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z-classes and rational conjugacy classes in alternating groups

Sushil Bhunia, Dilpreet Kaur and Anupam Singh

IISER Pune, Dr. Homi Bhabha Road, Pashan, Pune 411008, India
e-mail: sushilbhunia@gmail.com; dilpreetmaths@gmail.com; anupamk18@gmail.com

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Abstract. In this paper, we compute the number of z-classes (conjugacy classes of centralizers of elements) in the symmetric group S_n, when $n \geq 3$ and alternating group A_n when $n \geq 4$. It turns out that the difference between the number of conjugacy classes and the number of z-classes for S_n is determined by those restricted partitions of $n - 2$ in which 1 and 2 do not appear as its part. In the case of alternating groups, it is determined by those restricted partitions of $n - 3$ which has all its parts distinct, odd and in which 1 (and 2) does not appear as its part, along with an error term. The error term is given by those partitions of n which have distinct parts that are odd and perfect squares. Further, we prove that the number of rational-valued irreducible complex characters for A_n is same as the number of conjugacy classes which are rational.

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1. Introduction

Let G be a group. Two elements $x, y \in G$ are said to be z-conjugate if their centralizers $\mathcal{Z}_G(x)$ and $\mathcal{Z}_G(y)$ are conjugate in G. This defines an equivalence relation on G and the equivalence classes are called z-classes. Clearly if x and y are conjugate then they are also z-conjugate. Thus, in general, z-conjugacy is a weaker relation than conjugacy on G. In the theory of groups of Lie type, this is also called "types" (see [Gr]) and the number of z-classes of semisimple elements is called the genus number (see [Ca1,Ca2]). This has been studied explicitly for various groups of Lie type in several papers, see for example, [BS,Go,GK,Ku,Si]. In this work, we want to classify and count the number of z-classes for symmetric and alternating groups. For convenience we deal with these groups when they are non-commutative (the commutative cases can be easily calculated), i.e., we assume $n \geq 3$ while dealing with symmetric groups and $n \geq 4$ while dealing with alternating groups.

Let $\sigma \in S_n$. The conjugacy classes of elements in S_n are determined by their cycle structure which, in turn, is determined by a partition of n. Let $\lambda = \lambda_1^{e_1}\lambda_2^{e_2}\ldots\lambda_r^{e_r}$ be a partition of n, i.e., we have $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_r \leq n$, each $e_i > 0$ and $n = \sum_{i=1}^{r} \lambda_i e_i$. We may represent an element of S_n corresponding to a partition λ in cycle notation. We prove the following,

Theorem 1.1. *Suppose $n \geq 3$. Let ν be a restricted partition of $n - 2$ in which 1 and 2 do not appear as its part. Let $\lambda = 1^2\nu$ and $\mu = 2^1\nu$ be partitions of n obtained by extending ν. Then the conjugacy classes of λ and μ belong to the same z-class in S_n.*

Further, the converse is also true, i.e., the conjugacy class corresponding to $\lambda = \lambda_1^{e_1}\lambda_2^{e_2}\ldots\lambda_r^{e_r}$ is z-equivalent to another conjugacy class then $\lambda_1^{e_1} = 1^2$, and, in that case the other class corresponds to $2^1\lambda_2^{e_2}\ldots\lambda_r^{e_r}$.

Corollary 1.2. *The number of z-classes in S_n is $p(n) - \tilde{p}(n-2)$, where $p(n)$ is the number of partitions of n and $\tilde{p}(n-2)$ is the number of partitions of $n-2$ in which 1 and 2 do not appear as its part. Thus, the number of z-classes in S_n is equal to $p(n) - p(n-2) + p(n-3) + p(n-4) - p(n-5)$.*

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To prove this theorem, we need to understand the centralizers better which involves the generalised symmetric group. A group $S(a,b) = C_a \wr S_b \cong C_a^b \rtimes S_b$ is called a generalised symmetric group. We will briefly introduce this group in the following section. We remark that the centralizers could be isomorphic but not conjugate. For example, in S_6 the centralizers of $1^1 2^1 3^1$ and 6^1 are isomorphic but not conjugate.

Next we look at the problem of classifying z-classes in alternating groups A_n. Usually the conjugacy classes in A_n are studied as a restriction of that of S_n. First, it is easy to determine for what partitions $\lambda = \lambda_1^{e_1} \dots \lambda_r^{e_r}$ of n the corresponding element σ_λ is in A_n. This is precisely when $n - \sum e_i$ is even. We call such **partitions even**. Further, when $\sigma_\lambda \in A_n$, the conjugacy class of σ_λ in S_n splits in two conjugacy classes in A_n if and only if $\mathcal{Z}_{S_n}(\sigma_\lambda) = \mathcal{Z}_{A_n}(\sigma_\lambda)$, which is, if and only if the partition λ has all its parts distinct and odd, i.e., $e_i = 1$ and λ_i odd for all i. With this notation we have,

Theorem 1.3. *Suppose $n \geq 4$. Let $\lambda = \lambda_1^{e_1} \dots \lambda_r^{e_r}$ be an even partition of n. Then the following determines z-classes in A_n.*

(1) *Suppose $e_i = 1$ for all i and all λ_i are odd, i.e., λ corresponds to two distinct conjugacy classes in A_n. Then, λ corresponds to two distinct z-classes (corresponding to the two distinct conjugacy classes) if and only if all λ_i are square. Else, the two split conjugacy classes form a single z-class.*

(2) *Suppose either one of the $e_i \geq 2$ or at least one of the λ_i is even, i.e., λ corresponds to a unique conjugacy class in A_n. Then, λ is z-equivalent to another conjugacy class if and only if $\lambda = 1^3\nu$, where ν is a restricted partition of $n-3$, with all its parts distinct and odd, and in which 1 (and 2) does not appear as its part. Further the other equivalent class is $3^1\nu$.*

We remark that $3^1\nu$ could be of the first kind. For example, in A_8 the partitions $1^3 5^1$ and $3^1 5^1$ give same z-class. Further, the conjugacy class $3^1 5^1$ splits into two but both fall in a single z-class. We list few more examples (using GAP) in a table in Section 9. We also note that ν could have its first part 3, in that case while writing $3^1\nu$ we appropriately absorb the power of 3. We denote by $\epsilon(n)$, the number of partitions of n with all of its parts distinct, odd and square. We list the values of $\epsilon(n)$ for small values in a table in Section 9.

Corollary 1.4. *The number of z-classes in A_n is*

$$cl(A_n) - (q(n) + \tilde{q}(n-3)) + \epsilon(n),$$

where $cl(A_n) = \frac{p(n)+3q(n)}{2}$ is the number of conjugacy classes in A_n, $q(n)$ is the number of partitions of n which has all parts distinct and odd, $\tilde{q}(m)$ is the number of restricted partitions of m, with all parts distinct, odd and which do not have 1 (and 2) as its part.

Let G be a finite group. An element $g \in G$ is called rational if g is conjugate to g^m for all m with property $(m, o(g)) = 1$ where $o(g)$ is the order of g. Clearly if g is rational then all of its conjugates are rational. Thus a conjugacy class of G is said to be rational if it is a conjugacy class of a rational element. It is believed that, for a finite group G, the number of conjugacy classes which are rational is related to the number of rational-valued complex irreducible characters of the group G (for example, see Theorem A in [NT]). A group of which all elements are rational (and in that case, all complex irreducible characters are rational-valued) is called a rational group or $\mathbb{Q}$-group (see [Kl]). The alternating groups A_n play an important role in determining simple groups which are rational (see Theorem A [FS]). There is a related notion of rational class in a group which comes from an equivalence relation. For a finite group G, a rational class of an element g is a subset containing all elements of G that are conjugate to g^m, where $(m, o(g)) = 1$. Thus the rational class of g can be thought of as the conjugacy class of cyclic subgroup $\langle g \rangle$ of G. A conjugacy class which is rational is a rational class. However the converse need not be true. It is well known that, for a finite group G, the number of isomorphism classes of irreducible representations of G over $\mathbb{Q}$ is equal to the number of rational classes of G (see Corollary 1, Section 13.1 [Se]). The symmetric group S_n is rational. Alternating groups are not rational (see Corollary B.1 [FS,AO]). The rational-valued complex irreducible characters for A_n are discussed in [Br] and [Pr]. In this paper we determine conjugacy classes which are rational and the rational classes in alternating group. With notation as above,

Theorem 1.5. *Suppose $n \geq 4$. Let $\tilde{C}$ be a conjugacy class in A_n and corresponding partition be $\lambda = \lambda_1^{e_1} \dots \lambda_r^{e_r}$ of n.*

(1) *Then the conjugacy class $\tilde{C}$ is rational in A_n if and only if one of the following happens:*
   (a) *either one of the $e_i \geq 2$ or one of the λ_i is even, or,*
   (b) *all λ_i are distinct (i.e., $e_i = 1$ for all i) and odd, and the product $\prod_{i=1}^{r} \lambda_i$ is a perfect square. In this case, λ corresponds to two conjugacy classes in A_n and both are simultaneously rational (or non-rational).*

(2) *All conjugacy classes which are rational are rational classes. When $\tilde{C}$ is not a rational conjugacy class in A_n, the conjugacy class C in S_n containing $\tilde{C}$ is a rational class in A_n.*

We denote by $\delta(n)$, the number of partitions of n with all parts distinct, odd and the product of parts is a perfect square. We list the values of $\delta(n)$ for small values in a table in Section 9 which is also there in [Br].

Corollary 1.6. *For the alternating group A_n with $n \geq 4$,*

(1) *the number of conjugacy classes which are rational is $cl(A_n) - 2q(n) + 2\delta(n)$, and*
(2) *the number of rational classes is $cl(A_n) - q(n) + \delta(n)$.*

The character theory of A_n is well understood. We use the notation and results from [Pr] and conclude the following,

Theorem 1.7. *Suppose $n \geq 4$. Then, the number of conjugacy classes in A_n which are rational is same as the number of rational-valued complex irreducible characters.*

This theorem is proved in Section 8. We also acknowledge that we have used GAP [GAP] on several occasions to verify our computations and results.

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2. Restricted partitions

We require certain kind of restricted partitions which we introduce in this section. We denote by $p(m)$, the number of partitions of positive integer m. To set the notation clearly, a partition of m is $\lambda = m_1^{e_1} \ldots m_r^{e_r}$ where $1 \leq m_1 < \cdots < m_r \leq m$, $e_i \geq 1 \forall i$ and $m = \sum_{i=1}^{r} e_i m_i$. Sometimes this is also denoted as $\lambda \vdash m$ or $m_1^{e_1} \ldots m_r^{e_r} \vdash m$. We clarify that the partition written as $1^1 2^1$ is same as 1.2 but, in this case, latter notation is confusing if written without a dot. For us the significance of partitions is due to its one-one correspondence with conjugacy classes of the symmetric group S_m. Let $\tilde{p}(m)$ be the number of those partitions of m in which 1 and 2 do not appear as its part, i.e.,

$$\tilde{p}(m) = |\{\lambda = m_1^{e_1} \ldots m_r^{e_r} \vdash m \mid m_1 \geq 3\}|.$$

Here we list down values of $\tilde{p}(m)$ for some small values.

m	$\tilde{p}(m)$	m	$\tilde{p}(m)$	m	$\tilde{p}(m)$	m	$\tilde{p}(m)$
1	0	6	2	11	6	16	21
2	0	7	2	12	9	17	25
3	1	8	3	13	10	18	33
4	1	9	4	14	13	19	39
5	1	10	5	15	17	20	49

The generating function for $\tilde{p}(m)$ is

$$\prod_{i\geq 3} \frac{1}{1-x^i}$$

and a formula to compute $\tilde{p}(m)$ in terms of partition function is

$$\tilde{p}(m) = p(m) - p(m-1) - p(m-2) + p(m-3).$$

This is a well known sequence in OEIS database (see [OEIS]). This will be used in the study of z-classes of symmetric groups later.

Now we introduce the function $q(m)$. For a given integer m, the value of $q(m)$ is the number of those partitions of m which have all of its parts distinct and odd, i.e.,

$$q(m) = |\{\lambda = m_1^1 \dots m_r^1 \vdash m \mid m_i \text{ odd } \forall i\}|.$$

This number is same as the number of self-conjugate partitions. For us this would correspond to those partitions which give split conjugacy classes in A_n. Now we introduce $\tilde{q}(m)$ which is the number of partitions of m which have all its parts distinct, odd and 1 (and 2) does not appear as its part. The following table gives values of $\tilde{q}(m)$ for some values of m.

m	$q(m)$	$\tilde{q}(m)$	m	$q(m)$	$\tilde{q}(m)$	m	$q(m)$	$\tilde{q}(m)$	m	$q(m)$	$\tilde{q}(m)$
1	1	0	6	1	0	11	2	1	16	5	3
2	0	0	7	1	1	12	3	2	17	5	2
3	1	1	8	2	1	13	3	1	18	5	3
4	1	0	9	2	1	14	3	2	19	6	3
5	1	1	10	2	1	15	4	2	20	7	4

The generating function for $q(m)$ is $\prod_{i\geq 0}(1+x^{2i+1})$ and the generating function for $\tilde{q}(m)$ is $\prod_{i\geq 1}(1+x^{2i+1})$.

3. Symmetric groups

In this section we classify z-classes in S_n. Since the centralizers are a product of generalised symmetric groups, we begin with a brief introduction to them.

3.1 Generalised symmetric groups

The group $S(a,b) = C_a \wr S_b$, where C_a is a cyclic group and S_b is a symmetric group, is called a generalised symmetric group. This group is an example of wreath product and has been studied well in literature. Since the centralizer subgroups in the symmetric group are a product of generalised symmetric groups, we need to have more information about this group. For the sake of clarity, let us begin with defining this group. Consider the action of symmetric group S_b on the direct product $C_a^b = C_a \times \cdots \times C_a$ given by permuting the components:

$$\sigma(x_1, \dots, x_b) = (x_{\sigma(1)}, \dots, x_{\sigma(b)}).$$

Then the generalised symmetric group is $S(a,b) = C_a \wr S_b := C_a^b \rtimes S_b$. Hence the multiplication in this group is given as follows:

$$(x_1, \dots, x_b, \sigma)(y_1, \dots, y_b, \tau) = (x_1 y_{\sigma^{-1}(1)}, \dots, x_b y_{\sigma^{-1}(b)}, \sigma\tau).$$

This group has a monomial matrix (each row and each column has exactly one non-zero entry) representation where C_a^b is embedded in the diagonal matrices and the whole group is a subgroup of $GL_b(\mathbb{C})$, in particular as a subgroup of monomial group. Monomial group is well known in the study of $GL_b(\mathbb{C})$ as an algebraic group. This gives rise to the Weyl group and Bruhat decomposition. Let T be the diagonal maximal torus (set of all diagonal matrices), then the monomial group is the normaliser $N_{GL_b(\mathbb{C})}(T)$. The Weyl group is defined as $W = N_{GL_b(\mathbb{C})}(T)/T \cong S_b$.

Let D be the set of those diagonal matrices in $GL_b(\mathbb{C})$ of which each diagonal entry is an ath roots of unity, i.e., each diagonal entry is from the set $\{\zeta^i \mid 0 \leq i \leq a-1\}$ where ζ is an ath primitive root of unity. Assume $b > a$, then, $D \cong C_a^b$ and the group $S(a,b) \cong N_{GL_b(\mathbb{C})}(D)$. Thus, $S(a,b)$ is the set of those monomial matrices which have non-zero entries coming from ath roots of unity. The following can be easily verified:

(1) the center $\mathcal{Z}(S(a,b)) = \{\lambda.Id \mid \lambda^a = 1\} \cong C_a$ if $a \geq 2$ or $b \geq 3$.
(2) $N_{GL_b(\mathbb{C})}(D)/D \cong S_b$.

Representation theory of the generalised symmetric group has been studied by Osima [Os], Can [Ca], Mishra and Srinivasan [MS], just to mention a few.

3.2 z-classes in S_n

In this section we aim to prove Theorem 1.1. For $n = 3$ and 4 the conjugacy classes and z-classes are same. Thus, if necessary, we may assume $n \geq 5$ in this section. Let $\lambda = \lambda_1^{e_1}\lambda_2^{e_2}\ldots\lambda_r^{e_r}$ be a partition of n. Let us denote the partial sums as $n_i = \sum_{j=1}^{i}\lambda_j e_j$ and $n_0 = 0$. We may represent an element of S_n corresponding to λ as a product of cycles and we choose a representative of class denoted as $\sigma_\lambda = \sigma_{\lambda_1}\ldots\sigma_{\lambda_i}\ldots\sigma_{\lambda_r}$ where

$$\sigma_{\lambda_i} = \underbrace{(n_{i-1}+1,\ldots,n_{i-1}+\lambda_i)\ldots(n_{i-1}+(e_i-1)\lambda_i+1,\ldots,n_{i-1}+e_i\lambda_i)}_{e_i}$$

is a product of e_i many disjoint cycles, each of length λ_i. Then the centralizer of this element is (see [JK] Equation 4.1.19)

$$\mathcal{Z}_{S_n}(\lambda) := \mathcal{Z}_{S_n}(\sigma_\lambda) \cong \prod_{i=1}^{r} C_{\lambda_i} \wr S_{e_i},$$

where C_{λ_i} is a cyclic group of size λ_i and the size of the centralizer is given by the formula $|\mathcal{Z}_{S_n}(\lambda)| = \prod_{i=1}^{r}(\lambda_i^{e_i}.e_i!)$. Further, with the above chosen representative element the center of $\mathcal{Z}_{S_n}(\sigma_\lambda)$ is,

$$Z_\lambda = \mathcal{Z}(\mathcal{Z}_{S_n}(\sigma_\lambda)) = \begin{cases} \prod_{i=1}^{r}\langle\sigma_{\lambda_i}\rangle & \text{if } \lambda_1^{e_1} \neq 1^2 \\ \langle(1,2)\rangle \times \prod_{i=2}^{r}\langle\sigma_{\lambda_i}\rangle & \text{when } \lambda_1^{e_1} = 1^2. \end{cases}$$

Note that if $\lambda_1 = 1$ then the element $\sigma_{\lambda_1} = 1$.

Lemma 3.1. *Let $\lambda = \lambda_1^{e_1}\lambda_2^{e_2}\ldots\lambda_r^{e_r}$ be a partition of n. Then $\mathcal{Z}_{S_n}(\lambda)$ determines r uniquely.*

Proof. Consider the natural action of $G = \mathcal{Z}_{S_n}(\lambda)$ on the set $\{1,2,\ldots,n\}$ as a subgroup of S_n. Since $G \cong \prod_{i=1}^{r} C_{\lambda_i} \wr S_{e_i}$, the orbits are $\{\{1,\ldots,n_1\},\{n_1+1,\ldots,n_2\},\ldots\}$. The number of orbits is exactly r. □

Lemma 3.2. *Let $\lambda = \lambda_1^{e_1}\lambda_2^{e_2}\ldots\lambda_r^{e_r}$ be a partition of n and $\lambda_1^{e_1} \neq 1^2$. Let Z_λ be the center of $\mathcal{Z}_{S_n}(\lambda)$. Then Z_λ determines the partition λ uniquely.*

Proof. Let us make Z_λ act on the set $\{1,2,\ldots,n\}$. Then the orbits are of size λ_i and each of them occur e_i many times. This determines the partition λ. □

Proposition 3.3. *Let $\lambda = \lambda_1^{e_1}\lambda_2^{e_2}\dots\lambda_r^{e_r}$ and $\mu = \mu_1^{f_1}\mu_2^{f_2}\dots\mu_s^{f_s}$ be partitions of n. Then $\mathcal{Z}_{S_n}(\lambda)$ is conjugate to $\mathcal{Z}_{S_n}(\mu)$ if and only if*

(1) $r = s$,
(2) *for all $i \geq 2$, λ_i and μ_i are ≥ 3 and $\lambda_i^{e_i} = \mu_i^{f_i}$,*
(3) $\lambda_1^{e_1} = 1^2$ *and* $\mu_1^{f_1} = 2^1$ *or vice versa.*

Proof. Clearly if the three conditions are given we have $\lambda = 1^2\nu$ and $\mu = 2^1\nu$ where $\nu = \nu_1^{l_1}\dots\nu_k^{l_k}$ is a partition of $n-2$ with $\nu_1 > 2$. Thus the representative elements of the conjugacy classes are $\sigma_\mu = (12)\sigma_\lambda$ and σ_λ, where σ_λ has cycles each of length > 2. Thus centralizers of these two elements are same.

For the converse, we choose representative elements σ_λ and σ_μ and we are given that $\mathcal{Z}_{S_n}(\sigma_\lambda)$ and $\mathcal{Z}_{S_n}(\sigma_\mu)$ are conjugate. The Lemma 3.1 implies that $r = s$. Now we take the center of both of these groups Z_λ and Z_μ and make it act on the set $\{1, 2, \dots, n\}$. If $\lambda_1^{e_1}$ and $\mu_1^{f_1}$ are not both 1^2 then from Lemma 3.2 we get the required result. □

This proves Theorem 1.1.

4. Rational conjugacy classes in A_n

The group A_n is of index 2 in S_n. Thus, we usually think of conjugacy classes in A_n in terms of that of S_n. We have two kinds of conjugacy classes in A_n. Let σ_λ be a representative of a conjugacy class, corresponding to a partition λ, of S_n. Suppose $\sigma_\lambda \in A_n$, that is to say, λ is an even partition. Then the two kinds of conjugacy classes are,

a. **Split:** The conjugacy class of σ_λ in S_n splits into two conjugacy classes in A_n if and only if all parts of λ are distinct and odd, which happens, if and only if $\mathcal{Z}_{S_n}(\sigma_\lambda) = \mathcal{Z}_{A_n}(\sigma_\lambda)$.
b. **Non-split:** The conjugacy class of σ_λ remains a single conjugacy class in A_n if and only if either one of the $e_i \geq 2$ for some i or at least one of the λ_i is even, which is, if and only if $\mathcal{Z}_{A_n}(\sigma_\lambda) \subsetneq \mathcal{Z}_{S_n}(\sigma_\lambda)$.

While writing proofs in this section and later sections, we consider these two cases separately.

Let G be a finite group and $g \in G$. The Weyl group of an element g in G, denoted as, $W_G(g) := N_G(\langle g\rangle)/\mathcal{Z}_G(\langle g\rangle)$ where $\langle g\rangle$ is the subgroup generated by g. Using the map $\iota\colon N_G(\langle g\rangle) \to \mathrm{Aut}(\langle g\rangle)$ given by $\iota(x)(g^r) = xg^rx^{-1}$, one can show that, the element g in G is rational if and only if $W_G(g) \cong \mathrm{Aut}(\langle g\rangle)$. We need to understand Weyl group of elements σ in A_n. Since S_n is a rational group, we have, the Weyl group $W_{S_n}(\sigma) \cong \mathrm{Aut}(\langle\sigma\rangle)$. Thus, to understand if σ is rational in A_n, we need to understand $N_{A_n}(\langle\sigma\rangle)$. This is determined by Brison (see Theorem 4.3 [Br]) as follows,

Theorem 4.1. *Let $\sigma \in A_n$ and corresponding partition be $\lambda = \lambda_1^{e_1}\dots\lambda_r^{e_r}$. Then, $N_{S_n}(\langle\sigma\rangle) = N_{A_n}(\langle\sigma\rangle)$ if and only if λ satisfies the following,*

(1) *all parts of λ are distinct, i.e., $e_i = 1$ for all i,*
(2) *λ_i is odd for all i, and*
(3) *the product of parts $\prod_{i=1}^r \lambda_i \in \mathbb{Z}$ is a perfect square.*

Corollary 4.2. *Suppose n is odd and $w = (1, 2, \dots, n)$ is in A_n. Then, w is rational in A_n if and only if n is a perfect square (of odd number).*

Proof. We know w is rational in S_n. Thus $W_{S_n}(w) \cong \mathrm{Aut}(\langle w\rangle)$. Since n is odd the conjugacy class of w in S_n splits in A_n and $\mathcal{Z}_{A_n}(w) = \mathcal{Z}_{S_n}(w)$. Thus w is rational in A_n if and only if $N_{S_n}(\langle w\rangle) = N_{A_n}(\langle w\rangle)$ which is if and only if n is a perfect square (from Theorem 4.1 above). □

Now we determine which conjugacy classes are rational in A_n.

When the conjugacy class does not split, $C = \sigma_\lambda^{S_n} = \sigma_\lambda^{A_n}$ and $\mathcal{Z}_{A_n}(\sigma_\lambda) \subsetneq \mathcal{Z}_{S_n}(\sigma_\lambda)$ is of index 2. Then,

Proposition 4.3. *Let C be a non-split conjugacy class in A_n. Then, C is rational in A_n.*

Proof. For this, we need to prove σ_λ is conjugate to σ_λ^m for all m which is coprime to the order of σ_λ. Since S_n is rational we have $g \in S_n$ such that $g\sigma_\lambda g^{-1} = \sigma_\lambda^m$. If g is in A_n we are done. Else take $h \in \mathcal{Z}_{S_n}(\sigma_\lambda)$ which is not in $\mathcal{Z}_{A_n}(\sigma_\lambda)$. Now $gh \in A_n$ and $gh\sigma_\lambda h^{-1}g^{-1} = \sigma_\lambda^m$, and we are done. □

When the conjugacy class splits, let C be the conjugacy class of σ_λ in S_n where $\lambda = \lambda_1^1 \dots \lambda_r^1$ with all λ_i odd (and distinct). Let C_1 and C_2 be the conjugacy classes in A_n, which are obtained by splitting C. Then,

Proposition 4.4. *With the notation as above, both A_n conjugacy classes C_1 and C_2 are rational if and only if $\prod_{i=1}^r \lambda_i$ is a perfect square.*

Proof. In this case, we have $\mathcal{Z}_{A_n}(\sigma_\lambda) = \mathcal{Z}_{S_n}(\sigma_\lambda)$. Thus $W_{A_n}(\sigma_\lambda) = W_{S_n}(\sigma_\lambda) \cong \mathrm{Aut}(\langle\sigma_\lambda\rangle)$ if and only if $N_{A_n}(\langle\sigma_\lambda\rangle) = N_{S_n}(\langle\sigma_\lambda\rangle)$. Which is determined by Theorem 4.1. □

We remark that either both conjugacy classes C_1 and C_2 are rational or not rational simultaneously.

Proposition 4.5. *With the notation as above, suppose both conjugacy classes C_1 and C_2 are not rational. Then the subset $C = C_1 \cup C_2$ is a rational class in A_n.*

Proof. This follows easily because C is a rational conjugacy class in S_n. □

Proof of Theorem 1.5 Let $\tilde{C}$ be a conjugacy class in A_n. Consider the conjugacy class C in S_n containing $\tilde{C}$. Let $\lambda = \lambda_1^{e_1} \dots \lambda_r^{e_r}$ be the corresponding partition of C. Then either $\tilde{C} = C$ or $C = C_1 \cup C_2$, where $\tilde{C}$ is one of the C_1 or C_2. If $\tilde{C} = C$ it follows from Proposition 4.3 that it is always rational and this corresponds to the partitions where either $e_i \geq 2$ for some i or one of the λ_i is even.

Now suppose $C = C_1 \cup C_2$, where $\tilde{C}$ is one of the components. Then from Proposition 4.4, it follows that both C_1 and C_2, and hence $\tilde{C}$, are rational if and only if $\prod_{i=1}^r \lambda_i$ is a square. That is, in this case the partition λ has all parts distinct, odd and the product of parts is a square.

When $\tilde{C}$ is not a rational conjugacy class, Proposition 4.5 implies C is a rational class in A_n. This completes the proof. □

5. z-classes in A_n – when the conjugacy class splits

Since the z-equivalence is a relation on conjugacy classes we deal with split and non-split classes separately. We begin with a few Lemmas.

Lemma 5.1. *Let x, y be elements in A_n such that x and y are conjugate in S_n. If there exists $g \in A_n$ such that g is conjugate to y in A_n and $\mathcal{Z}_{A_n}(g) = \mathcal{Z}_{A_n}(x)$ then centralizers $\mathcal{Z}_{A_n}(x)$ and $\mathcal{Z}_{A_n}(y)$ are conjugate in A_n.*

Proof. Since g and y are conjugate in A_n, their centralizers $\mathcal{Z}_{A_n}(g)$ and $\mathcal{Z}_{A_n}(y)$ are conjugate in A_n. Hence centralizers $\mathcal{Z}_{A_n}(x)$ and $\mathcal{Z}_{A_n}(y)$ are conjugate in A_n. □

The following Lemma establishes partial converse to the above.

Lemma 5.2. *Let $\lambda = \lambda_1 \dots \lambda_r$ be a partition with all parts (distinct and) odd. Let x and y be elements in A_n representing the two distinct conjugacy classes corresponding to λ. Suppose $x = x_1x_2\dots x_r$ and $y = y_1y_2\dots y_r \in A_n$, where x_i and y_i are cycles of length λ_i and centralizers $\mathcal{Z}_{A_n}(x)$ and $\mathcal{Z}_{A_n}(y)$ are conjugate in A_n. Then, y is conjugate to $x_1^{i_1}x_2^{i_2}\dots x_r^{i_r}$ in A_n for some positive integers $i_1, \dots, i_r$, where i_j is coprime to λ_j (which is the order of x_j) for all j.*

Proof. Since x and y are z-conjugate in A_n, i.e., there exist $g \in A_n$ such that $\mathcal{Z}_{A_n}(x) = g\mathcal{Z}_{A_n}(y)g^{-1} = \mathcal{Z}_{A_n}(gyg^{-1})$. Now, we know that $\mathcal{Z}_{A_n}(x) = \mathcal{Z}_{S_n}(x) = \langle x_1, x_2, \dots, x_r\rangle \cong C_{\lambda_1} \times \dots \times C_{\lambda_r}$. Hence $gyg^{-1} = x_1^{i_1}x_2^{i_2}\dots x_r^{i_r}$ for some $i_1, \dots, i_r$. Therefore, y is conjugate to $x_1^{i_1}x_2^{i_2}\dots x_r^{i_r}$ in A_n. □

Now we prove the main proposition of this section.

Proposition 5.3. *Let $\lambda = \lambda_1 \dots \lambda_r$ be a partition with all parts (distinct and) odd. Let x and y be elements in A_n representing the two distinct conjugacy classes corresponding to λ. Suppose $x = x_1x_2\dots x_r$ and $y = y_1y_2\dots y_r$ written as a product of disjoint cycles where x_i and y_i are of length λ_i. Then, x and y are not z-conjugate in A_n if and only if each λ_i is a perfect square (of odd number) $\forall i = 1, \dots, r$.*

Proof. First, suppose there exists a k such that λ_k is not a perfect square of odd number. We define A_{λ_k} and S_{λ_k} to be the subgroups of A_n and S_n respectively, on the symbols involved in the cycle x_k. Corollary 4.2 implies that the element x_k is not a rational element of A_{λ_k}. Hence, there exists m with $(m, \lambda_k) = 1$ such that x_k is not conjugate to x_k^m in A_{λ_k}. In any case x_k is conjugate to x_k^m in S_{λ_k}, say, there exists $s \in S_{\lambda_k} \backslash A_{\lambda_k}$ such that $sx_ks^{-1} = x_k^m$. Thus, $sxs^{-1} = sx_1x_2 \ldots x_k \ldots x_rs^{-1} = x_1x_2 \ldots x_{k-1}x_k^m x_{k+1} \ldots x_r$. We claim that x is not conjugate to sxs^{-1} in A_n. Because any two such elements will differ by an element of $\mathcal{Z}_{A_n}(x)$ which, in this case, is equal to $\mathcal{Z}_{S_n}(x)$ thus all such elements would be even. This implies that x and sxs^{-1} are representatives of the two distinct conjugacy classes obtained by splitting that of x hence sxs^{-1} is conjugate to y. But $\mathcal{Z}_{A_n}(x) = \langle x_1, x_2, \ldots, x_r \rangle = \langle x_1, x_2, \ldots, x_{k-1}, x_k^m, x_{k+1}, \ldots, x_r \rangle = \mathcal{Z}_{A_n}(sxs^{-1})$, because of the structure of sxs^{-1}. Lemma 5.1 implies that $\mathcal{Z}_{A_n}(x)$ and $\mathcal{Z}_{A_n}(y)$ are conjugate in A_n.

Now, assume that each λ_i is a perfect square (of odd number), for all i. We define the subgroups A_{λ_i} of A_n on the symbols appearing in the cycle x_i for all i. Corollary 4.2 implies that x_i is rational in A_{λ_i}, hence x_i is conjugate to $x_i^{m_i}$ in A_{λ_i} for all m_i with $(m_i, \lambda_i) = 1$. Let $(j_1, \ldots, j_r)$ be a tuple where $(j_i, \lambda_i) = 1$. Then we can find $s_{j_i} \in A_{\lambda_i}$ such that $s_{j_i}x_is_{j_i}^{-1} = x_i^{j_i}$. Thus $s_{j_1}s_{j_2} \ldots s_{j_r}$ is in A_n and conjugates x to $x_1^{j_1}x_2^{j_2} \ldots x_r^{j_r}$. Hence, y can not be conjugate to $x_1^{j_1}x_2^{j_2} \ldots x_r^{j_r}$ for any tuple $(j_1, \ldots, j_r)$ where $(j_i, \lambda_i) = 1$. Lemma 5.2, implies that x and y can not be z-conjugate in A_n. □

6. The center of centralizers in A_n

In Lemma 3.2 we showed that, for the group S_n, the center of centralizers Z_λ determines the partition λ uniquely via its action on the set $\{1, 2, \ldots, n\}$ except in one case when $\lambda_1^{e_1} = 1^2$. For the alternating groups we employ similar strategy.

Let us begin with the case when the partition λ has only one part, say, $\lambda = a^b$. The representative element can be chosen as follows,

$$\sigma_\lambda = (1, 2, \ldots, a)(a + 1, a + 2, \ldots, 2a) \ldots ((b - 1)a + 1, (b - 1)a + 2, \ldots, ba)$$

which for convenience will be written as $\sigma_\lambda = \sigma_{\lambda,1}\sigma_{\lambda,2} \ldots \sigma_{\lambda,b}$ where $\sigma_{\lambda,i}$ are cycles of length a. And the centralizer is

$$\mathcal{Z}_{S_n}(\sigma_\lambda) = (\langle (1, 2, \ldots, a) \rangle \times \cdots \times \langle ((b - 1)a + 1, (b - 1)a + 2, \ldots, ba) \rangle) \rtimes S_b$$

where S_b permutes the various cyclic subgroups. To avoid confusion, we write the elements of S_b using roman numerals. For example, the element (I, II) in S_b would be actually $(1, a + 1)(2, a + 2) \ldots (a, 2a)$ in $\mathcal{Z}_{S_n}(\sigma_\lambda)$, similarly, the element $(I, II, \ldots, b)$ in S_b would be $(1, a + 1, \ldots, (b - 1)a + 1)(2, a + 2, \ldots, (b - 1)a + 2) \ldots (a, 2a, \ldots, ba)$. In general, the cycle $(I, II, \ldots, i)$ in S_b would be $(1, a + 1, \ldots, (i - 1)a + 1)(2, a + 2, \ldots, (i - 1)a + 2) \ldots (a, 2a, \ldots, ia)$ which is a product of a many disjoint cycles, each of length i. We can also compute $sgn((I, II \ldots, i)) = sgn((1, 2, \ldots, i))^a$ which will be useful to determine if $(I, II \ldots, i)$ belongs to A_n, when needed. We begin with,

Lemma 6.1. *If $\lambda = a^b$ is a partition of n and $b \geq 2$ then $\mathcal{Z}_{S_n}(\sigma_\lambda)$ contains at least one odd permutation.*

Proof. If a is even then the cycle $(1, 2, \ldots, a) \in \mathcal{Z}_{S_n}(\sigma_\lambda)$ is odd and we are done. Thus we may assume a is odd. From the computation above, $sgn((I, II)) = (-1)^a = -1$ hence (I, II) is odd. □

Lemma 6.2.

(1) *If $\tau = (I, II, \ldots, b) \in \mathcal{Z}_{S_n}(\sigma_\lambda)$ then $\mathcal{Z}_{\mathcal{Z}_{S_n}(\sigma_\lambda)}(\tau) = \langle \tau, \sigma_\lambda \rangle$,*

(2) *If $\tau = (I, II, \ldots, b - 1) \in \mathcal{Z}_{S_n}(\sigma_\lambda)$ then $\mathcal{Z}_{\mathcal{Z}_{S_n}(\sigma_\lambda)}(\tau) = \left\langle \tau, \sigma_{\lambda,b}, \prod_{i=1}^{b-1} \sigma_{\lambda,i} \right\rangle$.*

Proof. The proof is simple and follows from the multiplication defined on $S(a, b)$ in Section 3.1. □

We need to understand the center of centralizers of elements in A_n. Suppose $\lambda = \lambda_1^{e_1}\lambda_2^{e_2} \ldots \lambda_r^{e_r}$ is a partition of even type, i.e., $\sigma_\lambda \in A_n$. Recall the notation, $\sigma_\lambda = \sigma_{\lambda_1} \ldots \sigma_{\lambda_r}$, the centralizer is $\mathcal{Z}_{S_n}(\sigma_\lambda) \cong \prod_{i=1}^r \mathcal{Z}_{S_{e_i\lambda_i}}(\sigma_{\lambda_i})$ and its center is denoted as Z_λ.

Lemma 6.3. *Let $x \in A_n$. Then, $Z_\lambda \cap A_n = \mathcal{Z}(\mathcal{Z}_{S_n}(x)) \cap A_n \subseteq \mathcal{Z}(\mathcal{Z}_{A_n}(x))$.*

Proof. Let $g \in \mathcal{Z}(\mathcal{Z}_{S_n}(x)) \cap A_n$ then $g \in \mathcal{Z}_{A_n}(x)$. Now $\mathcal{Z}_{A_n}(x) = \mathcal{Z}_{S_n}(x) \cap A_n$ thus we get $g \in \mathcal{Z}(\mathcal{Z}_{A_n}(x))$. □

Now we need to decide when $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) \supsetneq Z_\lambda \cap A_n$. For the convenience of reader we draw a diagram of the subgroups involved in the proofs. We call the elements of Z_λ "diagonal elements" and the elements of $\mathcal{Z}_{S_n}(\sigma_\lambda)$ which are not central "non-diagonal elements".

$$\begin{tikzcd}
 & Z_{\lambda_1} \times \cdots \times Z_{\lambda_r} \arrow[d, dash, "\parallel"] & \mathcal{Z}_{S_{e_1\lambda_1}}(\sigma_{\lambda_1}) \times \cdots \times \mathcal{Z}_{S_{e_r\lambda_r}}(\sigma_{\lambda_r}) \arrow[d, dash, "\parallel"] & \\
Z_\lambda \arrow[r, dash, "="] \arrow[d, dash] & \mathcal{Z}(\mathcal{Z}_{S_n}(\sigma_\lambda)) \arrow[r, dash] & \mathcal{Z}_{S_n}(\sigma_\lambda) \arrow[r, dash] \arrow[d, dash] & S_n \arrow[d, dash] \\
Z_\lambda \cap A_n \arrow[r, dash, "\subset"] & \mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) \arrow[r, dash] & \mathcal{Z}_{A_n}(\sigma_\lambda) \arrow[r, dash] & A_n
\end{tikzcd}$$

The main theorem is as follows,

Theorem 6.4. *Let λ be a partition of n and $\sigma_\lambda \in A_n$. Then $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) \supsetneq Z_\lambda \cap A_n$ if and only if λ is one of the following:*

(1) $1^3\nu$; $2^2\nu$; $1^1 2^2 \nu$ *where* $\nu = \lambda_3 \dots \lambda_r$ *with all* $\lambda_i \geq 3$ *and odd.*
(2) $1^1\nu$; ν *where* $\nu = \lambda_3 \dots \lambda_{j-1}\lambda_j^2\lambda_{j+1} \dots \lambda_r$ *where* $\lambda_i \geq 3$ *and odd for all* i.

The rest of this section is devoted to the proof of this theorem.

Lemma 6.5. *Let $\lambda = \lambda_1^{e_1} \dots \lambda_r^{e_r}$ where at least two distinct e_i and e_j are ≥ 2. Then, $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) = Z_\lambda \cap A_n$.*

Proof. Let us first take the case when $\lambda_1 = 1$, $e_1 \geq 2$ and some other e_i is ≥ 2. We have $\mathcal{Z}_{S_n}(\sigma_\lambda) = S_{e_1} \times \mathcal{Z}_{S_{e_2\lambda_2}}(\sigma_{\lambda_2}) \times \cdots \times \mathcal{Z}_{S_{e_r\lambda_r}}(\sigma_{\lambda_r})$. We note that when $e_1 = 2$ the subgroup S_2 is central. Let $g = (g_1, \dots, g_r) \in \mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda))$ but $g \notin Z_\lambda$. That is, there exists some j such that g_j is non-diagonal element in $\mathcal{Z}_{S_{e_j\lambda_j}}(\sigma_{\lambda_j})$.

Suppose $j \neq 1$. Since g_j is non-diagonal there exists $h_j \in \mathcal{Z}_{S_{e_j\lambda_j}}(\sigma_{\lambda_j})$ such that $h_j g_j \neq g_j h_j$. Now define $h = (1, \dots, 1, h_j, 1, \dots, 1)$ if h_j is even else $h = ((1,2), 1, \dots, 1, h_j, 1, \dots, 1)$. Then $h \in A_n \cap \mathcal{Z}_{S_n}(\sigma_\lambda) = \mathcal{Z}_{A_n}(\sigma_\lambda)$ but $gh \neq hg$, a contradiction.

Now if $j = 1$ the element g_1 is non-diagonal in S_{e_1}, that is, $g_1 \neq 1$. We may also assume that all other g_i, other than the first one, are diagonal. However if $e_1 = 2$ the element $g = ((1,2), g_2, \dots g_r)$ is already in Z_λ, so we couldn't have assumed otherwise. Now if $e_1 \geq 3$, pick $h_1 \in S_{e_1}$ which does not commute with g_1. Now define $h = (h_1, 1, \dots, 1)$ if h_1 is even. Else define $h = (h_1, 1, \dots, 1, w, 1, \dots, 1)$ where $w \in \mathcal{Z}_{S_{e_i\lambda_i}}(\sigma_{\lambda_i})$ is an odd permutation guaranteed by Lemma 6.1. Then $h \in A_n$ but $gh \neq hg$, a contradiction.

The proof when $e_1 = 1$ and $\lambda_1 = 1$ or $\lambda_1 > 1$ follows similarly. Now two components i and j will have odd elements because of Lemma 6.1 which can be used to change the sign to get an appropriate h. □

This reduces drastically the number of cases we need to look at. Thus we may assume that at most one e_i is greater than 2 or none, i.e., $\lambda = \lambda_1 \dots \lambda_{i-1}\lambda_i^{e_i}\lambda_{i+1} \dots \lambda_r$ with $e_i \geq 1$. Let us deal with the case when $i = 1$ and $\lambda_1 = 1$.

Lemma 6.6. *Let $\lambda = 1^{e_1}\lambda_2 \dots \lambda_r$ be a partition of n. Then, $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) \supsetneq Z_\lambda \cap A_n$ if and only if $\lambda = 1^3\lambda_2 \dots \lambda_r$ where $\lambda_i > 1$ and odd for all i.*

Proof. Suppose λ is not of the form $1^3\lambda_2 \dots \lambda_r$ where $\lambda_i > 1$ and odd for all i. So, if $e_1 = 0, 1$ or 2 then $\mathcal{Z}_{S_n}(\sigma_\lambda)$ is Abelian and its subgroup $\mathcal{Z}_{A_n}(\sigma_\lambda)$ is also Abelian. Therefore,

$$\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) = \mathcal{Z}_{A_n}(\sigma_\lambda) = \mathcal{Z}_{S_n}(\sigma_\lambda) \cap A_n = Z_\lambda \cap A_n.$$

Thus, we assume $e_1 \geq 3$. Suppose at least one λ_j is even. Now σ_λ has e_1 fixed points. Hence, $\mathcal{Z}_{S_n}(\sigma_\lambda) = S_{e_1} \times \langle\sigma_{\lambda_2}\rangle \times \cdots \times \langle\sigma_{\lambda_r}\rangle$ and $Z_\lambda = \mathcal{Z}(\mathcal{Z}_{S_n}(\sigma_\lambda)) = \langle\sigma_{\lambda_2}\rangle \times \cdots \times \langle\sigma_{\lambda_r}\rangle$ since λ_i are distinct. Now let $g \in \mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) \subset \mathcal{Z}_{S_n}(\sigma_\lambda)$. Write $g = (g_1, g_2, \dots, g_r)$. If $g \notin Z_\lambda \cap A_n$ then $g_1 \neq 1$. But we can find $h_1 \in S_{e_1}$ such that $g_1 h_1 \neq h_1 g_1$.

Define $h = (h_1, 1, \dots, 1)$ if h_1 is even else $h = (h_1, 1, \dots, 1, \sigma_{\lambda_j}, 1, \dots, 1)$. Clearly $h \in A_n \cap \ddagger_{S_n}(\sigma_\lambda) = \mathcal{Z}_{A_n}(\sigma_\lambda)$ and $gh \neq hg$. This contradicts that $g \in \mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda))$, thus $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) = Z_\lambda \cap A_n$.

Now suppose $e_1 \geq 4$ and all λ_i are odd. In this case, $\mathcal{Z}_{A_n}(\sigma_\lambda) = A_{e_1} \times \langle \sigma_{\lambda_2} \rangle \times \cdots \times \langle \sigma_{\lambda_r} \rangle$ since all σ_{λ_i} are even. And $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) = \langle \sigma_{\lambda_2} \rangle \times \cdots \times \langle \sigma_{\lambda_r} \rangle$ which is equal to $Z_\lambda \cap A_n$.

For the converse, $\lambda = 1^3\lambda_2 \dots \lambda_r$ with all λ_i odd. Then $\mathcal{Z}_{S_n}(\sigma_\lambda) = S_3 \times \langle \sigma_{\lambda_2} \rangle \times \cdots \times \langle \sigma_{\lambda_r} \rangle$ and $Z_\lambda = \mathcal{Z}(\mathcal{Z}_{S_n}(\sigma_\lambda)) = \langle \sigma_{\lambda_2} \rangle \times \cdots \times \langle \sigma_{\lambda_r} \rangle \subset A_n$. Also $\mathcal{Z}_{A_n}(\sigma_\lambda) = A_3 \times \langle \sigma_{\lambda_2} \rangle \times \cdots \times \langle \sigma_{\lambda_r} \rangle = \mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda))$. Hence $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) \supsetneq Z_\lambda \cap A_n$. □

This also takes care of the case when all parts are distinct so we may assume $e_i \geq 2$. Thus, assume either $i \geq 2$ or $\lambda_1 \geq 2$. That is we can have at most one fixed point, if at all. If σ_λ has one fixed point, say, $\sigma_\lambda(n) = n$ then we may consider $\sigma_\lambda \in A_{n-1}$ with no fixed points. Further $\mathcal{Z}_{A_n}(\sigma_\lambda) = \mathcal{Z}_{A_{n-1}}(\sigma_\lambda)$. Therefore, it is enough to study the partitions which do not have 1 as its part, i.e., we have $\lambda_1 > 1$.

Lemma 6.7. *Let $\lambda = \lambda_1 \dots \lambda_{i-1}\lambda_i^{e_i}\lambda_{i+1} \dots \lambda_r$ be a partition with $\lambda_1 > 1$. Further suppose λ satisfies one of the followings,*

(1) *$e_i \geq 3$, or,*
(2) *$\lambda_i > 2$ and is even.*

Then, $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) = Z_\lambda \cap A_n$.

Proof. We prove (1) first. Let $g = (g_1, \dots, g_r) \in \mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda))$ but $g \notin Z_\lambda$. All g_j are diagonal except g_i which is non-diagonal. The element $g_i \in \mathcal{Z}_{S_{e_i\lambda_i}}(\sigma_{\lambda_i})$ where $\sigma_{\lambda_i} = \sigma_{\lambda_{i,1}} \dots \sigma_{\lambda_{i,e_i}}$. Recall the notation that $\sigma_{\lambda_{i,j}}$ are cycles of length λ_i as introduced in the beginning of Section 6. Now consider $\tau = (I, II, \dots, e_i)$ as in Lemma 6.2. Then $\mathcal{Z}_{\mathcal{Z}_{S_{e_i\lambda_i}}(\sigma_{\lambda_i})}(\tau) = \langle \sigma_{\lambda_i}, \tau \rangle$. If $g_i = \tau$ then it does not commute with $h_i = \sigma_{\lambda_{i,1}}\sigma_{\lambda_{i,2}}$ (remember that $e_i \geq 3$). Since all $\sigma_{\lambda_{i,j}}$ are of same length λ_i this is an element in A_n. Thus we get $h = (1, \dots, 1, h_i, 1 \dots, 1)$ in $\mathcal{Z}_{A_n}(\sigma_\lambda)$ which does not commute with g, a contradiction.

On other hand if $g_i \neq \tau$ then g_i does not commute with τ. We observe that τ is a product of λ_i many cycles, each of length e_i. If e_i is odd then τ is an even permutation. Further, if both e_i and λ_i are even then, also, τ is an even permutation. And in these cases we may take $h_i = \tau$ and get a contradiction as above.

Now let us assume that λ_i is odd and e_i is even and thus τ is an odd permutation. In this case instead of τ we make use of two elements $\tau_1, \tau_2 \in \mathcal{Z}_{S_{e_i\lambda_i}}(\sigma_{\lambda_i})$ as follows. The element $\tau_1 = (II, III, \dots, e_i)$ and $\tau_2 = (I, II, \dots, e_i - 1)$. Each of the τ_1 and τ_2 are product of λ_i many cycles, each of length $e_i - 1$ and hence even. Now we note that $\mathcal{Z}_{\mathcal{Z}_{S_{e_i\lambda_i}}(\sigma_{\lambda_i})}(\tau_1) = \langle \sigma_{\lambda_{i,1}}, \prod_{j=2}^{e_i} \sigma_{\lambda_{i,j}}, \tau_1 \rangle$ and $\mathcal{Z}_{\mathcal{Z}_{S_{e_i\lambda_i}}(\sigma_{\lambda_i})}(\tau_2) = \langle \sigma_{\lambda_{i,e_i}}, \prod_{j=1}^{e_i-1} \sigma_{\lambda_{i,j}}, \tau_2 \rangle$ (see Lemma 6.2). This gives us that $\mathcal{Z}_{\mathcal{Z}_{S_{e_i\lambda_i}}(\sigma_{\lambda_i})}(\tau_1) \cap \mathcal{Z}_{\mathcal{Z}_{S_{e_i\lambda_i}}(\sigma_{\lambda_i})}(\tau_2) = \langle \sigma_{\lambda_i} \rangle$. Since g_i is non-diagonal it does not commute with either τ_1 or τ_2 else it would be in the intersection of centralizers which is diagonal. Thus we may take h_i to be τ_1 or τ_2 as required, and get a contradiction.

For the proof of (2), let $g = (g_1, \dots, g_r) \in \mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda))$ but $g \notin Z_\lambda$. The component g_i is non-diagonal element in $\mathcal{Z}_{S_{e_i\lambda_i}}(\sigma_{\lambda_i})$. In this case $\sigma_{\lambda_i} = \sigma_{\lambda_{i,1}}\sigma_{\lambda_{i,2}}$. Take $\tau = (I, II)$ then τ is an even permutation as λ_i is even. If $\tau \neq g_i$ take $h_i = \tau$ and we are done. Else if $g_i = \tau$ then we take $\sigma_{\lambda_{i,1}}^2$. Since $\lambda_i > 2$, $\sigma_{\lambda_{i,1}}^2 \neq 1$ and it is even permutation. And now taking $h_i = \sigma_{\lambda_{i,1}}^2$ would lead to a contradiction. □

This leaves us with the following case now. The partition is $\lambda = \lambda_1 \dots \lambda_{i-1}\lambda_i^2\lambda_{i+1} \dots \lambda_r$ with $\lambda_1 > 1$ and either $\lambda_i = 2$ or λ_i is odd. And this is where all complication lies.

Lemma 6.8. *Let λ with $\lambda_1 > 1$ be one of the following,*

(1) *$\lambda_1 \dots \lambda_{i-1}\lambda_i^2\lambda_{i+1} \dots \lambda_r$, and suppose, λ_i is odd and λ_m even for some $m \neq i$, or,*
(2) *$2^2\lambda_2 \dots \lambda_r$ with some λ_m even.*

Then, $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) = Z_\lambda \cap A_n$.

Proof. For the proof of (1), let $g = (g_1, \dots, g_r) \in \mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda))$ but $g \notin Z_\lambda$. Then g_i is non-diagonal. Pick $h_i \in \mathcal{Z}_{S_{e_i\lambda_i}}(\sigma_{\lambda_i})$ such that $h_i g_i \neq g_i h_i$. If h_i is even then $h = (1, \dots, h_i, \dots, 1)$ would do the job. Else take $h = (1, \dots, h_i, 1, \dots, \sigma_{\lambda_m}, 1 \dots, 1)$ which is an even permutation, and does the job.

In the second case, we have $\sigma_\lambda = (1,2)(3,4)\sigma_{\lambda_2}\dots\sigma_{\lambda_r}$ and $\mathcal{Z}_{S_n}(\sigma_\lambda) = \mathcal{Z}_{S_4}((1,2)(3,4)) \times \langle\sigma_{\lambda_2}\rangle \times \cdots \times \langle\sigma_{\lambda_r}\rangle$. Let $g = (g_1,\dots,g_r) \in \mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda))$ but $g \notin Z_\lambda$. In this case g_1 has to be non-diagonal. Now we can do the same thing as above to get a contradiction. □

At this step we are left with the λ of following kinds, and its variant (see the discussion following Lemma 6.6) with exactly one fixed point,

(1) $\lambda_1\dots\lambda_{i-1}\lambda_i^2\lambda_{i+1}\dots\lambda_r$, where all λ_j are odd, and,
(2) $2^2\lambda_2\dots\lambda_r$, where all λ_j are odd.

Now we are ready to prove the main theorem of this section,

Proof of Theorem 6.4 Lemma 6.5, 6.6, 6.7 and 6.8 prove that if the partition λ is not of the type listed in the theorem then $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) = Z_\lambda \cap A_n$. Thus it remains to prove if λ is of one the kinds listed in the theorem then we do not get equality. Which we prove now case-by-case.

In case $\lambda = 1^3\lambda_3\dots\lambda_r$ and λ_i are odd for all i then the result follows from Lemma 6.6. Now, take $\lambda = 2^2\lambda_3\lambda_4\dots\lambda_r$ and $\lambda_3 \geq 2$ and odd for all i. Write $\sigma_\lambda = (1,2)(3,4)\sigma_{\lambda_3}\dots\sigma_{\lambda_r}$ then $\mathcal{Z}_{S_n}(\sigma_\lambda) = \{1, (1,2), (3,4), (1,2)(3,4), (1,3)(2,4), (1,3,2,4), (1,4,2,3), (1,4)(2,3)\} \times \langle\sigma_{\lambda_3}\rangle \cdots \times \langle\sigma_{\lambda_r}\rangle$. And $\mathcal{Z}_{A_n}(\sigma_\lambda) = \{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \times \langle\sigma_{\lambda_3}\rangle \cdots \times \langle\sigma_{\lambda_r}\rangle$ which is equal to its own center, being commutative. However the element $(1,3)(2,4) \notin Z_\lambda$. Thus we get strict inequality in this case. The argument is similar when $\lambda = 1^1 2^2\lambda_3\dots\lambda_r$.

Now suppose $\lambda = \lambda_3\dots\lambda_{i-1}\lambda_i^2\lambda_{i+1}\dots\lambda_r$ with $\lambda_3 \geq 3$ and all odd. In this case, $\sigma_\lambda = \sigma_{\lambda_3}\dots\sigma_{\lambda_i}\dots\sigma_{\lambda_r}$ where σ_{λ_j} is a cycle of length λ_j for $j \neq i$ and $\sigma_{\lambda_i} = \sigma_{\lambda_{i,1}}\sigma_{\lambda_{i,2}}$ is a product of two cycles, each of length λ_i. Then $\sigma_{\lambda_i,1}$ and $\sigma_{\lambda_i,2}$ both belong to $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda))$ but none of them belong to Z_λ instead their product belongs. A similar argument works for the case when $\lambda = 1^1\lambda_3\dots\lambda_{i-1}\lambda_i^2\lambda_{i+1}\dots\lambda_r$. □

7. z-classes in A_n – when the conjugacy class does not split

Our strategy for the proof is similar to that of S_n case. That is, we look at the action of $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda))$ on $\{1,2,\dots,n\}$ and decide when it determines the partition. This works in almost all cases. We continue to use notation from previous sections.

Proposition 7.1. *The action of $Z_\lambda \cap A_n$ on the set $\{1,2,\dots,n\}$ determines the partition λ uniquely except when $\lambda_1^{e_1} = 1^2$.*

Proof. We know that the action of Z_λ on the set $\{1,2,\dots,n\}$ determines the partition uniquely except when $\lambda_1^{e_1} = 1^2$ (see Lemma 3.2). We need to prove that if two points in $\{1,2,\dots,n\}$ are related under the action of Z_λ then they are so under the action of $Z_\lambda \cap A_n$.

Since $\sigma_\lambda = \sigma_{\lambda_1}\dots\sigma_{\lambda_r}$, we reorder σ_{λ_k}'s, if required, so that σ_{λ_k} for $1 \leq k \leq l$ are even permutations and σ_{λ_k} for $l < k \leq r$ are odd permutations. Since σ_λ is an even permutation, the number of odd permutations $r-l$ is even (including 0). If $r = l$ then $Z_\lambda = Z_\lambda \cap A_n$ and we are done. Else suppose $i \neq j$ are related under Z_λ. That is, there exists t such that $\sigma_{\lambda_t}^m(i) = j$ for some power m. If $\sigma_{\lambda_t}^m$ is even, we are done. So we may assume $\sigma_{\lambda_t}^m$ is odd. But since the number of odd permutations is assumed to be even we have another odd permutation σ_{λ_s} disjoint from this one. Thus, $\sigma_{\lambda_s}\sigma_{\lambda_t}^m$ will do the job. □

We record the following example of the exception case. Take $\lambda = 1^2 4^1 \vdash 6$ then $\mathcal{Z}_{S_6}(3,4,5,6) = \langle(1,2)\rangle \times \langle(3,4,5,6)\rangle = Z_{1^2 4^1}$. And $Z_{1^2 4^1} \cap A_5 = \langle(1,2)(3,4,5,6)\rangle$ which would determine the partition $2^1 4^1$. Now let us look at the case when $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) \neq Z_\lambda \cap A_n$. In this case we have the following,

Proposition 7.2. *If λ is one of the following with σ_λ in A_n,*

(1) $1^2\lambda_2^{e_2}\dots\lambda_r^{e_r}$, *where* $\lambda_2 \geq 2$, *or,*
(2) $1^1\nu, \nu$ *where* $\nu = \lambda_2\dots\lambda_{i-1}\lambda_i^2\lambda_{i+1}\dots\lambda_r$, *where* $\lambda_j \geq 3$ *and odd for all* j,

then, the action of $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda))$ on the set $\{1,2,\dots,n\}$ determines the partition λ uniquely.

Proof. The first case appears in S_n, where $\mathcal{Z}_{S_n}(\sigma_\lambda)$ determines all $\lambda_i > 2$ except for the first orbit which is $\{1, 2\}$. Thus there are two possibilities either 1^2 or 2^1. Since $\sigma_\lambda = \sigma_{\lambda_2} \dots \sigma_{\lambda_r} \in A_n$ we note that the partition $2^1\lambda_2^{e_2} \dots \lambda_r^{e_r}$ is not even because this would correspond to the element $(1, 2)\sigma_\lambda = (1, 2)\sigma_{\lambda_2} \dots \sigma_{\lambda_r}$ which is odd. Thus this leaves a unique choice for λ where the first part must be 1^2.

For the part (2), from the proof of Theorem 6.4, we see that

$$\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) = \langle \sigma_{\lambda_1}, \dots, \sigma_{\lambda_{i-1}}, \sigma_{\lambda_{i,1}}, \sigma_{\lambda_{i,2}}, \sigma_{\lambda_{i+1}}, \dots, \sigma_{\lambda_r} \rangle.$$

Clearly this determines the partition λ uniquely. □

Now, we prove the main proposition as follows.

Proposition 7.3. *Let $n \geq 4$. Let ν be a restricted partition of $n - 3$, with distinct and odd parts, in which 1 (and 2) does not appear as its part. Let $\lambda = 1^3\nu$ and $\mu = 3^1\nu$ be partitions of n obtained by extending ν. Then λ and μ belong to the same z-class in A_n. Conversely, if λ corresponds to a non-split class in A_n then it can be z-equivalent to at most one more class (possibly split), provided λ is of the form $1^3\nu$.*

Proof. When the partition $\lambda = 1^3\nu$ then $\mathcal{Z}_{S_n}(\sigma_\lambda) = S_3 \times \mathcal{Z}_{S_{n-3}}(\sigma_\nu)$ and its center is $Z_\lambda = \{1\} \times Z_\nu$. However $\mathcal{Z}_{A_n}(\sigma_\lambda) = A_3 \times \mathcal{Z}_{A_{n-3}}(\sigma_\nu)$ is Abelian and its action would give the partition $3^1\nu$. In this case, if we take partition $\lambda' = 3^1\nu$ then $\mathcal{Z}_{A_n}(\sigma_{\lambda'}) = \langle(1, 2, 3)\rangle \times \mathcal{Z}_{A_{n-3}}(\sigma_\nu)$ and $\mathcal{Z}_{A_n}(\sigma_{\lambda'}) = \mathcal{Z}_{A_n}(\sigma_\lambda)$ (in case λ' corresponds to a split class they are z-conjugate thus we may choose this representative). And thus σ_λ and $\sigma_{\lambda'}$ would be z-conjugate.

For the converse, if $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda)) = Z_\lambda \cap A_n$, then from Proposition 7.1, the action of $\mathcal{Z}(\mathcal{Z}_{A_n}(\sigma_\lambda))$ determines the partition λ of n uniquely, and we are done. Otherwise, we use Proposition 7.2 which implies that $\mathcal{Z}_{A_n}(\sigma_\lambda)$ determines the partition λ uniquely except in two cases. One of the cases is $1^3\nu$ where the centralizer is conjugate to that of $3^1\nu$ as required in the proposition. Thus we need to rule out the possibility when $\lambda = 2^2\nu$ and $1^12^2\nu$ where $\nu = \lambda_3\lambda_4 \dots \lambda_r$, $\lambda_i \geq 3$ and are odd for all i.

Let us deal with the case when $\lambda = 2^2\nu$, the other case is similar. The element $\sigma_\lambda = (1, 2)(3, 4)\sigma_{\lambda_2} \dots \sigma_{\lambda_r}$ and

$$\mathcal{Z}_{A_n}(\sigma_\lambda) = \langle(1, 2)(3, 4), (1, 3)(2, 4)\rangle \times \langle\sigma_{\lambda_2}\rangle \times \dots \times \langle\sigma_{\lambda_r}\rangle$$

which has size $4.\lambda_2.\dots.\lambda_r$. Since this is Abelian its center is itself which determines the partition λ except for the first orbit which is $\{1, 2, 3, 4\}$. Considering that λ is even, we have the possibilities of the first part being $1^4, 1^13^1, 2^2$. We claim that if $\lambda = 1^4\nu, 1^13^1\nu$ or $2^2\nu$ the size of centralizers is different and hence they can not be z-equivalent. We note that, $\mathcal{Z}_{A_n}(\sigma_{1^4\nu}) = \langle A_4\rangle \times \langle\sigma_{\lambda_2}\rangle \times \dots \times \langle\sigma_{\lambda_r}\rangle$ which has size $12.\lambda_2.\dots.\lambda_r$. And if $\lambda_2 > 3$, $\mathcal{Z}_{A_n}(\sigma_{1^13^1\nu}) = \langle(1, 2, 3)\rangle \times \langle\sigma_{\lambda_2}\rangle \times \dots \times \langle\sigma_{\lambda_r}\rangle$ of size $3.\lambda_2.\dots.\lambda_r$ and if $\lambda_2 = 3$, $\mathcal{Z}_{A_n}(\sigma_{1^13^2\lambda_3\dots\lambda_r}) = \langle(2, 3, 4), (5, 6, 7)\rangle \times \langle\sigma_{\lambda_3}\rangle \times \dots \times \langle\sigma_{\lambda_r}\rangle$ of size $3^2.\lambda_3.\dots.\lambda_r$. □

7.1 Proof of Theorem 1.3

Let C be a conjugacy class of S_n corresponding to the partition $\lambda_1\lambda_2 \dots \lambda_r$ of n with all λ_i distinct and odd. Then the conjugacy class C splits in two conjugacy classes, say, C_1 and C_2 in A_n. From Proposition 5.3 if each λ_i is a perfect square for all $1 \leq i \leq r$, then both the conjugacy classes C_1 and C_2 are distinct z-classes in A_n. Else $C_1 \cup C_2$ form a single z-class in A_n.

Now, when C does not split, it follows from Proposition 7.3, that except the partition $1^3\nu$ where ν is a partition of $n - 3$, with all parts odd and distinct without 1 as its part, all conjugacy classes remain distinct z-classes. And in the case when $\lambda = 1^3\nu$ its z-class can coincide with that of $3^1\nu$.

8. Rational-valued characters of A_n

We begin with recalling characters of the alternating group from [Pr]. First we note that, the number of partitions of n with distinct and odd parts is equal to the number of self-conjugate partitions of n (see Lemma 4.6.16 in [Pr]). In fact, these are in one-one correspondence via folding. This corresponds to the split conjugacy classes. The complex irreducible characters of A_n are given as follows (see Theorem 4.6.7 and 5.12.5 in [Pr]). For every partition μ of

n which is not self-conjugate (this corresponds to non-split conjugacy classes), the irreducible character χ_μ of S_n restricts to an irreducible character of A_n. Since all characters of S_n are integer-valued, these characters of A_n are rational-valued too. Now, for all partitions μ of n which are self-conjugate (these correspond to split conjugacy classes), there exists a pair of irreducible characters χ_μ^+ and χ_μ^-. The character values are given by the following formula. When $g \in A_n$ of cycle type λ with all parts distinct and odd, say, $\lambda = (2m_1+1, \ldots, 2m_l+1)$, and the folding corresponding to λ is the partition μ then

$$\chi_\mu^{\pm}(g_\lambda^+) = \frac{1}{2}(e_\lambda \pm \sqrt{e_\lambda |Z_\lambda|})$$

and $\chi_\mu^{\pm}(g_\lambda^-) = \chi_\mu^{\mp}(g_\lambda^+)$. Here g_λ^+ and g_λ^- denote the two split conjugacy classes in A_n and $e_\lambda = (-1)^{\sum_{i=1}^{l} m_i}$. Else $\chi_\mu^+(g) = \chi_\mu^-(g) = \frac{\chi_\mu(g)}{2}$. Clearly the characters $\chi_\mu^{\mp}$ are rational valued if and only if $e_\lambda = 1$ and $|Z_\lambda|$ is a perfect square.

Lemma 8.1. *For $\lambda = (2m_1+1, \ldots, 2m_l+1)$, if $|Z_\lambda|$ is a square then $e_\lambda = 1$.*

Proof. In this case, $|Z_\lambda| = \prod_{i=1}^{l}(2m_i+1) = (2a+1)^2$ for some a. Then $\sum_{i=1}^{l} m_i$ must be even. □

Proof of Theorem 1.7 From the discussion above, all characters of A_n corresponding to non-split conjugacy classes are rational-valued. And, both characters corresponding to split conjugacy classes are simultaneously rational-valued if and only if the partition λ has all its parts distinct and odd and the product of parts is a perfect square. Clearly this is same as the criteria determining conjugacy classes which are rational. □

Example 8.2. Let us work with A_{10}. In [AO] Section 2 Theorem 1, it is proved that the alternating groups are not Q-groups. However, in the proof for A_{10} case it is wrongly mentioned that $1^1 9^1$ is not rational. It is easy to see from our criteria that $1^1 9^1$ is a rational conjugacy class as the product $1.9 = 9$ is a square. In fact, the class corresponding to $3^1 7^1$ is not rational because of 1.5 part 1(b) as the product $3.7 = 21$ is not a square. We also note that the character value $\chi^{\pm}_{2^3 4^1}(g^+_{3^1 7^1}) = \frac{1}{2}(-1 \pm \sqrt{-21})$ is clearly not rational.

9. Some GAP calculations

In this work, we have come across two functions on natural numbers. The first one is ϵ defined as

$$\epsilon(n) = |\{n = m_1^2 + m_2^2 + \cdots + m_r^2 \mid 1 \le m_1 < m_2 < \cdots < m_r \le n, m_i \text{ odd } \forall i\}|$$

and its generating function is $\prod_{i=0}^{\infty}(1 + x^{(2i+1)^2})$. And another one is δ defined as,

$$\delta(n) = \left| \left\{ n = n_1 + \cdots + n_r \mid 1 \le n_1 < \cdots < n_r \le n, n_i \text{ odd } \forall i, \prod_{i=1}^{r} n_i \in \mathbb{N}^2 \right\} \right|.$$

Writing a natural number as a sum of squares is well studied problem in number theory. However, we could not find references to these functions. Clearly $\epsilon(n) \le \delta(n)$. The inequality could be strict, for example, $n = 78 = 3 + 75$ where $3.75 = 15^2$ but none of the components are square. This happens infinitely often. For example, let p_1 and p_2 be odd and distinct primes. Consider, $n = p_1 + p_2 + p_1 p_2$ and the partition of n given by $p_1^1 p_2^1 (p_1 p_2)^1$. Then $\epsilon(n) < \delta(n)$. We may also consider, for example, $m = p_1 + p_1 p_2^2$, i.e., we have the partition of m given as $p_1^1 (p_1 p_2^2)^1$. Then $\epsilon(m) < \delta(m)$. We make a table for the values of ϵ and δ for small values of n and also note down the partitions giving rise to the function δ. Some values of $\delta(n)$ are also given in [Br].

n	$\epsilon(n)$	$\delta(n)$	Partitions	n	$\epsilon(n)$	$\delta(n)$	Partitions
9	1	1	9^1	34	1	1	$9^1 25^1$
10	1	1	$1^1 9^1$	35	1	1	$1^1 9^1 25^1$
23	0	1	$3^1 5^1 15^1$	39	0	1	$3^1 9^1 27^1$
24	0	1	$1^1 3^1 5^1 15^1$	40	0	2	$1^1 3^1 9^1 27^1, 3^1 7^1 9^1 21^1$
25	1	1	25^1	41	0	1	$1^1 3^1 7^1 9^1 21^1$
26	1	1	$1^1 25^1$	47	0	3	$3^1 11^1 33^1, 5^1 7^1 35^1, 5^1 15^1 27^1$
30	0	1	$3^1 27^1$	48	0	5	$1^1 3^1 11^1 33^1, 1^1 5^1 7^1 35^1,$ $1^1 5^1 15^1 27^1, 5^1 7^1 15^1 21^1,$ $3^1 5^1 15^1 25^1$
31	0	2	$1^1 3^1 27^1, 3^1 7^1 21^1$	49	1	3	$49^1, 1^1 3^1 5^1 15^1 25^1,$ $1^1 5^1 7^1 15^1 21^1$
32	0	2	$1^1 3^1 7^1 21^1, 3^1 5^1 9^1 15^1$	50	1	2	$1^1 49^1, 5^1 45^1$
33	0	1	$1^1 3^1 5^1 9^1 15^1$	51	0	1	$1^1 5^1 45^1$

Next, we used GAP [GAP] to compute z-classes, rational conjugacy classes etc. Here we have some examples for A_n which verifies our theorem.

n	Number of conj. classes	Number of z-classes	Partitions
20	324	315	$\{1^3 3^1 5^1 9^1, 3^2 5^1 9^1\}, \{1^1 3^1 7^1 9^1, 1^1 3^1 7^1 9^1\}$ $\{1^1 3^1 5^1 11^1, 1^1 3^1 5^1 11^1\}, \{9^1 11^1, 9^1 11^1\}, \{1^1 19^1, 1^1 19^1\}$ $\{7^1 13^1, 7^1 13^1\}, \{5^1 15^1, 5^1 15^1\}, \{1^3 17^1, 3^1 17^1, 3^1 17^1\}$
27	1526	1506	$\{1^3 3^1 5^1 7^1 9^1, 3^2 5^1 7^1 9^1\}, \{1^1 3^1 5^1 7^1 11^1, 1^1 3^1 5^1 7^1 11^1\}$ $\{7^1 9^1 11^1, 7^1 9^1 11^1\}, \{5^1 9^1 13^1, 5^1 9^1 13^1\}$ $\{1^3 11^1 13^1, 3^1 11^1 13^1, 3^1 11^1 13^1\}, \{5^1 7^1 15^1, 5^1 7^1 15^1\}$ $\{1^3 9^1 15^1, 3^1 9^1 15^1, 3^1 9^1 15^1\}, \{1^1 11^1 15^1, 1^1 11^1 15^1\}$ $\{1^3 7^1 17^1, 3^1 7^1 17^1, 3^1 7^1 17^1\}, \{1^3 5^1 19^1, 3^1 5^1 19^1, 3^1 5^1 19^1\}$ $\{1^1 7^1 19^1, 1^1 7^1 19^1\}, \{1^1 9^1 17^1, 1^1 9^1 17^1\}$ $\{1^3 3^1 21^1, 3^2 21^1\}, \{1^1 5^1 21^1, 1^1 5^1 21^1\}$ $\{1^1 3^1 23^1, 1^1 3^1 23^1\}, \{27^1, 27^1\}$

The last column combines together the partitions which give same z-class and the repetition of a partition indicates a split conjugacy class.

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