag{3}$$

With $C = \beta\left(\mathcal{E}xt^\tau\left(\frac{\mathcal{E}_u}{B_u}, \mathcal{O}_X\right)\right)$, we obtain the exact sequence:

$$0 \longrightarrow C \longrightarrow \mathcal{E}xt^\tau(\mathcal{H}_u, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^{\tau+1}(B_{u-1}, \mathcal{O}_X) \longrightarrow 0 \tag{4}$$

Since $grade(\mathcal{H}_u(\mathcal{E}_\bullet)) = \tau$, $grade(C) \geq \tau$. Hence $\mathcal{E}xt^i(C, \mathcal{O}_X) = 0$ for all $i < \tau$. Let $0 \longrightarrow \mathcal{L}^*_{n+k-d} \longrightarrow \cdots \longrightarrow \mathcal{L}^*_{n+k} \longrightarrow C$ be a resolution of C, with $\mathcal{L}_i \in \mathscr{V}(X)$, of length at most d (we choose $\mathcal{L}^*_i = \mathcal{H}om(\mathcal{L}_i, \mathcal{O}_X)$ for some $\mathcal{L}_i \in \mathscr{V}(X)$). Denote the complex by $\mathcal{L}^\#_\bullet$. By Lemma 2.7, the surjective homomorphism $\mathcal{E}xt^\tau\left(\frac{\mathcal{E}_u}{B_u}, \mathcal{O}_X\right) \twoheadrightarrow C$ induces a map of complexes $\nu^\# : \mathcal{E}^\#_\bullet \longrightarrow \mathcal{L}^\#_\bullet$, in the derived category $\mathcal{D}^b(\mathscr{V}(X)$, which we denote by $\nu^\#$.

Dualizing, in $T^k\mathscr{D}^k$, we obtain a map of complexes $\nu : \mathcal{L}_\bullet \longrightarrow \mathcal{E}_\bullet$ in $\mathcal{D}^b(\mathscr{V}(X)$. We summarize all of it in the following diagram:

$$\begin{array}{ccccccc}
\mathcal{L}_{n+k} & \longrightarrow \cdots \longrightarrow & \mathcal{L}_u & \longrightarrow \cdots \longrightarrow & \mathcal{L}_{n+k-d} & \twoheadrightarrow & \mathcal{E}xt^d(C,\mathcal{O}_X) \\
\downarrow{\scriptstyle \nu_{n+k}} & & \downarrow{\scriptstyle \nu_u} & & & & \\
\mathcal{E}_{n+k} & \longrightarrow \cdots \longrightarrow & \mathcal{E}_u & \longrightarrow \cdots \longrightarrow & \mathcal{E}_{-n} & \longrightarrow & \mathcal{H}_{-n}(\mathcal{E}) \\
\downarrow{\scriptstyle \varphi_{n+k}} & & \downarrow{\scriptstyle \varphi_u} & & \downarrow{\scriptstyle \varphi_{-n}} & & \downarrow{\scriptstyle \wr} \\
\mathcal{E}^*_{-n} & \longrightarrow \cdots \longrightarrow & \mathcal{E}^*_{k-u} & \longrightarrow \cdots \longrightarrow & \mathcal{E}^*_{n+k} & \longrightarrow & \mathcal{E}xt^\tau\left(\frac{\mathcal{E}_u}{B_u},\mathcal{O}_X\right) \\
 & & & & \downarrow{\scriptstyle \nu^\#_{n+k}} & & \downarrow{\scriptstyle \beta} \\
\mathcal{L}^*_{n+k-d} & \longrightarrow \cdots \longrightarrow & \mathcal{L}^*_{k-u} & \longrightarrow \cdots \longrightarrow & \mathcal{L}^*_{n+k} & \longrightarrow & C
\end{array}$$

This diagram resides in $\mathcal{D}^b(\mathscr{V}(X))$.

Since $n+k-d > -n$ and $\mathcal{H}_i(\mathcal{L}^\#_\bullet) = 0 \ \forall\, i \neq -n$, it follows from (2.5) that $\nu^\#\varphi\nu = 0$ in $\mathcal{D}^b(\mathscr{V}(X))$. That means, ν is a sublangrangian. Consider the following diagram:

$$\begin{array}{ccccccc}
T^{-1}\mathcal{N}_\bullet & \xrightarrow{\nu_0} & \mathcal{L}_\bullet & \xrightarrow{\nu} & \mathcal{E}_\bullet & \xrightarrow{\nu_1} & \mathcal{N}_\bullet \\
\downarrow{\scriptstyle T^{-1}\mu_0^\#} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mu_0^\#} \\
T^{-1}\mathcal{L}^\#_\bullet & \longrightarrow & \mathcal{N}^\#_\bullet & \longrightarrow & \mathcal{E}^\#_\bullet & \longrightarrow & \mathcal{L}^\#_\bullet \\
 & & \downarrow{\scriptstyle \mu_1} & & & & \\
 & & \mathcal{R}_\bullet & & & & \\
 & & \downarrow{\scriptstyle \mu_2} & & & & \\
 & & T\mathcal{L}_\bullet & & & &
\end{array} \tag{5}$$

where $\mathcal{N}_\bullet$ is the cone of ν, the second line is the dual of the first line, μ_0 is a very good morphism (consult [B2]) and $\mathcal{R}_\bullet$ is the cone of μ_0. Since $\mathcal{L}^\#_\bullet$ has only one nonzero homology, at degree $-n$, it follows that $\mathcal{H}_i(\mathcal{N}^\#_\bullet) = \mathcal{H}_i(\mathcal{E}^\#_\bullet) = \mathcal{H}_i(\mathcal{E}_\bullet)$. for all $i > -n$. Further, consider the right tail of the homology sequence of the dual triangle:

$$0 \longrightarrow \mathcal{H}_{-n}(\mathcal{N}^\#_\bullet) \longrightarrow \mathcal{H}_{-n}(\mathcal{E}^\#_\bullet) \twoheadrightarrow C \longrightarrow \mathcal{H}_{-(n+1)}(\mathcal{N}^\#_\bullet) \longrightarrow 0.$$

Therefore, $\mathcal{H}_j(\mathcal{N}^\#_\bullet) = 0 \ \forall\, j < -n$. Now, $\mathcal{H}_i(\mathcal{L}_\bullet) = \mathcal{E}xt^{n+k-i}(C,\mathcal{O}_X)\ \forall\, i$. From grade consideration, $\forall\, i > u$, $\mathcal{H}_i(\mathcal{L}_\bullet) = \mathcal{E}xt^{n+k-i}(C,\mathcal{O}_X) = 0$ because $n+k-i < \tau$. Incorporating this information in the homology sequence of the vertical triangle in (5), we obtain the following:

1. First, $\mathcal{H}_i(\mathcal{R}_\bullet) = 0 \ \forall\, i \leq -(n+1)$.
2. So, by Lemma 2.8, we can cut the range of $\mathcal{R}_\bullet$ to $[n+k, -n]$ i.e. $\mathscr{R}(\mathcal{R}_\bullet) \subseteq [n+k, -n]$.
3. Now, $\mathcal{H}_{u+1}(\mathcal{R}_\bullet) \subseteq \mathcal{E}xt^\tau(C, \mathcal{O}_X)$. Hence $grade(\mathcal{H}_{u+1}(\mathcal{R}_\bullet)) \geq \tau$. By Lemma 2.4, $\mathcal{H}_{u+1}(\mathcal{R}_\bullet) = 0$, which we check locally.
4. *Claim.* $grade(\mathcal{H}_u(\mathcal{R}_\bullet)) \geq \tau + 1$. To see this, we can assume $X = Spec(A)$ is affine. Let $\wp \in Spec(A)$ with $height(\wp) = \tau$. Localizing the long exact sequence, we get an exact sequence

$$0 \longrightarrow \mathcal{E}xt^\tau(C, \mathcal{O}_X)_\wp \longrightarrow \mathcal{H}_u(\mathcal{E}_\bullet)_\wp \longrightarrow \mathcal{H}_u(\mathcal{R}_\bullet)_\wp \longrightarrow 0.$$

We will prove $\mathcal{H}_u(\mathcal{R}_\bullet)_\wp = 0$. If $\mathcal{H}_u(\mathcal{E}_\bullet)_\wp = 0$ then $\mathcal{H}_u(\mathcal{R}_\bullet)_\wp = 0$. So, assume $\mathcal{H}_u(\mathcal{E}_\bullet)_\wp \neq 0$. Since $grade(\mathcal{H}_u(\mathcal{E}_\bullet)_\wp) = \tau$, $\mathcal{H}_u(\mathcal{E}_\bullet)_\wp$ has finite length and $\dim_{\mathscr{V}}(\mathcal{H}_u(\mathcal{E}_\bullet)_\wp) = \tau$. From the exact sequence (4), we have $C_\wp \approx \mathcal{E}xt^\tau(\mathcal{H}_u(\mathcal{E}_\bullet), \mathcal{O}_X)_\wp$. So,

$$\mathcal{E}xt^\tau(C, \mathcal{O}_X)_\wp = \mathcal{E}xt^\tau(\mathcal{E}xt^\tau(\mathcal{H}_u(\mathcal{E}_\bullet), \mathcal{O}_X), \mathcal{O}_X)_\wp \approx \mathcal{H}_u(\mathcal{E}_\bullet)_\wp.$$

Now, it follows from the above exact sequence that $\mathcal{H}_u(\mathcal{R}_\bullet)_\wp = 0$. This establishes the claim.

Since $grade(\mathcal{H}_u(\mathcal{R}_\bullet)) \geq \tau + 1$, by Lemma 2.4, $\mathcal{H}_u(\mathcal{R}_\bullet) = 0$.
5. Now $\mathcal{H}_i(\mathcal{R}_\bullet) \approx \mathcal{E}xt^{\tau+i+1}(C, \mathcal{O}_X) \ \ \forall\ u - 1 \geq i > \upsilon$. Also since $\mathcal{H}_\upsilon(\mathcal{E}_\bullet) = 0$, we have $\mathcal{H}_\upsilon(\mathcal{R}_\bullet) \subseteq \mathcal{E}xt^{\tau+u-\upsilon+1}(C, \mathcal{O}_X)$

By downward induction, and by Lemma 2.4, $\mathcal{H}_i(\mathcal{R}_\bullet) = 0$ for all $i \geq \upsilon$. It follows from the sublagrangian theorem ([B2, 4.20]) that there is a symmetric form $\psi : \mathcal{R}_\bullet \xrightarrow{\sim} \mathcal{R}_\bullet^\#$ such that $[(\mathcal{E}_\bullet, \varphi)] = [(\mathcal{R}_\bullet, \varphi)]$ in $W^k(\mathscr{D}^k)$. The proof is complete. □

Now we clean up homologies on positive degrees.

Proposition 3.3. *Let X be a regular quasi-projective scheme over an affine scheme $Spec(A)$, as in (1.1). Suppose $(\mathcal{E}_\bullet, \varphi)$ is a symmetric form in $T^k\mathscr{D}^k(X)$, such that the range $\mathscr{R}(\mathcal{E}_\bullet) \subseteq [n+k, -n]$ for some integer $n \gg 0$. Then, there is a symmetric form $(\mathcal{R}_\bullet, \psi)$ such that*

1. *$(\mathcal{R}_\bullet, \psi)$ is Witt equivalent to $(\mathcal{E}_\bullet, \varphi)$*
2. *The range $\mathscr{R}(\mathcal{R}_\bullet, \psi) \subseteq [n+k, -n]$.*
3. *$\mathcal{H}_i(\mathcal{R}_\bullet) = 0\, \forall i \geq 1$.*
4. *Further, $\mathcal{H}_0(\mathcal{R}_\bullet) = \mathcal{H}_0(\mathcal{E}_\bullet)$.*

Proof. We use all the notations in the proof of Proposition 3.2. Let $\upsilon \geq 1$ such that $\mathcal{H}_\upsilon(\mathcal{E}_\bullet) \neq 0$ and $\mathcal{H}_i(\mathcal{E}_\bullet) = 0\, \forall\, i > \upsilon$. We show that there is a symmetric form $(\mathcal{R}_\bullet, \psi)$ such that

1. $(\mathcal{R}_\bullet, \psi)$ is Witt equivalent to $(\mathcal{E}_\bullet, \varphi)$.

2. The range $\mathscr{R}(\mathcal{R}_\bullet, \psi) \subseteq [n+k, -n]$.
3. $\mathcal{H}_i(\mathcal{R}_\bullet) = 0 \ \forall\, i \geq v$.

If $grade(\mathcal{H}_v(\mathcal{E}_\bullet)) = (n+k) - v =: t$, then, by Proposition 3.2, there is such a symmetric form $(\mathcal{R}_\bullet, \psi)$. Now assume $s := grade(\mathcal{H}_v(\mathcal{E}_\bullet)) < t$. We have $co\dim(Supp(\mathcal{H}_v(\mathcal{E}_\bullet)) = grade(\mathcal{H}_v(\mathcal{E}_\bullet)) = s \geq k$. By (2.9) there is a locally complete intersection subscheme Y with $co\dim Y = k$ and $Supp(\mathcal{H}_v(\mathcal{E}_\bullet)) \subseteq Y$. There is a surjective morphism $\mathcal{L}_v \twoheadrightarrow \mathcal{H}_v(\mathcal{E}_\bullet)$, where $\mathcal{L}_v \in \mathscr{V}(X)$. It follows that this homomorphism factors through $\mathcal{L}_{v|Y} \twoheadrightarrow \mathcal{H}_v(\mathcal{E}_\bullet)$. Write $\mathcal{F} := \mathcal{L}_{v|Y}$. Then, $\dim_{\mathscr{V}}(\mathcal{F}) = k$ and $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0$ $\forall\, i \neq k$. Further, the surjection $\mathcal{L}_v \twoheadrightarrow \mathcal{F}$ extends to a resolution

$$0 \longrightarrow \mathcal{L}_{v+k} \longrightarrow \cdots \longrightarrow \mathcal{L}_v \longrightarrow \mathcal{F} \longrightarrow 0.$$

Denote this complex by $\mathcal{L}_\bullet$, with $\mathcal{L}_i = 0$ unless $v + k \geq i \geq v$. The composition homomorphism $\mathcal{L}_v \twoheadrightarrow \mathcal{H}_v(\mathcal{E}_\bullet) \longrightarrow \frac{\mathcal{E}_v}{d_{n+1}(\mathcal{E}_{v+1})}$ induces, by Lemma 2.7, a map of complexes $\nu : \mathcal{L}_\bullet \longrightarrow \mathcal{E}_\bullet$ in the derived category $\mathcal{D}^b(\mathscr{V}(X))$. Now, $\mathcal{L}_\bullet^\#$ has only one nonzero homology, at degree $-v$. Since, $v \geq 1$, by (2.5), $\nu^\# \varphi \nu = 0$ in $T^k\mathscr{D}^k$. Hence ν is a sublagrangian. Consider the sublagrangian diagram, as above (5), and use the same notations.

Consider the homology sequence of the dual triangle of (5). It follows that, for all $i \neq -v, -(v+1)$, $\mathcal{H}_i(\mathcal{N}_\bullet^\#) \cong \mathcal{H}_i(\mathcal{E}_\bullet^\#) \cong \mathcal{H}_i(\mathcal{E}_\bullet)$ and

$$0 \longrightarrow \mathcal{H}_{-v}(\mathcal{N}_\bullet^\#) \longrightarrow \mathcal{H}_{-v}(\mathcal{E}_\bullet^\#) \longrightarrow \mathcal{E}xt^k(\mathcal{F}, A) \longrightarrow \mathcal{H}_{-(v+1)}(\mathcal{N}_\bullet^\#) \longrightarrow 0.$$

is exact. In particular, $\mathcal{H}_i(\mathcal{N}_\bullet^\#) = 0$ unless $v \geq i \geq -n$. Now, consider the homology sequence of the vertical triangle of (5). Since $\mathcal{L}_\bullet$ has only one nonzero homology $\mathcal{H}_v(\mathcal{L}_\bullet) = \mathcal{F}$, at degree v, it follows that, for $i \neq v, v-1$, $\mathcal{H}_i(\mathcal{R}_\bullet) = \mathcal{H}_i(\mathcal{N}_\bullet^\#)$, and

$$0 \longrightarrow \mathcal{H}_{v+1}(\mathcal{R}_\bullet) \longrightarrow \mathcal{F} \longrightarrow \mathcal{H}_v(\mathcal{N}_\bullet^\#) \longrightarrow \mathcal{H}_v(\mathcal{R}_\bullet) \longrightarrow 0$$

is exact. Since $\mathcal{H}_v(\mathcal{N}_\bullet^\#) \cong \mathcal{H}_v(\mathcal{E}_\bullet)$, the middle arrow is surjective. So, $\mathcal{H}_v(\mathcal{R}_\bullet) = 0$. Also, $\mathcal{H}_i(\mathcal{R}_\bullet) = 0$ for all $i \geq v+2$ and $i \leq -(n+1)$. So, we have the following:

1. By lemms 2.8, we can cut the range of $\mathcal{R}_\bullet$ to $[n+k, -n]$ i.e. $\mathscr{R}(\mathcal{R}_\bullet) \subseteq [n+k, -n]$.
2. $\mathcal{H}_i(\mathcal{R}_\bullet) = 0 \, \forall i = v, i \geq v+2$.
3. $grade(\mathcal{H}_{v+1}(\mathcal{R}_\bullet)) \geq k$.
4. In fact, $\mathcal{H}_i(\mathcal{R}_\bullet) = \mathcal{H}_i(\mathcal{E}_\bullet)$ if $i \neq \pm v, \pm(v+1)$. In particular, $\mathcal{H}_0(\mathcal{R}_\bullet) = \mathcal{H}_0(\mathcal{E}_\bullet)$.

Again, by ([B2, 4.20]), $(R_\bullet, \psi)$ is Witt equivalent to $(\mathcal{E}_\bullet, \varphi)$. We repeat the process which must stop, latest when $v = n$. Finally, proof of (3.3) is complete, by another application of Proposition 3.2. □

To establish (2) of Theorem 3.1, we further adjust so that $grade(\mathcal{H}_0(\mathcal{R}_\bullet)) = k$.

Proposition 3.4. *Let X be a regular quasi-projective scheme over $Spec(A)$, as in (1.1). Suppose $(\mathcal{E}_\bullet, \varphi)$ is a symmetric form in $T^k\mathscr{D}^k(X)$, such that the range $\mathscr{R}(\mathcal{E}_\bullet) \subseteq [n+k, -n]$ with $n \gg 0$. Then, there is a symmetric form $(\mathcal{R}_\bullet, \psi)$, such that (a) $(\mathcal{R}_\bullet, \psi)$ is Witt equivalent to $(\mathcal{E}_\bullet, \varphi)$, (b) the range $\mathscr{R}(\mathcal{R}_\bullet, \psi) \subseteq [n+k, -n]$, and (c) $\mathcal{H}_i(\mathcal{R}_\bullet) = 0\ \forall i \geq 1$ and $grade(\mathcal{H}_0(\mathcal{R}_\bullet)) = k$, unless $\mathcal{H}_0(\mathcal{R}_\bullet) = 0$.*

Proof. By (3.3), we can assume $\mathcal{H}_i(\mathcal{E}_\bullet)) = 0\ \forall\ i \geq 1$. Now, assume $grade(\mathcal{H}_0(\mathcal{E}_\bullet)) \geq k+1$. As in (3.3), we can construct (a) a coherent sheaf $\mathcal{F}$, with $\dim_{\mathscr{V}}(\mathcal{F}) = k+1$ and $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0\ \forall\ i \neq k+1$, (b) a surjective homomorphism $\mathcal{F} \twoheadrightarrow \mathcal{H}_0(\mathcal{E}_\bullet)$, (c) a resolution $0 \longrightarrow \mathcal{L}_{k+1} \longrightarrow \cdots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$, with $\mathcal{L}_i \in \mathscr{V}(X)$. Denote the complex $\mathcal{L}_\bullet$. The composition homomorphism $\mathcal{L}_0 \twoheadrightarrow \mathcal{H}_0(\mathcal{E}_\bullet) \longrightarrow \frac{\mathcal{E}_0}{d_1(\mathcal{E}_1)}$ induces, by Lemma 2.7, a map of complexes $\nu : \mathcal{L}_\bullet \longrightarrow \mathcal{E}_\bullet$ in the derived category $\mathcal{D}^b(\mathscr{V}(X))$.

The dual $\mathcal{L}_\bullet^\#$ has only one nonzero homology, at degree -1. Again, by (2.5), $\nu^\#\varphi\nu = 0$ in $T^k\mathscr{D}^k$. Consider the diagram (5) and use the same notations. Now consider the homology sequence of the dual triangle in (5). We obtain $\mathcal{H}_i(\mathcal{N}_\bullet^\#) = \mathcal{H}_i(\mathcal{E}_\bullet^\#) = \mathcal{H}_i(\mathcal{E}_\bullet) \forall\, i \neq -1, -2$ and the sequence

$$0 \longrightarrow \mathcal{H}_{-1}(\mathcal{N}_\bullet^\#) \longrightarrow \mathcal{H}_{-1}(\mathcal{E}_\bullet^\#) \longrightarrow \mathcal{E}xt^{k+1}(\mathcal{F}, \mathcal{O}_X)$$
$$\longrightarrow \mathcal{H}_{-2}(\mathcal{N}_\bullet^\#) \longrightarrow \mathcal{H}_{-2}(\mathcal{E}_\bullet^\#) \longrightarrow 0$$

is exact. Since $n \gg 0$, we have $\mathcal{H}_i(\mathcal{N}_\bullet^\#) = 0\ \forall\ i \leq -(n+1)$. Now consider the homology sequence for the vertical triangle in (2.5). Since, $\mathcal{L}_\bullet$ has only one nonzero homology, at degree zero, $\mathcal{H}_i(\mathcal{R}_\bullet) = \mathcal{H}_i(\mathcal{F}_\bullet^\#)\ \forall\ i \neq 0, 1$ and

$$0 \longrightarrow \mathcal{H}_1(\mathcal{R}_\bullet) \longrightarrow \mathcal{F} \longrightarrow\!\!\!\!\rightarrow \mathcal{H}_0(\mathcal{N}_\bullet^\#) \longrightarrow \mathcal{H}_0(\mathcal{R}_\bullet) \longrightarrow 0$$

is exact. Since $\mathcal{H}_0(\mathcal{N}_\bullet^\#) \cong \mathcal{H}_0(\mathcal{E}_\bullet)$, the middle arrow is surjective. So, $\mathcal{H}_0(\mathcal{R}_\bullet) = 0$. Also, $\mathcal{H}_i(\mathcal{R}_\bullet) = 0\ \forall\ i \leq -(n+1)$ and $\forall\ i \geq 0, i \neq 1$. Therefore, by (2.8), we can cut the range of $\mathcal{R}_\bullet$ to $[n+k, -n]$, i.e., $\mathscr{R}(\mathcal{R}_\bullet) \subseteq [n+k, -n]$. Again, by ([B2, 4.20]), there is a symmetric form $\psi : \mathcal{R}_\bullet \xrightarrow{\sim} \mathcal{R}_\bullet^\#$ such that $(\mathcal{R}_\bullet, \psi)$ is Witt equivalent to $(\mathcal{E}_\bullet, \varphi)$. To readjust the non-zero homology at degree one, apply (3.3) one more time. The proof is complete. □

Completing the proof of Theorem 3.1. As was stated before, the proof of Theorem 3.1 follows directly from Proposition 3.4. □

Completing the proof of Theorem 1.1. The point (1) of (1.1) was established in Theorem 3.1. We will prove point (2) regarding the range. We will assume that $(\mathcal{E}_\bullet, \varphi)$ satisfies assertion (1) of the theorem. Suppose the range $\mathscr{R}(\mathcal{E}_\bullet, \varphi) \subseteq [k + r + m, -(r + m)]$, where $m \geq 1$ and

$$r = \begin{cases} \frac{d-k}{2} & \text{if } d - k \text{ is even} \\ \frac{d-k-1}{2} & \text{if } d - k \text{ is odd} \end{cases}$$

We use downward induction on m. In either case, the total length of the range $= k + 2r + 2m \geq d + 1$. The form looks like:

$$\mathcal{E}_{k+r+m} \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \xrightarrow{d_0} \cdots \longrightarrow \mathcal{E}_{-r} \longrightarrow \cdots \longrightarrow \mathcal{E}_{-(r+m)}$$

ignoring the zeros on both ends. With $B_0 = image(d_1)$, the nonnegative part of $\mathcal{E}_\bullet$ is a resolution of $\frac{\mathcal{E}_0}{B_0}$. Hence,

$$\mathcal{H}_{-(r+m)}(\mathcal{E}_\bullet) \cong \mathcal{H}_{-(r+m)}(\mathcal{E}_\bullet^\#) \cong \mathcal{E}xt^{k+(r+m)}\left(\frac{\mathcal{E}_0}{B_0}, A\right)$$

has $grade \geq k + r + m$. Take a resolution, of width at most d:

$$0 \longrightarrow \mathcal{L}^*_{k+(r+m)-d} \longrightarrow \cdots \longrightarrow \mathcal{L}^*_{k+(r+m)} \twoheadrightarrow \mathcal{E}xt^{k+(r+m)}\left(\frac{\mathcal{E}_0}{B_0}, A\right)$$

where $\mathcal{L}_i^* = \mathcal{H}om(\mathcal{L}_i, \mathcal{O}_X)$ for some $\mathcal{L}_i \in \mathscr{V}(X)$. Denote this complex by $\mathcal{L}_\bullet^\#$, with $\mathcal{L}_i = 0$ unless $k + (r + m) \geq i \geq k + (r + m) - d$. By (2.7), the isomorphism $\mathcal{H}_{-(r+m)}(\mathcal{E}_\bullet^\#) \xrightarrow{\sim} \mathcal{E}xt^{k+(r+m)}\left(\frac{\mathcal{E}_0}{B_0}, A\right)$ induces a morphism $\nu^\# : \mathcal{E}_\bullet^\# \longrightarrow \mathcal{L}_\bullet^\#$ in the derived category $\mathcal{D}^b(\mathscr{V}(X))$, which we denote by $\nu^\#$. Now, dualizing, we get a morphism $\nu : \mathcal{L}_\bullet \longrightarrow \mathcal{E}_\bullet$. It follows, by (2.5), that $\nu\varphi\nu^\# = 0$ in $T^k\mathscr{D}^k$. Consider the diagram (5) and use the same notations.

Consider the homology exact sequence of the dual triangle in (5). Since $\mathcal{L}_\bullet^\#$ has only one nonzero homology, at degree $-(r + m)$, we obtain $\mathcal{H}_i(\mathcal{N}_\bullet^\#) \cong \mathcal{H}_i(\mathcal{E}_\bullet^\#) \cong \mathcal{H}_i(\mathcal{E}_\bullet)$, $\forall\, i \neq -(r + m), -(r + m) - 1$ and the sequence

$$0 \longrightarrow \mathcal{H}_{-(r+m)}(\mathcal{N}_\bullet^\#) \longrightarrow \mathcal{H}_{-(r+m)}(\mathcal{E}_\bullet^\#) \xrightarrow{\sim} \mathcal{H}_{-(r+m)}(\mathcal{L}_\bullet^\#) \longrightarrow \mathcal{H}_{-(r+m)-1}(\mathcal{N}_\bullet^\#) \longrightarrow 0$$

is exact. Since the middle arrow is an isomorphism, we have $\mathcal{H}_{-(r+m)-1}(\mathcal{N}_\bullet^\#) = \mathcal{H}_{-(r+m)}(\mathcal{N}_\bullet^\#) = 0$. It follows, $\mathcal{H}_i(\mathcal{N}_\bullet^\#) = 0$ unless $0 \geq i > -(r + m)$.

Write

$$E := \mathcal{E}xt^{k+(r+m)}\left(\frac{\mathcal{E}_0}{B_0}, \mathcal{O}_X\right) \cong \mathcal{H}_{-(r+m)}(\mathcal{E}_\bullet, \mathcal{O}_X).$$

Since $grade(E) \geq k + r + m, \mathcal{H}_i(\mathcal{L}_\bullet) = \mathcal{E}xt^{k+(r+m)-i}(E, A) = 0, \forall\, i \geq 1$ and $\forall\, i \leq -(r+m)$. Incorporating this information in the homology exact sequence of the vertical triangle in (5), we obtain $\mathcal{H}_i(\mathcal{R}_\bullet) \cong \mathcal{H}_i(\mathcal{N}_\bullet^\#)$ unless $1 \geq i \geq -(r+m)+1$ and the sequence

$$0 \longrightarrow \mathcal{H}_1(\mathcal{R}_\bullet) \longrightarrow \mathcal{E}xt^{k+(r+m)}(E, \mathcal{O}_X) \longrightarrow \mathcal{H}_0(\mathcal{N}_\bullet^\#)$$
$$\longrightarrow \mathcal{H}_0(\mathcal{R}_\bullet) \longrightarrow \mathcal{E}xt^{k+(r+m)+1}(E, \mathcal{O}_X)$$

is exact. Now it follows that $\mathcal{H}_i(\mathcal{R}_\bullet) = 0$ for $i \geq 2$ and $i \leq -(r+m)$. By (2.8), we cut the range of $\mathcal{R}_\bullet$ to $[k + (r+m) - 1, -((r+m) - 1)]$. Since $\mathcal{H}_1(\mathcal{R}_\bullet) \subseteq \mathcal{H}_0(\mathcal{L}_\bullet) = \mathcal{E}xt^{k+(r+m)}(E, \mathcal{O}_X)$, it follows that $grade(\mathcal{H}_1(\mathcal{R}_\bullet)) \geq (r+m)+k$, which is bigger than $k + (r+m) - 1$. So, by Lemma 2.4, $\mathcal{H}_1(\mathcal{R}_\bullet) = 0$. If $\mathcal{H}_0(\mathcal{E}_\bullet) = \mathcal{H}_0(\mathcal{N}_\bullet^\#) = 0$, by the same argument $\mathcal{H}_0(\mathcal{R}_\bullet) = 0$. Now, suppose $\mathcal{H}_0(\mathcal{E}_\bullet) = \mathcal{H}_0(\mathcal{N}_\bullet^\#) \neq 0$. In this case, $grade(\mathcal{H}_0(\mathcal{E}_\bullet)) = k$. We prove $grade(\mathcal{H}_0(\mathcal{R})) = k$, by checking locally. So, we assume $X = Spec(A)$ is affine. For $\wp \in Spec(A)$, with $height(\wp) = k - 1$, localizing the above exact sequence we have $\mathcal{H}_0(\mathcal{R}_\bullet)_\wp = 0$. Hence, $grade(\mathcal{H}_0(\mathcal{R})) = k$, in this case. Again, by ([B2, 4.20]) the proof is complete. □

3.1 The Witt groups $W^r(\mathscr{D}^k(X))$

In this section, we state all the results from the first set, describing the forms in all the shifted Witt groups $W^r(\mathscr{D}^k(X))$. First, we restate Theorem 1.1, in this list, for the convenience of the readers and completeness.

Theorem 3.5. *Let X be a regular quasi-projective scheme over an affine scheme $Spec(A)$, with $\dim X = d$, as in (2.3). Let $x = [(\mathcal{E}_\bullet, \varphi)] \in W^k(\mathscr{D}^k)$ or $x = [(\mathcal{E}_\bullet, \varphi)] \in W^k_-(\mathscr{D}^k)$. Then $(\mathcal{E}_\bullet, \varphi)$ is Witt equivalent to a form $(\mathcal{R}_\bullet, \psi)$ with the following properties:*

1. *$\mathcal{H}_i(\mathcal{R}_\bullet) = 0\ \forall\ i > 0$ and $grade(\mathcal{H}_0(\mathcal{R}_\bullet)) = k$, unless $\mathcal{H}_0(\mathcal{R}_\bullet) = 0$.*
2. *Further, the range*

$$\mathscr{R}(\mathcal{R}_\bullet, \psi) \subseteq \begin{cases} \left[k + \frac{d-k}{2}, -\frac{d-k}{2}\right] & \text{if } d-k \text{ is even} \\ \left[k + \frac{d-k-1}{2}, -\frac{d-k-1}{2}\right] & \text{if } d-k \text{ is odd} \end{cases}$$

In particular, the total length of the range $\leq d$.

With shift T^{k+2}, the structure of the forms is given by the following theorem.

Theorem 3.6. *Let X be a regular quasi-projective scheme over an affine scheme $Spec(A)$, with $\dim X = d$. Let $x = [(\mathcal{E}_\bullet, \varphi)] \in W^{k+2}(\mathscr{D}^k)$ or $x = [(\mathcal{E}_\bullet, \varphi)] \in W_-^{k+2}(\mathscr{D}^k)$. Then $(\mathcal{E}_\bullet, \varphi)$ is Witt equivalent to a form $(\mathcal{R}_\bullet, \psi)$ with the following properties:*

1. *$\mathcal{H}_i(\mathcal{R}_\bullet) = 0\ \forall i \geq 2$ and $grade(\mathcal{H}_1(\mathcal{R}_\bullet)) = k$, unless $\mathcal{H}_1(\mathcal{R}_\bullet) = 0$.*
2. *Further, the range*

$$\mathscr{R}(\mathcal{R}_\bullet, \psi) \subseteq \begin{cases} \left[(k+2) + \frac{d-k-2}{2}, -\frac{d-k-2}{2}\right] & \text{if } d-k \text{ is even} \\ \left[(k+2) + \frac{d-k-3}{2}, -\frac{d-k-3}{2}\right] & \text{if } d-k \text{ is odd} \end{cases}$$

Proof. The theorem follows from Theorem 3.5, by a shift T. □

The following theorem corresponds to the shift T^{k-1}.

Theorem 3.7. *Let X be a regular quasi-projective scheme over an affine scheme $Spec(A)$, with $\dim X = d$. Let $x = [(\mathcal{E}_\bullet, \varphi)] \in W^{k-1}(\mathscr{D}^k)$ or $x = [(\mathcal{E}_\bullet, \varphi)] \in W_-^{k-1}(\mathscr{D}^k)$. Then $(\mathcal{E}_\bullet, \varphi)$ is Witt equivalent to a form $(\mathcal{R}_\bullet, \psi)$ with the following properties:*

1. *$\mathcal{H}_i(\mathcal{R}_\bullet) = 0\ \forall i \geq 0$ and $grade(\mathcal{H}_{-1}(\mathcal{R}_\bullet)) = k+1$, unless $\mathcal{H}_{-1}(\mathcal{R}_\bullet) = 0$.*
2. *Further, the range*

$$\mathscr{R}(\mathcal{R}_\bullet, \psi) \subseteq \begin{cases} \left[k-1 + \frac{d-k+1}{2}, -\frac{d-k+1}{2}\right] & \text{if } d-k \text{ is odd} \\ \left[k-1 + \frac{d-k}{2}, -\frac{d-k}{2}\right] & \text{if } d-k \text{ is even} \end{cases}$$

In particular, the total length of the range $\leq d$.

Proof. The proof is similar to that of Theorem 3.5. □

With shift T^{k+1} the following is obtained.

Theorem 3.8. *Let X be a regular quasi-projective scheme over an affine scheme $Spec(A)$, with $\dim X = d$. Let $x = [(\mathcal{E}_\bullet, \varphi)] \in W^{k+1}(\mathscr{D}^k)$ or $x = [(\mathcal{E}_\bullet, \varphi)] \in W_-^{k+1}(\mathscr{D}^k)$. Then $(\mathcal{E}_\bullet, \varphi)$ is Witt equivalent to a form $(\mathcal{R}_\bullet, \psi)$ with the following properties:*

1. *$\mathcal{H}_i(\mathcal{R}_\bullet) = 0\ \forall i \geq 1$ and $grade(\mathcal{H}_0(\mathcal{R}_\bullet)) = k+1$, unless $\mathcal{H}_0(\mathcal{R}_\bullet) = 0$.*
2. *Further, the range*

$$\mathscr{R}(\mathcal{R}_\bullet, \psi) \subseteq \begin{cases} \left[k+1 + \frac{d-(k+1)}{2}, -\frac{d-(k+1)}{2}\right] & \text{if } d-k \text{ is odd} \\ \left[(k+1) + \frac{d-k-2}{2}, -\frac{d-k-2}{2}\right] & \text{if } d-k \text{ is even} \end{cases}$$

In particular, the total length of the range $\leq d$.

Proof. The theorem follows from Theorem 3.7, by a shift T. □

4. Foxby Sublagrangian

In this section, we drop the regularity condition and assume that X is a quasi-projective scheme over an affine scheme $Spec(A)$. Here we employ an extension of a construction of a morphism of chain complexes of modules, originally due to Foxby (see [F,FH]), to construct sublagrangians and reduce the length of the range of the symmetric forms. For our purposes, a version of Foxby's construction, for quasi-projective schemes was completed in ([M2]), which we quote below.

Theorem 4.1. *Suppose X is a noetherian quasi-projective scheme over a noetherian affine scheme $Spec(A)$, with* $\dim X = d$. *Let*

$$\mathcal{G}_{k+1} \longrightarrow \mathcal{G}_k \xrightarrow[\partial_k]{} \cdots \longrightarrow \mathcal{G}_r \xrightarrow[\partial_r]{} \mathcal{G}_{r-1} \longrightarrow \cdots \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G}_{-1}$$

be a complex of coherent $\mathcal{O}_X$-modules. Assume $\forall\, i \;\; grade(\mathcal{O}_{Y_i}) \geq k$, where $Y_i = Supp(\mathcal{H}_i(\mathcal{G}_\bullet)) \subseteq X$. Then, there is a morphism $\nu_\bullet : \mathcal{L}_\bullet \longrightarrow \mathcal{G}_\bullet$ where $\mathcal{L}_\bullet$:

$$0 \longrightarrow \mathcal{L}_k \longrightarrow \cdots \longrightarrow \mathcal{L}_r \xrightarrow[d_r]{} \mathcal{L}_{r-1} \longrightarrow \cdots \longrightarrow \mathcal{L}_0 \longrightarrow 0$$

is in $Ch^b(\mathscr{V}(X))$ such that

1. *$\mathcal{H}_i(\mathcal{L}_\bullet) = 0 \;\; \forall\, i \neq 0$ and $\mathcal{H}_0(\nu) : \mathcal{H}_0(\mathcal{L}_\bullet) \twoheadrightarrow \mathcal{H}_0(\mathcal{G}_\bullet)$ is surjective.*
2. *$\mathcal{E}xt^i\,(\mathcal{H}_0(\mathcal{L}_\bullet), \mathcal{O}_X) = 0 \;\; \forall\, i \neq 0$ and $\dim_{\mathscr{V}}(\mathcal{H}_0(\mathcal{L}_\bullet)) = k$. In fact, $\mathcal{L}_\bullet$ would be a direct sum of twisted Koszul complexes that resolves $\mathcal{H}_0(\mathcal{L}_\bullet)$.*

Before the statement of the main results in the section, define

$$\mathscr{D}^k_g(X) = \{\mathcal{E}_\bullet \in \mathcal{D}^b(\mathscr{V}(X)) : grade(\mathcal{H}_i(\mathcal{E}_\bullet)) \geq k \;\forall i\}.$$

Then, $\mathscr{D}^k_g(X)$ is a filtration of $\mathscr{D}^b(\mathscr{V}(X))$, by grade of the homologies. When X is not Cohen-Macaulay, this filtration differs from the usual filtration $\mathscr{D}^k(X)$, by co-dimension of support of the homologies, as defined in (2.3).

Theorem 4.2. *Suppose X is a quasi-projective scheme over an affine scheme $Spec(A)$, with* $\dim X = d$.

1. *Suppose $x \in W^k(\mathscr{D}^k_g(X))$ or $x \in W^k_-(\mathscr{D}^k_g(X))$. Then, $x = [(\mathcal{E}_\bullet, \varphi)]$ for some symmetric form $(\mathcal{E}_\bullet, \varphi)$ with the range*

$$\mathscr{R}(\mathcal{E}_\bullet, \psi) \subseteq \begin{cases} \left[k + \frac{k}{2}, -\frac{k}{2}\right] & \text{if } k \text{ is even} \\ \left[k + \frac{k-1}{2}, -\frac{k-1}{2}\right] & \text{if } k \text{ is odd.} \end{cases}$$

2. *Suppose $x \in W^{k+2}(\mathscr{D}_g^k(X))$ or $x \in W_-^{k+2}(\mathscr{D}_g^k(X))$. Then, $x = [(\mathcal{E}_\bullet, \varphi)]$ for some symmetric form $(\mathcal{E}_\bullet, \varphi)$ with the range*

$$\mathscr{R}(\mathcal{E}_\bullet, \psi) \subseteq \begin{cases} \left[(k+2) + \frac{k-2}{2}, -\frac{k-2}{2}\right] & \text{if } k \text{ is even} \\ \left[(k+2) + \frac{k-3}{2}, -\frac{k-3}{2}\right] & \text{if } k \text{ is odd.} \end{cases}$$

3. *Suppose $x \in W^{k-1}(\mathscr{D}_g^k(X))$ or $x \in W_-^{k-1}(\mathscr{D}_g^k(X))$. Then, $x = [(\mathcal{E}_\bullet, \varphi)]$ for some symmetric form $(\mathcal{E}_\bullet, \varphi)$ with the range*

$$\mathscr{R}(\mathcal{E}_\bullet, \psi) \subseteq \begin{cases} \left[(k-1) + \frac{k}{2}, -\frac{k}{2}\right] & \text{if } k \text{ is even} \\ \left[(k-1) + \frac{k+1}{2}, -\frac{k+1}{2}\right] & \text{if } k \text{ is odd.} \end{cases}$$

4. *Suppose $x \in W^{k+1}(\mathscr{D}_g^k(X))$ or $x \in W_-^{k+1}(\mathscr{D}_g^k(X))$. Then, $x = [(\mathcal{E}_\bullet, \varphi)]$ for some symmetric form $(\mathcal{E}_\bullet, \varphi)$ with the range*

$$\mathscr{R}(\mathcal{E}_\bullet, \psi) \subseteq \begin{cases} \left[(k+1) + \frac{k-2}{2}, -\frac{k-2}{2}\right] & \text{if } k \text{ is even} \\ \left[(k+1) + \frac{k-1}{2}, -\frac{k-1}{2}\right] & \text{if } k \text{ is odd.} \end{cases}$$

Proof. We will only prove (1), for plus duality. Suppose $(\mathcal{E}_\bullet, \varphi)$ is a symmetric form in $T^k\mathscr{D}_g^k(X)$. Assume that $\mathscr{R}(\mathcal{E}_\bullet) \subseteq [n+k, -n]$, $2n > k$ and $H_{-n}(\mathcal{E}_\bullet) \neq 0$. By (4.1), there is a chain complex morphism $\nu : \mathcal{L}_\bullet \longrightarrow \mathcal{E}_\bullet$ such that (1) $\mathcal{L}_\bullet$ is a locally free resolution of $\mathcal{H}_{-n}(\mathcal{L}_\bullet)$ and $\mathcal{L}_i = 0$ unless $-n+k \geq i \geq -n$, (2) ν induces a surjective homomorphism $\mathcal{H}_{-n}(\mathcal{L}_\bullet) \twoheadrightarrow \mathcal{H}_{-n}(\mathcal{E}_\bullet)$, and (3) $\mathcal{E}xt^i\ (\mathcal{H}_{-n}(\mathcal{L}_\bullet), \mathcal{O}_X) = 0$ for all $i \neq k$.

Since $2n > k$, it follows that $\nu^{\#}\varphi\nu = 0$. So, ν is a sublagrangian. Consider the diagram (5) of triangles and use the same notations. By going through the same methods of reduction of length of symmetric forms, as in section 3, we obtain a symmetric form $(R_\bullet, \psi)$ which is Witt equivalent to $(\mathcal{E}_\bullet, \varphi)$ in $\mathscr{D}_g^k(X)$ and $\mathscr{R}(R_\bullet) \subseteq [(n-1)+k, -(n-1)]$. The process can be repeated, while $2n > k$. This completes the proof. □

Corollary 4.3. *The corresponding version of (4.2) for $\mathscr{D}^k_{\mathbb{H}(X),g}(X)$ would also be valid, where $\mathscr{D}^k_{\mathbb{H}(X),g}(X)$ denotes the subcategory of $\mathscr{D}_g^k(X)$ consisting of complexes with homologies in $\mathbb{H}(X)$.*

Proof. As in the proof of (4.2), consider the diagram (5). Arguing locally, by ([M1, 4.4]) it follows, $grade(H_n(\mathcal{N}_\bullet^{\#})) \geq k+1$ and hence by (2.4) $H_n(\mathcal{N}_\bullet^{\#}) = 0$. Now, it follows $\mathcal{H}_i(\mathcal{N}_\bullet^{\#}) \in \mathbb{H}(X)\ \forall\, i$. Now it also varifies that $\mathcal{H}_i(\mathcal{R}_\bullet) \in \mathbb{H}(X)\ \forall\, i$. The proof is complete. □

Remark 4.4. *It would be interesting to point out that, for $k = 0$, the sublagrangian used in the proof of (4.2) coincides with that used in ([B1, 4.1]) designed to kill homologies of the symmetric forms, starting from right tail of the form.*

5. Some consequences

As an immediate consequence we obtain the following.

Corollary 5.1. *Let X be a quasi-projective regular scheme over an affine scheme $Spec(A)$. The twelve term exact sequence* ([B2]) *corresponding to the inclusion $\mathscr{D}^{k+1} \subseteq \mathscr{D}^k$ splits into two six term exact sequences terminating, for integers $k \geq 0$, at the surjective homomorphims*

$$\begin{cases} W^{k+1}(\mathscr{D}^{k+1}) \twoheadrightarrow W^{k+1}(\mathscr{D}^k), \\ W^{k+3}(\mathscr{D}^{k+1}) \twoheadrightarrow W^{k+3}(\mathscr{D}^k). \end{cases}$$

The same holds for skew duality.

Proof. Immediate from 3.5, 3.7, 3.8, 3.6. □

While we obtain (5.1) by elementary methods, in deed by [BW], the following terms in the twelve term sequence $W^{k+1}(\frac{\mathscr{D}^k}{\mathscr{D}^{k+1}}) = W^{k+3}(\frac{\mathscr{D}^k}{D^{k+1}}) = 0$ and they are five term exact sequences. Next, we would like to interpret the main theorems for $k = d-1$. Before that, we record the following homomorphisms.

Proposition 5.2. *Let X be a noetherian scheme X and $k, r \geq 0$ be integers. (We are mainly interested in the case $r = 0$.) There are natural homomorphisms of Witt groups,*

$$\begin{cases} W(\mathcal{A}(X, k+2r)) \longrightarrow W^k(\mathscr{D}^k_g(X)) & \textit{induced by } \zeta_r, \\ W_-(\mathcal{A}(X, k+2r)) \longrightarrow W^k_-(\mathscr{D}^k_g(X)) & \textit{induced by } \zeta_r \\ W(\mathcal{A}(X, k+2r+1)) \longrightarrow W^{k-1}(\mathscr{D}^k_g(X)) & \textit{induced by } \zeta_{r+1} \\ W_-(\mathcal{A}(X, k+2r+1)) \longrightarrow W_-(\mathscr{D}^k_g(X)) & \textit{induced by } \zeta_{r+1} \end{cases}$$

where $\zeta_n = T^{-n}\zeta_0$, with the standard signed translation T.

Proof. Obvious. □

The Witt groups $W^r(\mathscr{D}^d(X))$ were computed in ([BW]) in the regular case, which was extended to non-regular situation in ([M1,MS]). The following is some comments on the groups $W^r(\mathscr{D}^{d-1}(X))$.

Corollary 5.3. *Let X be a quasi-projective regular scheme over a noetherian affine scheme Spec(A), with* $\dim X = d$. *Then,*

$$\begin{cases} W(\mathcal{A}(X, d-1)) \twoheadrightarrow W^{d-1}(\mathcal{D}^{d-1}(X)) & \text{is surjective,} \\ W_-(\mathcal{A}(X, d-1)) \twoheadrightarrow W_-^{d-1}(\mathcal{D}^{d-1}(X)) & \text{is surjective and,} \\ W^{d-2}(\mathcal{D}^{d-1}(X)) = 0. \end{cases}$$

Proof. The surjectivity of these homomorphisms follow from Theorem 3.5 and Lemma 2.4. To prove the vanishing statement, consider the exact sequence corresponding to the inclusion

$$\mathcal{D}^{d-1} \subseteq \mathcal{D}^d : \ W^{d-2}(\mathcal{D}^d) \longrightarrow W^{d-2}(\mathcal{D}^{d-1}) \longrightarrow W^{d-2}\left(\frac{\mathcal{D}^{d-1}}{\mathcal{D}^d}\right).$$

Since the first and the last term are zero ([BW, 6.1]; the middle term $W^{d-2}(\mathcal{D}^{d-1}) = 0$. The proof is complete. □

Remark 5.4. In fact, because of the the exact sequence

$$W^{d-1}\left(\frac{\mathcal{D}^{d-1}}{\mathcal{D}^d}\right) \xrightarrow{\partial} W^d(\mathcal{D}^d) \longrightarrow W^d(\mathcal{D}^{d-1}) \longrightarrow W^d\left(\frac{\mathcal{D}^{d-1}}{\mathcal{D}^d}\right)$$

and since the last term is zero ([BW]), $W^d(\mathcal{D}^{d-1}) = 0$ only when ∂ is surjective, which means when $H^d(X, W) = 0$. □

The following are some comments on $W^k(\mathcal{D}^{d-2}(X))$.

Corollary 5.5. *Let X be as in (5.3). Then,*

$$\begin{cases} W(\mathscr{A}(X, d-1)) \twoheadrightarrow W^{d-1}(\mathcal{D}^{d-2}(X)) & \text{is surjective,} \\ W_-(\mathscr{A}(X, d-1)) \twoheadrightarrow W_-^{d-1}(\mathcal{D}^{d-2}(X)) & \text{is surjective.} \end{cases}$$

Proof. Follows from (3.8) and (2.4). □

In a similar way, we can apply Theorem 4.2 for the Witt groups $W^r(\mathcal{D}_g^0(X))$ and $W^r(\mathcal{D}_g^1(X))$, as follows.

Corollary 5.6. *Let X be quasi-projective scheme over a noetherian affine scheme Spec(A), as in (4.2). Then,*

1. *We have the following:*

$$\begin{cases} W(\mathscr{A}(X, 0)) \xrightarrow{\sim} W^0(\mathcal{D}^0_{\mathbb{H}(X)}(X), g) \xrightarrow{\sim} W^0(\mathcal{D}_g^0(X)) \\ \quad \text{are isomorphisms.} \\ W_-(\mathscr{A}(X, 0)) \xrightarrow{\sim} W^2(\mathcal{D}^1_{\mathbb{H}(X),g}(X)) \xrightarrow{\sim} W^2(\mathcal{D}_g^0(X)) \\ \quad \text{are isomorphisms.} \\ W^0(\mathcal{D}_g^1(X)) = 0 \quad \text{and} \\ W^0(\mathcal{D}_g^3(X)) = 0. \end{cases}$$

2. *Also,*

$$\begin{cases} W(\mathscr{A}(X,1)) \twoheadrightarrow W^1(\mathscr{D}^1_g(X)) & \text{is surjective, and} \\ W_-(\mathscr{A}(X,1) \twoheadrightarrow W^3(\mathscr{D}^1_g(X)) & \text{is surjective.} \end{cases}$$

Proof. In fact $\mathscr{D}^0_g(X) = \mathcal{D}^b(\mathscr{V}(X))$. To see the first isomorphisms of (1), note that the composition of these two map is an isomorphism (([B3]). So, the first homomorphism $W(\mathscr{A}(X,0)) \xrightarrow{\sim} W^0(\mathscr{D}^0_{\mathbb{H}(X),g}(X))$ is injective. The surjectivity of this homomorphism follows from Corollary 4.3. So, it is an isomorphism. The second isomorphism is the skew duality version of the first one. The last two statements of (1) on vanishing follow from Theorem 4.2. The surjectivity of the first homomorphism of (2) follows from Theorem 4.2 and (2.4). The latter homomorphism is the skew duality version of the first one. □

Remark 5.7. The structure of the forms in $W^r(\mathscr{D}^k(X))$ is likely to have some use in the context of Gersten-Witt complexes (see [B4,BW,Bet]) of regular quasi-projective schemes over affine schemes $Spec(A)$. In particular, the first term of the exact sequence

$$0 \longrightarrow W^{d-1}(\mathscr{D}^{d-1}(X)) \longrightarrow W^{d-1}\left(\frac{\mathscr{D}^{d-1}(X)}{\mathscr{D}^d(X)}\right) \longrightarrow W^d(\mathscr{D}^{d-1}(X))$$

is given by the surjective map $W(\mathscr{A}(X,d-1)) \twoheadrightarrow W^{d-1}(\mathscr{D}^{d-1})$. If and when $W(\mathscr{A}(X,d-1))$ is generated by Koszul complexes, so is $W^{d-1}(\mathscr{D}^{d-1}(X))$. In this case, one can prove that $W^{d-1}(\mathscr{D}^{d-2}(X)) = 0$. This would suffice to establish the exactness of the Gersten-Witt complex at codimension $d-1$. Likewise, one can make a similar statement for codimension $d-2$. There are similar results in K-theory (see [Mo, 0.3]) in this direction.

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