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The growth of fine Selmer groups

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Abstract. Let A be an abelian variety defined over a number field F. In this paper, we will investigate the growth of the p-rank of the fine Selmer group in three situations. In particular, in each of these situations, we show that there is a strong analogy between the growth of the p-rank of the fine Selmer group and the growth of the p-rank of the class groups.

1. Introduction

In the study of rational points on Abelian varieties, the Selmer group plays an important role. In Mazur's fundamental work [Maz], the Iwasawa theory of Selmer groups was introduced. Using this theory, Mazur was able to describe the growth of the size of the p-primary part of the Selmer group in $\mathbb{Z}_p$-towers. Recently, several authors have initiated the study of a certain subgroup, called the fine Selmer group. This subgroup, as well as the 'fine' analogues of the Mordell-Weil group and Shafarevich-Tate group, seem to have stronger finiteness properties than the classical Selmer group (respectively, Mordell-Weil or Shafarevich-Tate groups). The fundamental paper of Coates and Sujatha [CS] explains some of these properties.

Let F be a number field and p an odd prime. Let A be an Abelian variety defined over F and let S be a finite set of primes of F including the infinite primes, the primes where A has bad reduction, and the primes of F over p.

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Fix an algebraic closure $\overline{F}$ of F and denote by F_S the maximal subfield of $\overline{F}$ containing F which is unramified outside S. We set $G = \mathrm{Gal}(\overline{F}/F)$ and $G_S = \mathrm{Gal}(F_S/F)$.

The usual p^∞-Selmer group of A is defined by

$$\mathrm{Sel}_{p^\infty}(A/F) = \ker\left(H^1(G, A[p^\infty]) \longrightarrow \bigoplus_v H^1(F_v, A)[p^\infty] \right).$$

Here, v runs through all the primes of F and as usual, for a G-module M, we write $H^*(F_v, M)$ for the Galois cohomology of the decomposition group at v. Following [CS], the p^∞-fine Selmer group of A is defined by

$$R_{p^\infty}(A/F) = \ker\left(H^1(G_S(F), A[p^\infty]) \longrightarrow \bigoplus_{v\in S} H^1(F_v, A[p^\infty]) \right).$$

This definition is in fact independent of the choice of S as can be seen from the exact sequence (Lemma 4.1)

$$0 \longrightarrow R_{p^\infty}(A/F) \longrightarrow \mathrm{Sel}_{p^\infty}(A/F) \longrightarrow \bigoplus_{v|p} A(F_v)\otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

Coates and Sujatha study this group over a field F_∞ contained in F_S and for which $\mathrm{Gal}(F_\infty/F)$ is a p-adic Lie group. They set

$$R_{p^\infty}(A/F_\infty) = \varinjlim R_{p^\infty}(A/L)$$

where the inductive limit ranges over finite extensions L of F contained in F_∞. When $F_\infty = F^{cyc}$ is the cyclotomic $\mathbb{Z}_p$-extension of F, they conjecture that the Pontryagin dual $Y_{p^\infty}(A/F_\infty)$ is a finitely generated $\mathbb{Z}_p$-module. This is known *not* to be true for the dual of the classical Selmer group. A concrete example of such is the elliptic curve $E/\mathbb{Q}$ of conductor 11 which is given by

$$y^2 + y = x^3 - x^2 - 10x - 20.$$

For the prime $p = 5$, it is known that the Pontryagin dual of the 5^∞-Selmer group over $\mathbb{Q}^{\mathrm{cyc}}$ is not finitely generated over $\mathbb{Z}_5$ (see [Maz, §10, Example 2]). On the other hand it is expected to be true if the Selmer group is replaced by a module made out of class groups. Thus, in some sense, the fine Selmer group seems to approximate the class group. One of the themes of our paper is to give evidence of that by making the relationship precise in three instances.

Coates and Sujatha also study extensions for which $G_\infty = \mathrm{Gal}(F_\infty/F)$ is a p-adic Lie group of dimension larger than 1 containing F^{cyc}. They make a striking conjecture that the dual $Y_{p^\infty}(A/F_\infty)$ of the fine Selmer group is pseudo-null as a module over the Iwasawa algebra $\Lambda(G_\infty)$. While we have nothing to say about this conjecture, we investigate the growth of p-ranks of fine Selmer groups in some pro-p towers that are *not* p-adic analytic.

2. Outline of the paper

Throughout this paper, p will always denote an odd prime. In the first situation, we study the growth of the fine Selmer groups over certain $\mathbb{Z}_p$-extensions. It was first observed in [CS] that over a cyclotomic $\mathbb{Z}_p$-extension, the growth of the fine Selmer group of an abelian variety in a cyclotomic $\mathbb{Z}_p$-extension exhibits phenomena parallel to the growth of the p-part of the class groups over a cyclotomic $\mathbb{Z}_p$-extension. Subsequent papers [A,JhS,Lim] in this direction has further confirmed this observation. (Actually, in [JhS,Lim], they have also considered the variation of the fine Selmer group of a more general p-adic representation. In this article, we will only be concerned with the fine Selmer groups of abelian varieties.) In this paper, we will show that the growth of the p-rank of fine Selmer group of an abelian variety in a certain class of $\mathbb{Z}_p$-extension is determined by the growth of the p-rank of ideal class groups in the $\mathbb{Z}_p$-extension in question (see Theorem 5.1) and vice versa. We will also specialize our theorem to the cyclotomic $\mathbb{Z}_p$-extension to recover a theorem of Coates-Sujatha [CS, Theorem 3.4].

In the second situation, we investigate the growth of the fine Selmer groups over $\mathbb{Z}/p$-extensions of a fixed number field. We note that it follows from an application of the Grunwald-Wang theorem that the p-rank of the ideal class groups grows unboundedly in $\mathbb{Z}/p$-extensions of a fixed number field. Recently, many authors have made analogous studies in this direction replacing the ideal class group by the classical Selmer group of an abelian variety (see [Ba,Br,Ce,Mat2]). In this article, we investigate the analogous situation for the fine Selmer group of an abelian variety, and we show that the p-rank of the fine Selmer group of the abelian variety grows unboundedly in $\mathbb{Z}/p$-extensions of a fixed number field (see Theorem 6.1). Note that the fine Selmer group is a subgroup of the classical Selmer group, and therefore, our results will also recover some of their results.

In the last situation, we consider the growth of the fine Selmer group in an infinite unramified pro-p extensions. It is known that the p-rank of the class groups is unbounded in such tower under suitable assumptions. Our result will again show that we have the same phenomenon for the p-rank of fine Selmer groups (see Theorem 7.2). As above, our result will also imply some of the main results in [LM,Ma,MO], where analogous studies in this direction have been made for the classical Selmer group of an abelian variety.

3. p-rank

In this section, we record some basic results on Galois cohomology that will be used later. For an abelian group N, we define its p-rank to be the $\mathbb{Z}/p$-dimension of $N[p]$ which we denote by $r_p(N)$. If G is a pro-p group, we write $h_i(G) = r_p\big(H^i(G, \mathbb{Z}/p)\big)$. We now state the following lemma which gives an estimate of the p-rank of the first cohomology group.

Lemma 3.1. *Let G be a pro-p group, and let M be a discrete G-module which is cofinitely generated over $\mathbb{Z}_p$. If $h_1(G)$ is finite, then $r_p(H^1(G,M))$ is finite, and we have the following estimates for $r_p(H^1(G,M))$*

$$\begin{aligned} &h_1(G)r_p(M^G) - r_p((M/M^G)^G) \\ &\quad \le r_p(H^1(G,M)) \le h_1(G)(\mathrm{corank}_{\mathbb{Z}_p}(M) + \log_p(|M/M_{\mathrm{div}}|)). \end{aligned}$$

Proof. See [LM, Lemma 3.2]. □

We record another useful estimate.

Lemma 3.2. *Let*

$$W \longrightarrow X \longrightarrow Y \longrightarrow Z$$

be an exact sequence of cofinitely generated abelian groups. Then we have

$$|r_p(X) - r_p(Y)| \le 2r_p(W) + r_p(Z).$$

Proof. It suffices to show the lemma for the exact sequence

$$0 \longrightarrow W \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

We break up the exact sequence into two short exact sequences

$$0 \longrightarrow W \longrightarrow X \longrightarrow C \longrightarrow 0,$$
$$0 \longrightarrow C \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

From these short exact sequences, we obtain two exact sequences of finite dimensional $\mathbb{Z}/p$-vector spaces (since W, X, Y and Z are cofinitely generated abelian groups)

$$0 \longrightarrow W[p] \longrightarrow X[p] \longrightarrow C[p] \longrightarrow P \longrightarrow 0,$$
$$0 \longrightarrow C[p] \longrightarrow Y[p] \longrightarrow Q \longrightarrow 0,$$

where $P \subseteq W/p$ and $Q \subseteq Z[p]$. It follows from these two exact sequences and a straightforward calculation that we have

$$r_p(X) - r_p(Y) = r_p(W) - r_p(P) - r_p(Q).$$

The inequality of the lemma is immediate from this. □

4. Fine Selmer groups

As before, p will denote an odd prime. Let A be an abelian variety over a number field F. Let S be a finite set of primes of F which contains the primes above p, the primes of bad reduction of A and the archimedean primes. Denote by F_S the maximal algebraic extension of F unramified outside S. We will write $G_S(F) = \mathrm{Gal}(F_S/F)$.

As stated in the introduction and following [CS], the fine Selmer group of A is defined by

$$R(A/F) = \ker\left(H^1(G_S(F), A[p^\infty]) \longrightarrow \bigoplus_{v \in S} H^1(F_v, A[p^\infty]) \right).$$

(Note that we have dropped the subscript p^∞ on $R(A/F)$ as p is fixed.)

To facilitate further discussion, we also recall the definition of the classical Selmer group of A which is given by

$$\mathrm{Sel}_{p^\infty}(A/F) = \ker\left(H^1(F, A[p^\infty]) \longrightarrow \bigoplus_{v} H^1(F_v, A)[p^\infty] \right),$$

where v runs through all the primes of F. (Note the difference of the position of the "$[p^\infty]$" in the local cohomology groups in the definitions.)

At first viewing, it will seem that the definition of the fine Selmer group depends on the choice of the set S. We shall show that this is not the case.

Lemma 4.1. *We have an exact sequence*

$$0 \longrightarrow R(A/F) \longrightarrow \mathrm{Sel}_{p^\infty}(A/F) \longrightarrow \bigoplus_{v \mid p} A(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

In particular, the definition of the fine Selmer group does not depend on the choice of the set S.

Proof. Let S be a finite set of primes of F which contains the primes above p, the primes of bad reduction of A and the archimedean primes. Then by [Mi, Chap. I, Corollary 6.6], we have the following description of the Selmer group

$$\mathrm{Sel}_{p^\infty}(A/F) = \ker\left(H^1(G_S(F), A[p^\infty]) \longrightarrow \bigoplus_{v \in S} H^1(F_v, A)[p^\infty] \right).$$

Combining this description with the definition of the fine Selmer group and an easy diagram-chasing argument, we obtain the required exact sequence (noting that $A(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$ for $v \nmid p$). □

Remark 4.2. In [Wu], Wuthrich used the exact sequence in the lemma for the definition of the fine Selmer group.

We end the section with the following simple lemma which gives a lower bound for the p-rank of the fine Selmer group in terms of the p-rank of the S-class group. This will be used in Sections 6 and 7.

Lemma 4.3. *Let A be an abelian variety defined over a number field F. Suppose that $A(F)[p] \neq 0$. Then we have*

$$r_p(R(A/F)) \geq r_p(\mathrm{Cl}_S(F))r_p(A(F)[p]) - 2d,$$

where d denotes the dimension of the abelian variety A.

Proof. Let H_S be the p-Hilbert S-class field of F which, by definition, is the maximal abelian unramified p-extension of F in which all primes in S split completely. Consider the following diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & R(A/F) & \longrightarrow & H^1(G_S(F), A[p^\infty]) & \longrightarrow & \bigoplus_{v\in S} H^1(F_v, A[p^\infty]) \\
& \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
0 \longrightarrow & R(A/H_S) & \longrightarrow & H^1(G_S(H_S), A[p^\infty]) & \longrightarrow & \bigoplus_{v\in S}\bigoplus_{w|v} H^1(H_{S,w}, A[p^\infty])
\end{array}$$

with exact rows. Here the vertical maps are given by the restriction maps. Write $\gamma = \oplus_v \gamma_v$, where

$$\gamma_v : H^1(F_v, A[p^\infty]) \longrightarrow \bigoplus_{w|v} H^1(H_{S,w}, A[p^\infty]).$$

It follows from the inflation-restriction sequence that $\ker \gamma_v = H^1(G_v, A(H_{S,v})[p^\infty])$, where G_v is the decomposition group of $\mathrm{Gal}(H_S/F)$ at v. On the other hand, by the definition of H_S, all the primes of F in S split completely in H_S, and therefore, we have $G_v = 1$ which in turn implies that $\ker \gamma = 0$. Similarly, the inflation-restriction sequence gives the equality $\ker \beta = H^1(\mathrm{Gal}(H_S/F), A(H_S)[p^\infty])$. Therefore, we obtain an injection

$$H^1(\mathrm{Gal}(H_S/F), A(H_S)[p^\infty]) \hookrightarrow R(A/F)$$

It follows from this injection that we have

$$r_p(R(A/F)) \geq r_p(H^1(\mathrm{Gal}(H_S/F), A(H_S)[p^\infty])).$$

By Lemma 3.1, the latter quantity is greater or equal to

$$h_1(\mathrm{Gal}(H_S/F))r_p(A(F)[p^\infty]) - 2d.$$

By class field theory, we have $\mathrm{Gal}(H_S/F) \cong \mathrm{Cl}_S(F)$, and therefore,

$$h_1(\mathrm{Gal}(H_S/F)) = r_p(\mathrm{Cl}_S(F)/p) = r_p(\mathrm{Cl}_S(F)),$$

where the last equality follows from the fact that $\mathrm{Cl}_S(F)$ is finite. The required estimate is now established (and noting that $r_p(A(F)[p]) = r_p(A(F)[p^\infty])$). □

Remark 4.4. Since the fine Selmer group is contained in the classical Selmer group (cf. Lemma 4.1), the above estimate also gives a lower bound for the classical Selmer group.

5. Growth of fine Selmer groups in a $\mathbb{Z}_p$-extension

As before, p denotes an odd prime. In this section, F_∞ will always denote a fixed $\mathbb{Z}_p$-extension of F. We will denote F_n to be the subfield of F_∞ such that $[F_n : F] = p^n$. If S is a finite set of primes of F, we denote by S_f the set of finite primes in S.

We now state the main theorem of this section which compares the growth of the fine Selmer groups and the growth of the class groups in the $\mathbb{Z}_p$-extension of F. To simplify our discussion, we will assume that $A[p] \subseteq A(F)$.

Theorem 5.1. *Let A be a d-dimensional abelian variety defined over a number field F. Let F_∞ be a fixed $\mathbb{Z}_p$-extension of F such that the primes of F above p and the bad reduction primes of A decompose finitely in F_∞/F. Furthermore, we assume that $A[p] \subseteq A(F)$. Then we have*

$$|r_p(R(A/F_n)) - 2dr_p(\mathrm{Cl}(F_n))| = O(1).$$

In preparation for the proof of the theorem, we require a few lemmas.

Lemma 5.2. *Let F_∞ be a $\mathbb{Z}_p$-extension of F and let F_n be the subfield of F_∞ such that $[F_n : F] = p^n$. Let S be a given finite set of primes of F which contains all the primes above p and the archimedean primes. Suppose that all the primes in S_f decompose finitely in F_∞/F. Then we have*

$$|r_p(\mathrm{Cl}(F_n)) - r_p(\mathrm{Cl}_S(F_n))| = O(1).$$

Proof. For each F_n, we write $S_f(F_n)$ for the set of finite primes of F_n above S_f. For each n, we have the following exact sequence (cf. [NSW, Lemma 10.3.12])

$$\mathbb{Z}^{|S_f(F_n)|} \longrightarrow \mathrm{Cl}(F_n) \twoheadrightarrow \mathrm{Cl}_S(F_n) \longrightarrow 0.$$

Denote by C_n the kernel of $\mathrm{Cl}(F_n) \longrightarrow \mathrm{Cl}_S(F_n)$. Note that C_n is finite, since it is contained in $\mathrm{Cl}(F_n)$. Also, it is clear from the above exact sequence that $r_p(C_n) \leq |S_f(F_n)|$ and $r_p(C_n/p) \leq |S_f(F_n)|$. By Lemma 3.2, we have

$$|r_p(\mathrm{Cl}(F_n)) - r_p(\mathrm{Cl}_S(F_n))| \leq 3|S_f(F_n)| = O(1),$$

where the last equality follows from the assumption that all the primes in S_f decompose finitely in F_∞/F. □

Before stating the next lemma, we introduce the p-fine Selmer group of an abelian variety A. Let S be a finite set of primes of F which contains the primes above p, the primes of bad reduction of A and the archimedean primes. Then the p-fine Selmer group (with respect to S) is defined to be

$$R_S(A[p]/F) = \ker\left(H^1(G_S(F), A[p]) \longrightarrow \bigoplus_{v \in S} H^1(F_v, A[p]) \right).$$

Note that the p-fine Selmer group may be dependent on S. In fact, as we will see in the proof of Theorem 5.1 below, when $F = F(A[p])$, we have $R_S(A[p]/F) = \mathrm{Cl}_S(F)[p]^{2d}$, where the latter group is clearly dependent on S.

We can now state the following lemma which compare the growth of $r_p(R_S(A[p]/F_n))$ and $r_p(R(A/F_n))$.

Lemma 5.3. *Let F_∞ be a $\mathbb{Z}_p$-extension of F and let F_n be the subfield of F_∞ such that $[F_n : F] = p^n$. Let A be an abelian variety defined over F. Let S be a finite set of primes of F which contains the primes above p, the primes of bad reduction of A and the archimedean primes. Suppose that all the primes in S_f decompose finitely in F_∞/F. Then we have*

$$|r_p(R_S(A[p]/F_n)) - r_p(R(A/F_n))| = O(1).$$

Proof. We have a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & R_S(A[p]/F_n) & \longrightarrow & H^1(G_S(F_n), A[p]) & \longrightarrow & \bigoplus_{v_n \in S(F_n)} H^1(F_{n,v_n}, A[p]) \\
& \downarrow s_n & & \downarrow h_n & & \downarrow g_n \\
0 \longrightarrow & R(A/F_n)[p] & \longrightarrow & H^1(G_S(F_n), A[p^\infty])[p] & \longrightarrow & \bigoplus_{v_n \in S(F_n)} H^1(F_{n,v_n}, A[p^\infty])[p]
\end{array}$$

with exact rows. It is an easy exercise to show that the maps h_n and g_n are surjective, that $\ker h_n = A(F_n)[p^\infty]/p$ and that

$$\ker g_n = \bigoplus_{v_n \in S(F_n)} A(F_{n,v_n})[p^\infty]/p.$$

Since we are assuming p is odd, we have $r_p(\ker g_n) \le 2d|S_f(F_n)|$. By an application of Lemma 3.2, we have

$$\begin{aligned}|r_p(R_S(A[p]/F_n))) - r_p(R(A/F_n))| &\le 2r_p(\ker s_n) + r_p(\operatorname{coker} s_n)\\ &\le 2r_p(\ker h_n) + r_p(\ker g_n)\\ &\le 4d + 2d|S_f(F_n)| = O(1),\end{aligned}$$

where the last equality follows from the assumption that all the primes in S_f decompose finitely in F_∞/F. □

We are in the position to prove our theorem.

Proof of Theorem 5.1 Let S be the finite set of primes of F consisting precisely of the primes above p, the primes of bad reduction of A and the archimedean primes. By the hypothesis $A[p] \subseteq A(F)$ ($\subseteq A(F_n)$) of the theorem, we have $A[p] \cong (\mathbb{Z}/p)^{2d}$ as $G_S(F_n)$-modules. Therefore, we have $H^1(G_S(F_n), A[p]) = \operatorname{Hom}(G_S(F_n), A[p])$. We have similar identification for the local cohomology groups, and it follows that

$$R_S(A[p]/F_n) = \operatorname{Hom}(\mathrm{Cl}_S(F_n), A[p]) \cong \mathrm{Cl}_S(F_n)[p]^{2d}$$

as abelian groups. Hence we have $r_p(R_S(A[p]/F_n)) = 2dr_p(\mathrm{Cl}_S(F_n))$. The conclusion of the theorem is now immediate from this equality and the above two lemmas. □

Corollary 5.4. *Retain the notations and assumptions of Theorem 5.1. Then we have*

$$r_p(R(A/F_n)) = O(1)$$

if and only if

$$r_p(\mathrm{Cl}(F_n)) = O(1).$$

For the remainder of the section, F_∞ will be taken to be the cyclotomic $\mathbb{Z}_p$-extension of F. As before, we denote by F_n the subfield of F_∞ such that $[F_n : F] = p^n$. Denote by X_∞ the Galois group of the maximal abelian unramified pro-p extension of F_∞ over F_∞. A well-known conjecture of Iwasawa asserts that X_∞ is finitely generated over $\mathbb{Z}_p$ (see [Iw1,Iw2]). We will call this conjecture the *Iwasawa μ-invariant conjecture* for F_∞. By [Wa, Proposition 13.23], this is also equivalent to saying that $r_p(\mathrm{Cl}(F_n)/p)$ is bounded independently of n. Now, by the finiteness of class groups, we have $r_p(\mathrm{Cl}(F_n)) = r_p(\mathrm{Cl}(F_n)/p)$. Hence the Iwasawa μ-invariant conjecture is equivalent to saying that $r_p(\mathrm{Cl}(F_n))$ is bounded independently of n.

We consider the analogous situation for the fine Selmer group. Define $R(A/F_\infty) = \varinjlim_n R(A/F_n)$ and denote by $Y(A/F_\infty)$ the Pontryagin dual of $R(A/F_\infty)$. We may now recall the following conjecture which was first introduced in [CS].

Conjecture A. For any number field F, $Y(A/F_\infty)$ is a finitely generated $\mathbb{Z}_p$-module, where F_∞ is the cyclotomic $\mathbb{Z}_p$-extension of F.

We can now give the proof of [CS, Theorem 3.4]. For another alternative approach, see [JhS,Lim].

Theorem 5.5. *Let A be a d-dimensional abelian variety defined over a number field F and let F_∞ be the cyclotomic $\mathbb{Z}_p$-extension of F. Suppose that $F(A[p])$ is a finite p-extension of F. Then Conjecture A holds for A over F_∞ if and only if the Iwasawa μ-invariant conjecture holds for F_∞.*

Proof. Now if L'/L is a finite p-extension, it follows from [Iw1, Theorem 3] that the Iwasawa μ-invariant conjecture holds for L_∞ if and only if the Iwasawa μ-invariant conjecture holds for L'_∞. On the other hand, it is not difficult to show that the map

$$Y(A/L'_\infty)_G \longrightarrow Y(A/L_\infty)$$

has finite kernel and cokernel, where $G = \mathrm{Gal}(L'/L)$. It follows from this observation that Conjecture A holds for A over L_∞ if and only if $Y(A/L'_\infty)_G$ is finitely generated over $\mathbb{Z}_p$. Since G is a p-group, $\mathbb{Z}_p[G]$ is local with a unique maximal (two-sided) ideal $p\mathbb{Z}_p[G] + I_G$, where I_G is the augmentation ideal (see [NSW, Proposition 5.2.16(iii)]). It is easy to see from this that

$$Y(A/L'_\infty)/\mathfrak{m} \cong Y(A/L'_\infty)_G/pY(A/L'_\infty)_G.$$

Therefore, Nakayama's lemma for $\mathbb{Z}_p$-modules tells us that $Y(A/L'_\infty)_G$ is finitely generated over $\mathbb{Z}_p$ if and only if $Y(A/L'_\infty)/\mathfrak{m}$ is finite. On the other hand, Nakayama's lemma for $\mathbb{Z}_p[G]$-modules tells us that $Y(A/L'_\infty)/\mathfrak{m}$ is finite if and only if $Y(A/L'_\infty)$ is finitely generated over $\mathbb{Z}_p[G]$. But since G is finite, the latter is equivalent to $Y(A/L'_\infty)$ being finitely generated over $\mathbb{Z}_p$. Hence we have shown that Conjecture A holds for A over L_∞ if and only if Conjecture A holds for A over L'_∞.

Therefore, replacing F by $F(A[p])$, we may assume that $A[p] \subseteq A(F)$. Write $\Gamma_n = \mathrm{Gal}(F_\infty/F_n)$. Consider the following commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & R(A/F_n) & \longrightarrow & H^1(G_S(F_n), A[p^\infty]) & \longrightarrow & \bigoplus_{v_n \in S(F_n)} H^1(F_{n,v_n}, A[p^\infty]) \\
& \downarrow r_n & & \downarrow f_n & & \downarrow \gamma_n \\
0 \longrightarrow & R(A/F_\infty)^{\Gamma_n} & \longrightarrow & H^1(G_S(F_\infty), A[p^\infty])^{\Gamma_n} & \longrightarrow & \left(\varinjlim_n \bigoplus_{v_n \in S(F_n)} H^1(F_{n,v_n}, A[p^\infty])\right)^{\Gamma_n}
\end{array}$$

with exact rows, and the vertical maps given by the restriction maps. It is an easy exercise to show that $r_p(\ker f_n) \leq 2d$, $r_p(\ker \gamma_n) \leq 2d|S_f(F_n)|$, and that f_n and γ_n are surjective. It then follows from these estimates and Lemma 3.2 that we have

$$|r_p(R(A/F_n))) - r_p(R(A/F_\infty)^{\Gamma_n})| = O(1).$$

Combining this observation with [Wa, Lemma 13.20], we have that Conjecture A holds for A over F_∞ if and only if $r_p(R(A/F_n)) = O(1)$. The conclusion of the theorem is now immediate from Corollary 5.4. □

6. Unboundedness of fine Selmer groups in $\mathbb{Z}/p$-extensions

In this section, we will study the question of unboundedness of fine Selmer groups in $\mathbb{Z}/p$-extensions. We first recall the case of class groups. Since for a number field L, the S-class group $\mathrm{Cl}_S(L)$ is finite, we have $r_p(\mathrm{Cl}_S(L)) = \dim_{\mathbb{Z}/p}(\mathrm{Cl}_S(L)/p)$.

Proposition 6.1. *Let S be a finite set of primes of F which contains all the the archimedean primes. Then there exists a sequence $\{L_n\}$ of distinct number fields such that each L_n is a $\mathbb{Z}/p$-extension of F and such that*

$$r_p(\mathrm{Cl}_S(L_n)) \geq n$$

for every $n \geq 1$.

Proof. Denote r_1 (resp. r_2) be the number of real places of F (resp. the number of the pairs of complex places of F). Let S_1 be a set of primes of F which contains S and such that

$$|S_1| \geq |S| + r_1 + r_2 + \delta + 1.$$

Here $\delta = 1$ if F contains a primitive p-root of unity, and 0 otherwise. By the theorem of Grunwald-Wang (cf. [NSW, Theorem 9.2.8]), there exists a $\mathbb{Z}/p$-extension L_1 of F such that L_1/F is ramified at all the finite primes of S_1 and unramified outside S_1. By [NSW, Proposition 10.10.3], we have

$$r_p(\mathrm{Cl}_S(L_1)) \geq |S_1| - |S| - r_1 - r_2 - \delta \geq 1.$$

Choose S_2 to be a set of primes of F which contains S_1 (and hence S_0) and which has the property that

$$|S_2| \geq |S_1| + 1 \geq |S| + r_1 + r_2 + \delta + 2.$$

By the theorem of Grunwald-Wang, there exists a $\mathbb{Z}/p$-extension L_2 of F such that L_2/F is ramified at all the finite primes of S_2 and unramified outside S_2. In particular, the fields L_1 and L_2 are distinct. By an application of [NSW, Proposition 10.10.3] again, we have

$$r_p(\mathrm{Cl}_S(L_2)) \geq |S_2| - |S| - r_1 - r_2 - \delta \geq 2.$$

Note that since there are infinitely many primes in F, we can always continue the above process iteratively. Also, it is clear from our choice of L_n, they are mutually distinct. Therefore, we have the required conclusion. □

For completeness and for ease of later comparison, we record the following folklore result.

Theorem 6.2. *Let F be a number field. Then we have*

$$\sup\{r_p(\mathrm{Cl}(L)) | L/F \textit{ is a cyclic extension of degree } p\} = \infty$$

Proof. Since $\mathrm{Cl}(L)$ surjects onto $\mathrm{Cl}_S(L)$, the theorem follows from the preceding proposition. □

We now record the analogous statement for the fine Selmer groups.

Theorem 6.3. *Let A be an abelian variety defined over a number field F. Suppose that $A(F)[p] \neq 0$. Then we have*

$$\sup\{r_p(R(A/L)) | L/F \textit{ is a cyclic extension of degree } p\} = \infty$$

Proof. This follows immediately from combining Lemma 4.3 and Proposition 6.1. □

In the case, when $A(F)[p] = 0$, we have the following weaker statement.

Corollary 6.4. *Let A be a d-dimensional abelian variety defined over a number field F. Suppose that $A(F)[p] = 0$. Define*

$$m = \min\{[K : F] | A(K)[p] \neq 0\}.$$

Then we have

$$\sup\{r_p(R(A/L)) | L/F \textit{ is an extension of degree } pm\} = \infty$$

Proof. This follows from an application of the previous theorem to the field K. □

Remark 6.5. Clearly $1 < m \leq |\mathrm{GL}_{2d}(\mathbb{Z}/p)| = (p^{2d} - 1)(p^{2d} - p) \cdots (p^{2d} - p^{2d-1})$. In fact, we can even do better[1]. Write $G = \mathrm{Gal}(F(A[p])/F)$. Note that this is a subgroup of $\mathrm{GL}_{2d}(\mathbb{Z}/p)$. Let P be a nontrivial point in $A[p]$ and denote by H the subgroup of G which fixes P. Set $K = F(A[p])^H$. It is easy to see that $[K : F] = [G : H] = |O_G(P)|$, where $O_G(P)$ is the orbit of P under the action of G. Since $O_G(P)$ is contained in $A[p]\backslash\{0\}$, we have $m \leq [K : F] = |O_G(P)| \leq p^{2d} - 1$.

[1] We thank Christian Wuthrich for pointing this out to us.

As mentioned in the introductory section, analogous result to the above theorem for the classical Selmer groups have been studied (see [Ba,Br,Ce,K,KS,Mat1,Mat2]). Since the fine Selmer group is contained in the classical Selmer group (cf. Lemma 4.1), our result recovers the above mentioned results (under our hypothesis). We note that the work of [Ce] also considered the cases of a global field of positive characteristic. We should also mention that in [ClS,Cr], they have even established the unboundness of $Ш(A/L)$ over $\mathbb{Z}/p$-extensions L of F in certain cases (see also [K,Mat2] for some other related results in this direction). In view of these results on $Ш(A/L)$, one may ask for analogous results for a 'fine' Shafarevich-Tate group.

Wuthrich [Wu] introduces such a group as follows. One first defines a 'fine' Mordell-Weil group $M_{p^\infty}(A/L)$ by the exact sequence

$$0 \longrightarrow M_{p^\infty}(A/L) \longrightarrow A(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \bigoplus_{v|p} A(L_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

Then, the 'fine' Shafarevich-Tate group is defined by the exact sequence

$$0 \longrightarrow M_{p^\infty}(A/L) \longrightarrow R_{p^\infty}(A/L) \longrightarrow Ж_{p^\infty}(A/L) \longrightarrow 0.$$

In fact, it is not difficult to show that $Ж(A/L)$ is contained in the (p-primary) classical Shafarevich-Tate group (see loc. cit.). One may therefore think of $Ж(A/L)$ as the 'Shafarevich-Tate part' of the fine Selmer group.

With this definition in hand, one is naturally led to the following question for which we do not have an answer at present.

Question. *Retaining the assumptions of Theorem 6.3, do we also have*

$$\sup\{r_p(Ж_{p^\infty}(A/L)) \mid L/F \textit{ is a cyclic extension of degree } p\} = \infty?$$

7. Growth of fine Selmer groups in infinite unramified pro-p extensions

We introduce an interesting class of infinite unramified extensions of F. Let S be a finite set (possibly empty) of primes in F. As before, we denote the S-ideal class group of F by $\mathrm{Cl}_S(F)$. For the remainder of the section, F_∞ will denote the maximal unramified p-extension of F in which all primes in S split completely. Write $\Sigma = \Sigma_F = \mathrm{Gal}(F_\infty/F)$, and let $\{\Sigma_n\}$ be the derived series of Σ. For each n, the fixed field F_{n+1} corresponding to Σ_{n+1} is the p-Hilbert S-class field of F_n.

Denote by S_∞ the collection of infinite primes of F, and define δ to be 0 if $\mu_p \subseteq F$ and 1 otherwise. Let $r_1(F)$ and $r_2(F)$ denote the number of real and complex places of F respectively. It is known that if the inequality

$$r_p(\mathrm{Cl}_S(F)) \geq 2 + 2\sqrt{r_1(F) + r_2(F) + \delta + |S \backslash S_\infty|}$$

holds, then Σ is infinite (see [GS], and also [NSW, Chap. X, Theorem 10.10.5]). Stark posed the question on whether $r_p(\mathrm{Cl}_S(F_n))$ tends to infinity in an infinite p-class field tower as n tends to infinity. By class field theory, we have $r_p(\mathrm{Cl}_S(F_n)) = h_1(\Sigma_n)$. It then follows from the theorem of Lubotzsky and Mann [LuM] that Stark's question is equivalent to whether the group Σ is p-adic analytic. By the following conjecture of Fontaine-Mazur [FM], one does not expect Σ to be an analytic group if it is infinite.

Conjecture. (Fontaine-Mazur). For any number field F, the group Σ_F has no infinite p-adic analytic quotient.

Without assuming the Fontaine-Mazur Conjecture, we have the following unconditional (weaker) result, proven by various authors.

Theorem 7.1. *Let F be a number field. If the following inequality*

$$r_p(\mathrm{Cl}_S(F)) \geq 2 + 2\sqrt{r_1(F) + r_2(F) + \delta + |S \backslash S_\infty|}$$

holds, then the group Σ_F is not p-adic analytic.

Proof. When S is the empty set, this theorem has been proved independently by Boston [B] and Hajir [Ha]. For a general nonempty S, this is proved in [Ma, Lemma 2.3]. □

Collecting all the information we have, we obtain the following result which answers an analogue of Stark's question, namely the growth of the p-rank of the fine Selmer groups.

Theorem 7.2. *Let A be an Abelian variety of dimension d defined over F and let S be a finite set of primes which contains the primes above p, the primes of bad reduction of A and the archimedean primes. Let F_∞ be the maximal unramified p-extension of F in which all primes of the given set S split completely, and let F_n be defined as above. Suppose that*

$$r_p(\mathrm{Cl}_S(F)) \geq 2 + 2\sqrt{r_1(F) + r_2(F) + \delta + |S \backslash S_\infty|}$$

holds, and suppose that $A(F)[p] \neq 0$. Then the p-rank of $R(A/F_n)$ is unbounded as n tends to infinity.

Proof. By Lemma 4.3, we have

$$r_p(R(A/F_n)) \geq r_p(\mathrm{Cl}_S(F_n)) r_p(A(F)[p]) - 2d.$$

Now by the hypothesis of the theorem, it follows from Theorem 7.1 that Σ_F is not p-adic analytic. By the theorem of Lubotzsky and Mann [LuM], this in turn implies that $r_p(\mathrm{Cl}_S(F_n))$ is unbounded as n tends to infinity. Hence we also have that $r_p(R(A/F_n))$ is unbounded as n tends to infinity (note here we also make use of the fact that $r_p(A(F)[p]) \neq 0$ which comes from the hypothesis that $A(F)[p] \neq 0$). □

Remark 7.3.

(1) The analogue of the above result for the classical Selmer group has been established in [LM,Ma]. In particular, our result here refines (and implies) those proved there.

(2) Let A be an abelian variety defined over F with complex multiplication by K, and suppose that $K \subseteq F$. Let $\mathfrak{p}$ be a prime ideal of K above p. Then one can define a $\mathfrak{p}$-version of the fine Selmer group replacing $A[p^\infty]$ by $A[\mathfrak{p}^\infty]$ in the definition of the fine Selmer group. The above arguments carry over to establish the fine version of the results in [MO].

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