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# The units-Picard complex of a reductive group scheme

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**Abstract.** Let $S$ be a locally noetherian regular scheme. We compute the units-Picard complex of a reductive $S$-group scheme $G$ in terms of the dual algebraic fundamental complex of $G$. To this end, we establish a units-Picard-Brauer exact sequence for a torsor under a smooth $S$-group scheme.

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*In memory of Jost van Hamel*

## 1. Introduction

Let $S$ be a non-empty scheme and let $D^b(S_{\text{ét}})$ denote the derived category of the category of bounded complexes of abelian etale sheaves on $S$. A morphism of schemes $f: X \to S$ induces a morphism $f^{\natural}: \mathbb{G}_{m,S} \to \tau_{\leq 1}\mathbb{R}f_*\mathbb{G}_{m,X}$ in $D^b(S_{\text{ét}})$ that factors through the canonical morphism (of etale sheaves on $S$) $f^{\flat}: \mathbb{G}_{m,S} \to f_*\mathbb{G}_{m,X}$. The (relative) *units-Picard complex of $X$ over $S$* is the object $\text{UPic}_{X/S} = C^{\bullet}(f^{\natural})[1]$ of $D^b(S_{\text{ét}})$, where $C^{\bullet}(f^{\natural})$ is the mapping cone of any morphism of complexes that represents $f^{\natural}$. There exists a distinguished triangle in $D^b(S_{\text{ét}})$

$$\text{Pic}_{X/S}[-1] \to \text{RU}_{X/S}[1] \to \text{UPic}_{X/S} \to \text{Pic}_{X/S},$$

where $\text{Pic}_{X/S}$ is the etale relative Picard functor of $X$ over $S$ and $\text{RU}_{X/S} = C^{\bullet}(f^{\flat})$ is the complex of relative units of $X$ over $S$. Except for a shift, $\text{UPic}_{X/S}$ was originally introduced by Borovoi and van Hamel in the case where $S$ is the spectrum of a field of characteristic zero [BvH]. The more general object $\text{UPic}_{X/S}$ discussed in this paper was introduced in [GA3, §3], where the reader can find proofs of some of its main properties. Borovoi and van Hamel, and later Harari and Skorobogatov [HSk], showed that $\text{UPic}_{X/S}$ is well-suited for simplifying and generalizing various classical constructions [San81,CTS]. Further, the author has shown in [GA3] that $\text{UPic}_{X/S}$ is important in the study of the Brauer group of $X$.

We now explain the contents of the paper.

Let $k$ be a field, let $G$ be a (connected) reductive $k$-group scheme [SGA3$_{\text{new}}$, XIX, 1.6 and 2.7] and let $\widetilde{G}$ be the simply connected central cover of the (semisimple) derived group of $G$. Let $T$ be a maximal $k$-torus in $G$, set $\widetilde{T} = T \times_G \widetilde{G}$ and let $T^*$ and $\widetilde{T}^*$ denote the $k$-group schemes of characters of $T$ and $\widetilde{T}$, respectively. The *dual algebraic fundamental complex of $G$* is the object $\pi_1^D(G)$ of $D^b(k_{\text{ét}})$ represented by the mapping cone of the morphism $T^* \to \widetilde{T}^*$ induced by the canonical morphism $\widetilde{G} \to G$. Up to isomorphism, $\pi_1^D(G)$ is independent of the choice of $T$. The main theorem of [BvH] establishes the existence of an isomorphism in $D^b(k_{\text{ét}})$ that is functorial in $G$:

$$\text{UPic}_{G/k} \xrightarrow{\sim} \pi_1^D(G) \tag{1.1}$$

(if the characteristic of $k$ is 0, the reductivity assumption on $G$ can be dropped). Over a general base scheme $S$, $\pi_1^D(G)$ can be defined in terms of a *t-resolution* of $G$, i.e., a central extension of $S$-group schemes $1 \to T \to H \to G \to 1$,

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where $T$ is an $S$-torus and $H$ is a reductive $S$-group scheme whose derived subgroup $H^{\text{der}}$ is simply connected [BGA, Definition 2.1]. In effect, if $R = H^{\text{tor}} = H/H^{\text{der}}$ is the largest quotient of $H$ which is an $S$-torus, then $\pi_1^D(G)$ is (represented by) the cone of the morphism of etale twisted constant $S$-group schemes $R^* \to T^*$ induced by $T \to H$. If $G$ contains a maximal $S$-torus $T$, then there exists a $t$-resolution $1 \to \widetilde{T} \to H \to G \to 1$ of $G$ where $H^{\text{tor}}$ is canonically isomorphic to $T$ and we recover the Borovoi-van Hamel definition of $\pi_1^D(G)$. Up to isomorphism, $\pi_1^D(G)$ is independent of the choice of a $t$-resolution of $G$. The aim of this paper is to establish the following generalization of (1.1):

**Theorem 1.1 (=Theorem 4.20).** *Let $S$ be a locally noetherian regular scheme and let $G$ be a reductive $S$-group scheme. Then there exists an isomorphism in $D^b(S_{\text{ét}})$*

$$\text{UPic}_{G/S} \xrightarrow{\sim} \pi_1^D(G)$$

*which is functorial in $G$.*

As in [BvH], the main ingredients of the proof of Theorem 1.1 are certain long exact sequences of abelian groups of the form

$$\cdots \to G_2^* \to G_1^* \to \text{Pic}\, G_3 \to \text{Pic}\, G_2 \to \dots \tag{1.2}$$

induced by short exact sequences of $S$-group schemes $1 \to G_1 \to G_2 \to G_3 \to 1$. In this paper we derive all the sequences of the form (1.2) that we need for the proof of Theorem 1.1 from the *units-Picard-Brauer exact sequence of a torsor* given in Proposition 3.11. The latter is a broad generalization of an exact sequence of Sansuc [San81, Proposition 6.10, p. 43], which itself generalizes a well-known exact sequence of Fossum and Iversen [FI, Proposition 3.1]. On the other hand, in constrast to [BvH] (and [CT08, Appendix B]), we avoid working with explicit presentations of $\text{UPic}_{G/S}$ (in terms of rational functions and divisors on $G$) since such presentations are not needed for the proof of Theorem 1.1 (as already suggested in [BvH, Remark 4.6]).

In [BvH], the isomorphism (1.1) plays an important role in the study of the so-called *elementary obstruction* to the existence of $k$-rational points on $G$-torsors over $k$. Theorem 1.1 above has similar applications to the existence of sections on $G$-torsors over an arbitrary locally noetherian and regular scheme $S$, as we hope to show in a subsequent publication.

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## 2. Preliminaries

### 2.1 Generalities

If $X$ is an object of a category, the identity morphism of $X$ will be denoted by $1_X$. An *exact and commutative diagram* is a commutative diagram with exact rows and columns. The category of abelian groups will be denoted by **Ab**.

The *maximal points* of a topological space are the generic points of its irreducible components. If $f\colon X \to S$ is a morphism of schemes, a *maximal fiber of $f$* is a fiber of $f$ over a maximal point of $S$. Recall also that $f$ is called *schematically dominant* if the canonical morphism (of Zariski sheaves on $S$) $f^\#\colon \mathscr{O}_S \to f_*\mathscr{O}_X$ is injective [EGA $\text{I}_{\text{new}}$, §5.4]. By [Pic, Proposition 52, p. 10], a faithfully flat morphism is schematically dominant.

A *stricly local* scheme is a scheme which is isomorphic to the spectrum of a local henselian ring with separably closed residue field [EGA, $\text{IV}_4$, Definition 18.8.2].

**Lemma 2.1.** *Let $A \xrightarrow{u} B \xrightarrow{v} C$ be morphisms in an abelian category $\mathscr{A}$. Then there exists a canonical exact sequence in $\mathscr{A}$*

$$0 \to \text{Ker}\, u \to \text{Ker}(v \circ u) \to \text{Ker}\, v \to \text{Coker}\, u \to \text{Coker}(v \circ u) \to \text{Coker}\, v \to 0.$$

*Proof.* See, for example, [Bey, §1.2]. □

If $S$ is a scheme (which is tacitly assumed to be non-empty throughout the paper), we will write $S_{\text{fl}}$ for the small fppf site over $S$. Thus $S_{\text{fl}}$ is the category of $S$-schemes that are flat and locally of finite presentation over $S$ equipped with the fppf topology, i.e., a covering of $(X \to S) \in S_{\text{fl}}$ is a family $\varphi_i : U_i \to X$ of flat $S$-morphisms locally of finite presentation such that $X = \bigcup \varphi_i(U_i)$. We will also need the small etale site $S_{\text{ét}}$, which is defined as above by writing *étale* in place of *flat and locally of finite presentation.*

A sequence of $S$-group schemes $1 \to G_1 \to G_2 \to G_3 \to 1$ will be called *exact* if the corresponding sequence of (representable) sheaves of groups on $S_{\text{fl}}$ is exact.

We will write $S_{\text{ét}}^{\sim}$ for the category of abelian sheaves on $S_{\text{ét}}$ and $C^b(S_{\text{ét}})$ for the category of bounded complexes of objects of $S_{\text{ét}}^{\sim}$. The corresponding derived category will be denoted by $D^b(S_{\text{ét}})$. The (mapping) *cone* of a morphism $u\colon A^\bullet \to B^\bullet$ in $C^b(S_{\text{ét}})$ is the complex $C^\bullet(u)$ whose $n$-th component is $C^n(u) = A^{n+1} \oplus B^n$, with differential $d^n_{C(u)}(a, b) = (-d_A^{n+1}(a), u(a) + d_B^n(b))$, where $n \in \mathbb{Z}, a \in A^{n+1}$ and $b \in B^n$. If $u\colon A \to B$ is a morphism in $S_{\text{ét}}^{\sim}$ and $A$ and $B$ are regarded as complexes concentrated in degree 0, then $C^\bullet(u) = (A \overset{u}{\to} B)$, where $A$ and $B$ are placed in degrees $-1$ and 0, respectively. Thus $H^{-1}(C^\bullet(u)) = \operatorname{Ker} u$ and $H^0(C^\bullet(u)) = \operatorname{Coker} u$. The distinguished triangle in $D^b(S_{\text{ét}})$ corresponding to $u$ is

$$C^\bullet(u)[-1] \overset{v}{\to} A \overset{u}{\to} B \to C^\bullet(u), \tag{2.1}$$

where the map $v\colon C^\bullet(u)[-1] = (A \overset{u}{\to} B)[-1] \to A$ is the negative of the canonical projection. We also recall that, if $A^\bullet$ is a bounded-below complex of objects of $S_{\text{ét}}^{\sim}$ and $n \geq 0$ is an integer, then the *n-th truncation* $\tau_{\leq n} A^\bullet$ of $A^\bullet$ is the following object of $C^b(S_{\text{ét}})$:

$$\tau_{\leq n} A^\bullet = \cdots \to A^{n-2} \to A^{n-1} \to \operatorname{Ker}[A^n \to A^{n+1}] \to 0.$$

For every $n \in \mathbb{N}$, there exists a distinguished triangle in $D^b(S_{\text{ét}})$

$$\tau_{\leq n-1} A^\bullet \overset{i^n}{\to} \tau_{\leq n} A^\bullet \to H^n(A^\bullet)[-n] \to (\tau_{\leq n-1} A^\bullet)[1]. \tag{2.2}$$

Next we recall the definition of the separable index of a scheme over a field. If $k$ is a field with fixed separable algebraic closure $k^s$ and $X$ is a $k$-scheme such that $X(k') \neq \emptyset$ for some finite subextension $k'/k$ of $k^s/k$, then the *separable index of $X$ over $k$* is the integer

$$I(X) = \gcd\{[k' : k] : k'/k \subset k^s/k \text{ finite and } X(k') \neq \emptyset\}.$$

The positive integer $I(X)$ is defined if $X$ is geometrically reduced and locally of finite type over $k$ (see [Liu, §3.2, Proposition 2.20, p. 93] for the finite type case and note that the more general case can be obtained by applying the indicated reference to a nonempty open affine subscheme of $X$). The above is a particular case of the following definition: if $f : X \to S$ is a morphism of schemes, the *étale index* $I(f)$ of $f$ is the greatest common divisor of the degrees of all finite etale quasi-sections of $f$ of constant degree, if any exist (recall that an *étale quasi-section* of $f$ is an etale and surjective morphism $T \to S$ such that there exists an $S$-morphism $T \to X$). Note that $I(f)$ is defined (and is equal to 1) if $f$ has a section.

## 2.2 Torsors

Let $G$ be an $S$-group scheme and let $X$ (respectively, $Y$) be a right (respectively, left) $G$-scheme. Then $X \times_S Y$ is a right $G$-scheme under the action $(X \times_S Y) \times_S G \to X \times_S Y, (x, y, g) \mapsto (xg, g^{-1}y)$. The corresponding quotient fppf sheaf of sets is denoted by $X \wedge^G Y$. If $G''$ is another $S$-group scheme which acts on $Y$ from the right compatibly with the given left action of $G$, then $X \wedge^G Y$ is naturally a right $G''$-sheaf. For example, if $Y = G''_r$ is a right $G''$-scheme via right translations and a left $G$-scheme via an $S$-homomorphism $v\colon G \to G''$, then $X \wedge^{G,v} G''_r$ is a right $G''$-sheaf [Gi, Proposition 1.3.6, p. 116]. If $X = G'$ is a third $S$-group scheme regarded as a right $G$-scheme via an $S$-homomorphism $u\colon G \to G'$, then $G' \wedge^{u,G,v} G''_r$ is the well-known pushout of $u$ and $v$. We will write

$H^1(S_{\text{fl}}, G)$ for the pointed set of isomorphism classes of right (sheaf) $G$-torsors over $S$ relative to the fppf topology on $S$, where the distinguished point is the isomorphism class of the trivial right $G$-torsor $G_r$. An $S$-homomorphism $u\colon G \to G'$ induces a map of pointed sets [Gi, Proposition 1.4.6(i), p. 119]

$$u^{(1)}\colon H^1(S_{\text{fl}}, G) \to H^1(S_{\text{fl}}, G'), [X] \mapsto [X \wedge^{G,u} G'_r]. \tag{2.3}$$

The latter map is functorial in $S$, i.e., for every morphism of schemes $T \to S$, there exists a (canonical) isomorphism of $G_T$-torsors over $T$

$$(X \wedge^{G,u} G'_r) \times_S T \simeq X_T \wedge^{G_T, u_T} G'_{T,r}. \tag{2.4}$$

See [Gi, 1.5.1.2, p. 316]. Now, if

$$1 \to H \overset{i}{\to} G \overset{q}{\to} K \to 1 \tag{2.5}$$

is an exact sequence of $S$-group schemes, then the right action $G \times_K (H \times_S K) \to G$, $(g, (h, q(g))) \mapsto gi(h)$, endows $G$ with the structure of a right $H_K$-torsor over $K$ and we obtain a class $[G] \in H^1(K_{\text{fl}}, H_K)$. If $L$ is an $S$-group scheme and $u\colon H \to L$ is an $S$-homomorphism, let $P = G \wedge^{i,H,u} L$ be the pushout of $i$ and $u$. Then there exists an exact and commutative diagram of fppf sheaves of groups on $S$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \overset{i}{\longrightarrow} & G & \overset{q}{\longrightarrow} & K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle u} & & \downarrow & & \| & & \\ 1 & \longrightarrow & L & \overset{j}{\longrightarrow} & P & \overset{p}{\longrightarrow} & K & \longrightarrow & 1, \end{array} \tag{2.6}$$

where the maps $j$ and $p$ are defined as follows: if $\pi_L\colon L \to S$ is the structural morphism of $L$ and $\varepsilon_G\colon S \to G$ and $\varepsilon_K\colon S \to K$ are the unit sections of $G$ and $K$, respectively, then $j$ and $p$ are induced, respectively, by $(\varepsilon_G \circ \pi_L, 1_L)_S\colon L \to G \times_S L$ and $m \circ (q \times_S (\varepsilon_K \circ \pi_L))\colon G \times_S L \to K \times_S K \to K$, where $m\colon K \times_S K \to K$ is the product morphism of $K$. As noted previously, the bottom row of diagram (2.6) equips $P$ with the structure of a right $L_K$-torsor over $K$. Using the explicit definitions of the maps $j$ and $p$ given above, it is not difficult to check that the canonical isomorphism of $S$-schemes $G \times_S L \overset{\sim}{\to} G \times_K L_K$ induces an isomorphism of right $L_K$-torsors over $K$

$$G \wedge^{i,H,u} L \overset{\sim}{\to} G \wedge^{H_K, u_K} L_{K,r}. \tag{2.7}$$

Thus $[P] = u_F^{(1)}([G])$, where $u_F^{(1)}\colon H^1(K_{\text{fl}}, H_K) \to H^1(K_{\text{fl}}, L_K)$ is the map (2.3) induced by the morphism of $K$-group schemes $u_K\colon H_K \to L_K$.

## 2.3 Units, Picard groups and Brauer groups

Let $f\colon X \to S$ be a morphism of schemes and let $f^\flat\colon \mathbb{G}_{m,S} \to f_*\mathbb{G}_{m,X}$ be the morphism of abelian sheaves on $S_{\text{ét}}$ induced by $f$. The *complex of relative units of $X$ over $S$* is the complex

$$\text{RU}_{X/S} = C^\bullet(f^\flat) = (\mathbb{G}_{m,S} \overset{f^\flat}{\to} f_*\mathbb{G}_{m,X}), \tag{2.8}$$

where $\mathbb{G}_{m,S}$ and $f_*\mathbb{G}_{m,X}$ are placed in degrees $-1$ and 0, respectively.

If $X \overset{g}{\to} Y \overset{h}{\to} S$ are morphisms of schemes, then $(h \circ g)^\flat\colon \mathbb{G}_{m,S} \to h_*(g_*\mathbb{G}_{m,X})$ factors as

$$\mathbb{G}_{m,S} \overset{h^\flat}{\to} h_*\mathbb{G}_{m,Y} \overset{h_*(g^\flat)}{\longrightarrow} h_*(g_*\mathbb{G}_{m,X}). \tag{2.9}$$

Consequently, $h_*(g^\flat)$ induces a morphism $\text{RU}_S(g)\colon \text{RU}_{Y/S} \to \text{RU}_{X/S}$ in $C^b(S_{\text{ét}})$. Thus (2.8) defines a contravariant functor

$$\text{RU}_S\colon (\text{Sch}/S) \to C^b(S_{\text{ét}}), (X \to S) \mapsto \text{RU}_{X/S}.$$

Now set

$$\text{U}_{X/S} = H^0(\text{RU}_{X/S}) = \text{Coker}\, f^\flat \tag{2.10}$$

and consider the contravariant functor

$$\mathrm{U}_S\colon (\mathrm{Sch}/S) \to S_{\text{ét}}^{\sim}, (X \to S) \mapsto \mathrm{U}_{X/S}.$$

If $f$ is schematically dominant, then $f^{\flat}$ is injective [GA2, Lemma 2.4] and $\mathrm{RU}_{X/S} = \mathrm{U}_{X/S}$ (placed in degree 0).

**Lemma 2.2.** *Let $X \xrightarrow{g} Y \xrightarrow{h} S$ be morphisms of schemes, where $g$ is schematically dominant. Then the canonical morphism $\mathrm{U}_S(g)\colon \mathrm{U}_{Y/S} \to \mathrm{U}_{X/S}$ is injective.*

*Proof.* Apply Lemma 2.1 to the pair of morphisms (2.9) using the injectivity of $h_*(g^{\flat})\colon \mathbb{G}_{m,Y} \to h_*(g_*\mathbb{G}_{m,X})$. □

If $G$ is an $S$-group scheme, the presheaf of groups

$$G^* = \underline{\mathrm{Hom}}_{S\text{-gr}}(G, \mathbb{G}_{m,S}) \tag{2.11}$$

is a sheaf on $S_{\text{ét}}$ such that

$$G^*(S) = \mathrm{Hom}_{S\text{-gr}}(G, \mathbb{G}_{m,S}). \tag{2.12}$$

See [SGA3$_{\text{new}}$, IV, Corollary 4.5.13 and Proposition 6.3.1(iii)].

**Lemma 2.3.** *Let $S$ be a reduced scheme and let $G$ be a flat $S$-group scheme locally of finite presentation with smooth and connected maximal fibers. Then there exists a canonical isomorphism of étale sheaves on $S$*

$$\omega_G\colon \mathrm{U}_{G/S} \xrightarrow{\sim} G^*, \tag{2.13}$$

*where $\mathrm{U}_{G/S}$ and $G^*$ are given by* (2.10) *and* (2.11), *respectively.*

*Proof.* See [GA2, Lemma 4.8]. □

If $X$ is a scheme, the etale cohomology group $\mathrm{Pic}\, X = H^1(X_{\text{ét}}, \mathbb{G}_{m,X})$ will be identified with the group of isomorphism classes of right $\mathbb{G}_{m,X}$-torsors over $X$ with respect to either the etale or fppf topologies on $X$. See [MiEt, Theorem 4.9, p. 124]. A morphism of schemes $g\colon X \to Y$ induces a morphism of abelian groups

$$\mathrm{Pic}\, g\colon \mathrm{Pic}\, Y \to \mathrm{Pic}\, X, [E] \mapsto [E \times_Y X], \tag{2.14}$$

where $[E]$ denotes the isomorphism class of the right $\mathbb{G}_{m,Y}$-torsor $E$ over $Y$ and $[E \times_Y X]$ denotes the isomorphism class of the right $\mathbb{G}_{m,X}$-torsor $E \times_Y X$ over $X$.

Next we write $f^{(p)}\colon \mathbb{G}_{m,S} \to f_*\mathbb{G}_{m,X}$ for the morphism of abelian presheaves on $S_{\text{fl}}$ such that, for any object $T \to S$ of $S_{\text{fl}}$, $f^{(p)}(T)\colon \mathbb{G}_{m,S}(T) \to (f_*\mathbb{G}_{m,X})(T) = \mathbb{G}_{m,S}(X_T)$ is the canonical map induced by the projection $X_T \to T$. Now consider the following abelian presheaf on $S_{\text{fl}}$

$$\mathcal{U}_{X/S} = \mathrm{Coker}\left[\mathbb{G}_{m,S} \xrightarrow{f^{(p)}} f_*\mathbb{G}_{m,X}\right]$$

and set

$$\mathcal{U}_S(X) = \mathcal{U}_{X/S}(S) = \mathrm{Coker}[\mathbb{G}_{m,S}(S) \to \mathbb{G}_{m,S}(X)]. \tag{2.15}$$

Thus we obtain an abelian presheaf on $(\mathrm{Sch}/S)$

$$\mathcal{U}_S\colon (\mathrm{Sch}/S) \to \mathbf{Ab},\ (X \to S) \mapsto \mathcal{U}_S(X). \tag{2.16}$$

**Remark 2.4.** If $\mathcal{U}_{X/S}$ is regarded as an abelian presheaf on $S_{\text{ét}}$, then the etale sheaf on $S$ associated to $\mathcal{U}_{X/S}$ is $\mathrm{U}_{X/S}$ (2.10).

If $f$ is schematically dominant, then $f^{(p)}$ is an injective morphism of abelian presheaves on $S_{\mathrm{fl}}$ [GA2, proof of Lemma 2.4]. Thus, in this case, there exists a canonical exact sequence of abelian presheaves on $S_{\mathrm{fl}}$

$$1 \to \mathbb{G}_{m,S} \xrightarrow{f^{(p)}} f_*\mathbb{G}_{m,X} \to \mathcal{U}_{X/S} \to 1. \tag{2.17}$$

Further, by [GA2, comments after Lemma 2.5], there exists a canonical exact sequence of abelian groups

$$0 \to \mathbb{G}_{m,S}(S) \to \mathbb{G}_{m,S}(X) \to \mathrm{U}_{X/S}(S) \to \operatorname{Pic} S \xrightarrow{\operatorname{Pic} f} \operatorname{Pic} X,$$

where $\mathrm{U}_{X/S}$ is the etale sheaf (2.10). The preceding sequence induces an exact sequence of abelian groups

$$0 \to \mathcal{U}_S(X) \to \mathrm{U}_{X/S}(S) \to \operatorname{Pic} S \xrightarrow{\operatorname{Pic} f} \operatorname{Pic} X.$$

Thus, if $\operatorname{Pic} f$ is injective (e.g., $f$ has a section), then $\mathcal{U}_S(X) = \mathrm{U}_{X/S}(S)$. In particular, if $G$ is an $S$-group scheme, then $\mathcal{U}_S(G) = \mathrm{U}_{G/S}(S)$ and the following statement is immediate from Lemma 2.3.

**Lemma 2.5.** *Let $S$ be a reduced scheme and let $G$ be a flat $S$-group scheme locally of finite presentation with smooth and connected maximal fibers. Then there exists a canonical isomorphism of abelian groups*

$$\mathcal{U}_S(G) \xrightarrow{\sim} G^*(S),$$

*where the groups $\mathcal{U}_S(G)$ and $G^*(S)$ are given by* (2.15) *and* (2.12), *respectively.*

Now let $\mathrm{Pic}_{X/S}$ be the (etale) *relative Picard functor of $X$ over $S$*, i.e., the etale sheaf on $S$ associated to the abelian presheaf $(\mathrm{Sch}/S) \to \mathbf{Ab}$, $(T \to S) \mapsto \operatorname{Pic} X_T$. Then

$$\mathrm{Pic}_{X/S} = R^1_{\text{ét}} f_*\mathbb{G}_{m,X}.$$

If $g\colon X \to Y$ is a morphism of $S$-schemes and $T$ is an $S$-scheme, then the canonical maps $\operatorname{Pic} g_T\colon \operatorname{Pic} Y_T \to \operatorname{Pic} X_T$ (2.14) induce a morphism $\operatorname{Pic}_S(g)\colon \mathrm{Pic}_{Y/S} \to \mathrm{Pic}_{X/S}$ in $S^{\sim}_{\text{ét}}$. Thus we obtain a contravariant functor

$$\operatorname{Pic}_S\colon (\mathrm{Sch}/S) \to S^{\sim}_{\text{ét}},\ (X \to S) \mapsto \mathrm{Pic}_{X/S}.$$

We will also need to consider the abelian group

$$\mathrm{NPic}(X/S) = \operatorname{Coker}\left[\operatorname{Pic} S \overset{\operatorname{Pic} f}{\to} \operatorname{Pic} X\right] \tag{2.18}$$

and the associated abelian presheaf on $(\mathrm{Sch}/S)$

$$\mathrm{NPic}_S\colon (\mathrm{Sch}/S) \to \mathbf{Ab},\ (X \to S) \mapsto \mathrm{NPic}(X/S). \tag{2.19}$$

**Remark 2.6.** If $S$ is a strictly local scheme and $X \to S$ is quasi-compact and quasi-separated, then $\mathrm{Pic}_{X/S}(S) = \operatorname{Pic} X$. See [T, Lemma 6.2.3, p. 124, and Theorem 6.4.1, p. 128].

Next we recall from [GA3, §3] the definition of the units-Picard complex of a morphism of schemes $f\colon X \to S$. Let $A^\bullet$ be any representative of $\mathbb{R}f_*\mathbb{G}_{m,X} \in D(S_{\text{ét}})$ and consider the following composition of morphisms in $C^b(S_{\text{ét}})$:

$$f^\natural\colon \mathbb{G}_{m,S} \overset{f^\flat}{\to} f_*\mathbb{G}_{m,X} \simeq \tau_{\le 0} A^\bullet \xrightarrow{i^1} \tau_{\le 1} A^\bullet,$$

where $i^1$ is the first map in the distinguished triangle (2.2) for $n = 1$. The *relative units-Picard complex of $X$ over $S$* is the following (well-defined) object of $D^b(S_{\text{ét}})$:

$$\mathrm{UPic}_{X/S} = C^\bullet(f^\natural)[1]. \tag{2.20}$$

There exists a distinguished triangle in $D^b(S_{\text{ét}})$

$$\mathrm{Pic}_{X/S}[-1] \to \mathrm{RU}_{X/S}[1] \to \mathrm{UPic}_{X/S} \to \mathrm{Pic}_{X/S}, \tag{2.21}$$

where $\mathrm{RU}_{X/S}$ is the complex (2.8).

If $f: X \to S$ is schematically dominant, then $f^{\flat}$ is injective and therefore $\mathrm{RU}_{X/S} = \mathrm{U}_{X/S}$ in $D^b(S_{\text{ét}})$. In this case (2.21) is a distinguished triangle

$$\mathrm{Pic}_{X/S}[-1] \to \mathrm{U}_{X/S}[1] \to \mathrm{UPic}_{X/S} \to \mathrm{Pic}_{X/S}. \tag{2.22}$$

Consequently, $H^r(\mathrm{UPic}_{X/S}) = 0$ for $r \neq -1, 0$,

$$H^{-1}(\mathrm{UPic}_{X/S}) = \mathrm{U}_{X/S} \tag{2.23}$$

and

$$H^0(\mathrm{UPic}_{X/S}) = \mathrm{Pic}_{X/S}. \tag{2.24}$$

If $X \xrightarrow{g} Y \xrightarrow{h} S$ are morphisms of schemes, then $g^{\flat}: \mathbb{G}_{m,Y} \to g_*\mathbb{G}_{m,X}$ induces a morphism $g': \tau_{\leq 1}\mathbb{R}h_*\mathbb{G}_{m,Y} \to \tau_{\leq 1}\mathbb{R}(h \circ g)_*\mathbb{G}_{m,X}$ in $D^b(S_{\text{ét}})$ such that $g' \circ h^{\natural} = (h \circ g)^{\natural}$. Thus $g'$ induces a morphism $\mathrm{UPic}_S(g): \mathrm{UPic}_{Y/S} \to \mathrm{UPic}_{X/S}$ in $D^b(S_{\text{ét}})$ and we obtain a contravariant functor

$$\mathrm{UPic}_S: (\mathrm{Sch}/S) \to D^b(S_{\text{ét}}), (X \to S) \to \mathrm{UPic}_{X/S},$$

such that $H^{-1}(\mathrm{UPic}_S) = H^0(\mathrm{RU}_S) = \mathrm{U}_S$ by (2.21). Further, if $h$ and $h \circ g$ are schematically dominant, then the following diagram in $D^b(S_{\text{ét}})$ commutes

$$\begin{array}{ccc} \mathrm{U}_{Y/S}[1] & \xrightarrow{\mathrm{U}_S(g)[1]} & \mathrm{U}_{X/S}[1] \\ \downarrow & & \downarrow \\ \mathrm{UPic}_{Y/S} & \xrightarrow{\mathrm{UPic}_S(g)} & \mathrm{UPic}_{X/S}, \end{array} \tag{2.25}$$

where the vertical arrows are those in (2.22).

Next let $\mathrm{Br}\, X = H^2(X_{\text{ét}}, \mathbb{G}_{m,X})$ be the cohomological Brauer group of $X$. We will write $\mathrm{Br}_{X/S}$ for the etale sheaf on $S$ associated to the presheaf $(T \to S) \mapsto \mathrm{Br}\, X_T$, i.e.,

$$\mathrm{Br}_{X/S} = R^2_{\text{ét}} f_*\mathbb{G}_{m,X}.$$

There exists a canonical morphism of abelian groups $\mathrm{Br}\, X \to \mathrm{Br}_{X/S}(S)$[1] and we set

$$\mathrm{Br}_1(X/S) = \mathrm{Ker}\left[\mathrm{Br}\, X \to \mathrm{Br}_{X/S}(S)\right]. \tag{2.26}$$

Note that, if $f = 1_S: S \to S$, then $\mathrm{Br}_1(S/S) = \mathrm{Br}\, S$. We obtain an abelian presheaf on $(\mathrm{Sch}/S)$

$$\mathrm{Br}_{1,S}: (\mathrm{Sch}/S) \to \mathbf{Ab}, (X \to S) \mapsto \mathrm{Br}_1(X/S).$$

If $h$ is a morphism of $S$-schemes, we will write $\mathrm{Br}_1 h$ for $\mathrm{Br}_{1,S}(h)$.

Next, since $f: X \to S$ is a morphism of $S$-schemes and $\mathrm{Br}_1(S/S) = \mathrm{Br}\, S$, we may consider

$$\mathrm{Br}_{\mathrm{a}}(X/S) = \mathrm{Coker}\left[\mathrm{Br}\, S \xrightarrow{\mathrm{Br}_1 f} \mathrm{Br}_1(X/S)\right] \tag{2.27}$$

and the corresponding abelian presheaf on $(\mathrm{Sch}/S)$

$$\mathrm{Br}_{\mathrm{a},S}: (\mathrm{Sch}/S) \to \mathbf{Ab}, (X \to S) \mapsto \mathrm{Br}_{\mathrm{a}}(X/S). \tag{2.28}$$

If $h$ is a morphism of $S$-schemes, we will write $\mathrm{Br}_{\mathrm{a}} h$ for $\mathrm{Br}_{\mathrm{a},S}(h)$. Note that

$$\mathrm{Br}_{\mathrm{a}}(S/S) = 0. \tag{2.29}$$

[1] This is an instance of the canonical adjoint morphism $P(S) \to P^{\#}(S)$, where $P$ is a presheaf of abelian groups on $(\mathrm{Sch}/S)$ and $P^{\#}$ is its associated etale sheaf [T, Remark, p. 46].

The groups (2.26) and (2.27) are related by an exact sequence

$$\operatorname{Br} S \xrightarrow{\operatorname{Br}_1 f} \operatorname{Br}_1(X/S) \xrightarrow{c_{(X)}} \operatorname{Br}_{\mathrm{a}}(X/S) \to 0, \tag{2.30}$$

where $c_{(X)}$ is the canonical projection.

If $f$ has a section $\sigma : S \to X$, then $\operatorname{Br}_1\sigma : \operatorname{Br}_1(X/S) \to \operatorname{Br}_1(S/S) = \operatorname{Br} S$ is a retraction of $\operatorname{Br}_1 f$ that splits (2.30). Thus, if we define

$$\operatorname{Br}_\sigma(X/S) = \operatorname{Ker}\Big[\operatorname{Br}_1(X/S) \xrightarrow{\operatorname{Br}_1\sigma} \operatorname{Br} S\Big], \tag{2.31}$$

then

$$\operatorname{Br}_1(X/S) = \operatorname{Im}\operatorname{Br}_1 f \oplus \operatorname{Br}_\sigma(X/S) \tag{2.32}$$

and the restriction of $c_{(X)}$ (2.30) to $\operatorname{Br}_\sigma(X/S) \subseteq \operatorname{Br}_1(X/S)$ is an isomorphism of abelian groups

$$c_{(X),\sigma} : \operatorname{Br}_\sigma(X/S) \xrightarrow{\sim} \operatorname{Br}_{\mathrm{a}}(X/S). \tag{2.33}$$

Next let $g : Y \to S$ be another morphism of schemes with section $\tau : S \to Y$ and let $h : X \to Y$ be an $S$-morphism such that $h \circ \sigma = \tau$. Then the restriction of $\operatorname{Br}_1 h$ to $\operatorname{Im}\operatorname{Br}_1 g \subseteq \operatorname{Br}_1(Y/S)$ is an isomorphism of abelian groups

$$\operatorname{Im}\operatorname{Br}_1 g \xrightarrow{\sim} \operatorname{Im}\operatorname{Br}_1 f \tag{2.34}$$

which fits into a commutative diagram

$$\begin{array}{ccc} \operatorname{Im}\operatorname{Br}_1 g & \xrightarrow{\simeq} & \operatorname{Im}\operatorname{Br}_1 f \\ & \nwarrow_{\simeq} \quad {}_{\simeq}\nearrow & \\ & \operatorname{Br}' S & \end{array}$$

On the other hand, the restriction of $\operatorname{Br}_1 h$ to $\operatorname{Br}_\tau(Y/S) \subseteq \operatorname{Br}_1(Y/S)$ induces a map

$$\operatorname{Br}_{\sigma,\tau} h : \operatorname{Br}_\tau(Y/S) \to \operatorname{Br}_\sigma(X/S) \tag{2.35}$$

such that (by (2.32) and (2.34))

$$\operatorname{Ker}\operatorname{Br}_{\sigma,\tau} h = \operatorname{Ker}\operatorname{Br}_1 h \tag{2.36}$$

and the following diagram commutes

$$\begin{array}{ccc} \operatorname{Br}_\tau(Y/S) & \xrightarrow{\operatorname{Br}_{\sigma,\tau} h} & \operatorname{Br}_\sigma(X/S) \\ {\scriptstyle c_{(Y),\tau}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle c_{(X),\sigma}} \\ \operatorname{Br}_{\mathrm{a}}(Y/S) & \xrightarrow{\operatorname{Br}_{rma} h} & \operatorname{Br}_{\mathrm{a}}(X/S), \end{array} \tag{2.37}$$

where the vertical maps are the isomorphisms (2.33).

We now discuss products. If $f : X \to S$ and $g : Y \to S$ are morphisms of schemes, then the canonical projection morphisms $p_X : X \times_S Y \to X$ and $p_Y : X \times_S Y \to Y$ define a morphism of abelian groups

$$\mathbb{G}_{m,S}(X) \oplus \mathbb{G}_{m,S}(Y) \to \mathbb{G}_{m,S}(X \times_S Y), \; (u, v) \mapsto p_X^{(0)}(u) \cdot p_Y^{(0)}(v), \tag{2.38}$$

where $p_X^{(0)} : \mathbb{G}_{m,S}(X) \to \mathbb{G}_{m,S}(X \times_S Y)$ is the pullback map and $p_Y^{(0)}$ is defined similarly. Further, there exist canonical morphisms of abelian groups

$$\operatorname{Pic} X \oplus \operatorname{Pic} Y \to \operatorname{Pic}(X \times_S Y), \; ([E], [F]) \mapsto (\operatorname{Pic} p_X)[E] + (\operatorname{Pic} p_Y)[F],$$

and

$$\operatorname{Br} X \oplus \operatorname{Br} Y \to \operatorname{Br}'(X \times_S Y), \; (a, b) \mapsto (\operatorname{Br} p_X)(a) + (\operatorname{Br} p_Y)(b).$$

The preceding maps induce morphisms of abelian groups

$$\mathcal{U}_S(X) \oplus \mathcal{U}_S(Y) \to \mathcal{U}_S(X \times_S Y), \tag{2.39}$$

$$\mathrm{NPic}(X/S) \oplus \mathrm{NPic}(Y/S) \to \mathrm{NPic}(X \times_S Y/S), \tag{2.40}$$

$$\mathrm{Br}_1(X/S) \oplus \mathrm{Br}_1(Y/S) \to \mathrm{Br}_1(X \times_S Y/S) \tag{2.41}$$

and

$$\mathrm{Br}_{\mathrm{a}}(X/S) \oplus \mathrm{Br}_{\mathrm{a}}(Y/S) \to \mathrm{Br}_{\mathrm{a}}(X \times_S Y/S). \tag{2.42}$$

Further, there exists a canonical morphism in $D^b(S_{\text{ét}})$ (see [GA3, beginning of §4])

$$\mathrm{UPic}_{X/S} \oplus \mathrm{UPic}_{Y/S} \to \mathrm{UPic}_{X \times_S Y/S}. \tag{2.43}$$

Now, if $g\colon Y \to S$ has a section $\tau\colon S \to Y$ and $\mathrm{Br}_\tau(Y/S) \subseteq \mathrm{Br}_1(Y/S)$ is the group (2.31) associated to $\tau$, we write

$$\zeta_{X,Y}\colon \mathrm{Br}_1(X/S) \oplus \mathrm{Br}_\tau(Y/S) \to \mathrm{Br}_1(X \times_S Y/S) \tag{2.44}$$

for the restriction of (2.41) to $\mathrm{Br}_1(X/S) \oplus \mathrm{Br}_\tau(Y/S) \subseteq \mathrm{Br}_1(X/S) \oplus \mathrm{Br}_1(Y/S)$.

$$\text{If} \quad i_{\mathrm{Br}_1(X/S)}\colon \mathrm{Br}_1(X/S) \to \mathrm{Br}_1(X/S) \oplus \mathrm{Br}_\tau(Y/S),\, a \mapsto (a, 0),$$

is the canonical embedding, then

$$\mathrm{Br}_1\, p_X = \zeta_{X,Y} \circ i_{\mathrm{Br}_1(X/S)}. \tag{2.45}$$

Now we define

$$\zeta^0_{X,Y}\colon \mathrm{Br}_1(X/S) \oplus \mathrm{Br}_{\mathrm{a}}(Y/S) \to \mathrm{Br}_1(X \times_S Y/S) \tag{2.46}$$

by the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{Br}_1(X/S) \oplus \mathrm{Br}_{\mathrm{a}}(Y/S) & \xrightarrow{\ \zeta^0_{X,Y}\ } & \mathrm{Br}_1(X \times_S Y/S), \\
{\scriptstyle (1_{\mathrm{Br}_1(X/S)},\, c_{(Y),\tau})}\uparrow \simeq & \nearrow{\scriptstyle \zeta_{X,Y}} & \\
\mathrm{Br}_1(X/S) \oplus \mathrm{Br}_\tau(Y/S) & &
\end{array}$$

where $c_{(Y),\tau}\colon \mathrm{Br}_\tau(Y/S) \xrightarrow{\sim} \mathrm{Br}_{\mathrm{a}}(Y/S)$ is the map (2.33) defined by $\tau$. Clearly, $\zeta^0_{X,Y}$ is an isomorphism if, and only if, $\zeta_{X,Y}$ is an isomorphism. If this is the case, then (2.45) shows that

$$p_{\mathrm{Br}_1(X/S)} \circ \zeta^{-1}_{X,Y} \circ \mathrm{Br}_1\, p_X = 1_{\mathrm{Br}_1(X/S)} \tag{2.47}$$

$$p_{\mathrm{Br}_\tau(Y/S)} \circ \zeta^{-1}_{X,Y} \circ \mathrm{Br}_1\, p_X = 0, \tag{2.48}$$

where $p_{\mathrm{Br}_1(X/S)}\colon \mathrm{Br}_1(X/S) \oplus \mathrm{Br}_\tau(Y/S) \to \mathrm{Br}_1(X/S)$ is the canonical projection and $p_{\mathrm{Br}_\tau(Y/S)}$ is defined similarly.

**Lemma 2.7.** *If the map* $\zeta^0_{X,Y}$ (2.46) *is an isomorphism, then the map* (2.42)

$$\mathrm{Br}_{\mathrm{a}}(X/S) \oplus \mathrm{Br}_{\mathrm{a}}(Y/S) \to \mathrm{Br}_{\mathrm{a}}(X \times_S Y/S)$$

*is an isomorphism as well.*

*Proof.* The composition

$$\mathrm{Br}\, S \overset{(\mathrm{Br}_1 f, 0)}{\to} \mathrm{Br}_1(X/S) \oplus \mathrm{Br}_{\mathrm{a}}(Y/S) \overset{\zeta^0_{X,Y}}{\to} \mathrm{Br}_1(X \times_S Y/S)$$

equals $(\mathrm{Br}_1\, p_X) \circ (\mathrm{Br}_1 f) = \mathrm{Br}_1(f \times_S g)$. Thus Lemma 2.1 applied to the above pair of maps yields an exact sequence of abelian groups

$$\mathrm{Ker}\, \zeta^0_{X,Y} \to \mathrm{Br}_{\mathrm{a}}(X/S) \oplus \mathrm{Br}_{\mathrm{a}}(Y/S) \to \mathrm{Br}_{\mathrm{a}}(X \times_S Y/S) \to \mathrm{Coker}\, \zeta^0_{X,Y},$$

where the middle arrow is the map (2.42). The lemma is now clear. □

2.4 Additivity theorems

**Proposition 2.8.** *Let $S$ be a reduced scheme and let $f: X \to S$ and $g: Y \to S$ be faithfully flat morphisms locally of finite presentation with reduced and connected maximal geometric fibers. Assume that $f$ has an étale quasi-section and $g$ has a section or, symmetrically, that $f$ has a section and $g$ has an étale quasi-section. Then the canonical map* (2.39)

$$\mathcal{U}_S(X) \oplus \mathcal{U}_S(Y) \to \mathcal{U}_S(X \times_S Y)$$

*is an isomorphism of abelian groups.*

*Proof.* Since $f, g$ and $f \times_S g$ are schematically dominant, the following diagram of abelian groups, whose exact rows are induced by (2.17), commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_{m,S}(S) \oplus \mathbb{G}_{m,S}(S) & \xrightarrow{\iota} & \mathbb{G}_{m,S}(X) \oplus \mathbb{G}_{m,S}(Y) & \longrightarrow & \mathcal{U}_S(X) \oplus \mathcal{U}_S(Y) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle (\cdot)} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{G}_{m,S}(S) & \longrightarrow & \mathbb{G}_{m,S}(X \times_S Y) & \longrightarrow & \mathcal{U}_S(X \times_S Y) & \longrightarrow & 1. \end{array}$$

The left-hand vertical map in the above diagram is the (surjective) multiplication homomorphism. The middle and right-hand vertical arrows are the maps (2.38) and (2.39), respectively. Now, since $f \times_S g$ has an etale quasi-section, the map labeled $\iota$ in the above diagram induces an isomorphism between the kernels of the first two vertical arrows. See [GA2, Corollary 4.5 and its proof]. Further, the cokernel of the middle vertical map is canonically isomorphic to $\operatorname{Ker}\operatorname{Pic} f \cap \operatorname{Ker}\operatorname{Pic} g \subseteq \operatorname{Pic} S$. Thus the diagram yields an exact sequence of abelian groups

$$1 \to \mathcal{U}_S(X) \oplus \mathcal{U}_S(Y) \to \mathcal{U}_S(X \times_S Y) \to \operatorname{Ker}\operatorname{Pic} f \cap \operatorname{Ker}\operatorname{Pic} g \to 1.$$

Since either $f$ or $g$ has a section, either $\operatorname{Ker}\operatorname{Pic} f$ or $\operatorname{Ker}\operatorname{Pic} g$ is zero, whence the proposition follows. □

The next statement collects together and reformulates results obtained in [GA3], with the exception of assertion (c) (v″), which is new and generalizes a result of Sansuc [San81, Lemma 6.6(ii)] (the separable indices $I(X_\eta)$, $I(Y_\eta)$ and the etale index $I(f \times_S g)$ that appear below have been defined at the end of Subsection 2.1).

**Proposition 2.9.** *Let $S$ be a locally noetherian normal scheme and let $f: X \to S$ and $g: Y \to S$ be faithfully flat morphisms locally of finite type. Assume that the following conditions hold:*

(i) *$f \times_S g$ has an étale quasi-section,*
(ii) *for every étale and surjective morphism $T \to S$, $X_T$, $Y_T$ and $X_T \times_T Y_T$ are locally factorial,*
(iii) *for every point $s \in S$ of codimension $\leq 1$, the fibers $X_s$ and $Y_s$ are geometrically integral, and*
(iv) *for every maximal point $\eta$ of $S$, $\gcd(I(X_\eta), I(Y_\eta)) = 1$ and*

$$\operatorname{Pic}(X_\eta^{\mathrm{s}} \times_{k(\eta)^{\mathrm{s}}} Y_\eta^{\mathrm{s}})^{\Gamma(\eta)} = (\operatorname{Pic} X_\eta^{\mathrm{s}})^{\Gamma(\eta)} \oplus (\operatorname{Pic} Y_\eta^{\mathrm{s}})^{\Gamma(\eta)}, \tag{2.49}$$

*where $\Gamma(\eta) = \operatorname{Gal}(k(\eta)^{\mathrm{s}}/k(\eta))$.*

*Then*

(a) *the canonical map* (2.40)

$$\operatorname{NPic}(X/S) \oplus \operatorname{NPic}(Y/S) \to \operatorname{NPic}(X \times_S Y/S)$$

*is an isomorphism of abelian groups,*

(b) *the canonical morphism* (2.43)

$$\operatorname{UPic}_{X/S} \oplus \operatorname{UPic}_{Y/S} \to \operatorname{UPic}_{X \times_S Y/S}$$

*is an isomorphism in $D^b(S_{\text{ét}})$, and*

(c) *if, in addition, either*

(v) $H^3(S_{\text{ét}}, \mathbb{G}_{m,S}) = 0$, *or*

(v′) *the étale index* $I(f \times_S g)$ *is defined and is equal to* 1, *or*

(v″) $g$ *has a section,*

*then the canonical map* (2.42)

$$\mathrm{Br_a}(X/S) \oplus \mathrm{Br_a}(Y/S) \to \mathrm{Br_a}(X \times_S Y/S)$$

*is an isomorphism of abelian groups.*

*Proof.* [Cf. [San81, proof of Lemma 6.6]] For (a) and (b), see [GA3, Propositions 2.7 and 4.4, respectively]. By [GA3, Corollary 4.5], is suffices to check that (2.42) is an isomorphism of abelian groups when (v″) holds, i.e., $g$ has a section $\tau: S \to Y$. In this case we will show that the map $\zeta^0_{X,Y}$ (2.46) is an isomorphism, which will show that (2.42) is an isomorphism by Lemma 2.7.

By [GA3, Proposition 3.6(iv)], there exists a canonical exact sequence of abelian groups

$$\mathrm{Pic}_{Y/S}(S) \to H^2(S_{\text{ét}}, \mathrm{U}_{Y/S}) \to \mathrm{Br_a}(Y/S) \to H^1(S_{\text{ét}}, \mathrm{Pic}_{Y/S}) \to H^3(S_{\text{ét}}, \mathrm{U}_{Y/S}). \tag{2.50}$$

Further, if $(h, Z) = (f, X)$ or $(f \times_S g, X \times_S Y)$, then the Cartan-Leray spectral sequence associated to $h: Z \to S$

$$H^r(S_{\text{ét}}, R^s h_* \mathbb{G}_{m,Z}) \Rightarrow H^{r+s}(Z_{\text{ét}}, \mathbb{G}_{m,Z})$$

induces an exact sequence of abelian groups [MiEt, p. 309, line 8]

$$\mathrm{Pic}_{Z/S}(S) \to H^2(S_{\text{ét}}, h_*\mathbb{G}_{m,Z}) \to \mathrm{Br}'_1(Z/S) \to H^1(S_{\text{ét}}, \mathrm{Pic}_{Z/S}) \to H^3(S_{\text{ét}}, h_*\mathbb{G}_{m,Z}). \tag{2.51}$$

Next, by [GA2, Corollary 4.4], the given section $\tau$ of $g$ induces an isomorphism of etale sheaves on $S$

$$f_*\mathbb{G}_{m,X} \oplus \mathrm{U}_{Y/S} \xrightarrow{\sim} (f \times_S g)_*\mathbb{G}_{m,X\times_S Y}. \tag{2.52}$$

We now use the sequences (2.50) and (2.51) to form the following 5-column diagram of abelian groups with exact rows

$$\begin{array}{ccccc}
\mathrm{Pic}_{X/S}(S) \oplus \mathrm{Pic}_{Y/S}(S) & \longrightarrow & H^2(S_{\text{ét}}, f_*\mathbb{G}_{m,X}) \oplus H^2(S_{\text{ét}}, \mathrm{U}_{Y/S}) & \longrightarrow & \mathrm{Br}_1(X/S) \oplus \mathrm{Br_a}(Y/S) \\
\alpha \downarrow \simeq & & \beta \downarrow \simeq & & \downarrow \zeta^0_{X,Y} \\
\mathrm{Pic}_{X\times_S Y/S}(S) & \longrightarrow & H^2(S_{\text{ét}}, (f \times_S g)_*\mathbb{G}_{m,X\times_S Y}) & \longrightarrow & \mathrm{Br}_1(X \times_S Y/S)
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow H^1(S_{\text{ét}}, \mathrm{Pic}_{X/S}) \oplus H^1(S_{\text{ét}}, \mathrm{Pic}_{Y/S}) & \longrightarrow & H^3(S_{\text{ét}}, f_*\mathbb{G}_{m,X}) \oplus H^3(S_{\text{ét}}, \mathrm{U}_{Y/S}) \\
\gamma \downarrow \simeq & & \delta \downarrow \simeq \\
\longrightarrow H^1(S_{\text{ét}}, \mathrm{Pic}_{X\times_S Y/S}) & \longrightarrow & H^3(S_{\text{ét}}, (f \times_S g)_*\mathbb{G}_{m,X\times_S Y}).
\end{array}$$

The maps $\alpha$ and $\gamma$ are isomorphisms since, by (b), $\mathrm{Pic}_{X/S} \oplus \mathrm{Pic}_{Y/S} \to \mathrm{Pic}_{X\times_S Y/S}$, i.e., $H^0(\mathrm{UPic}_{X/S} \oplus \mathrm{UPic}_{Y/S}) \to H^0(\mathrm{UPic}_{X\times_S Y/S})$, is an isomorphism of etale sheaves on $S$. The maps $\beta$ and $\delta$ are induced by the isomorphism of etale sheaves (2.52). It follows from the definition of $\zeta^0_{X,Y}$ (2.46) and the proof of [GA2, Corollary 4.4] that the above diagram commutes. Thus the five lemma applied to the diagram shows that $\zeta^0_{X,Y}$ is an isomorphism, as claimed. □

**Remark 2.10.**

(a) The proof shows that if hypotheses (i)–(iv) and (v″) of the proposition hold, then the fact that

$$\mathrm{Br_a}(X/S) \oplus \mathrm{Br_a}(Y/S) \to \mathrm{Br_a}(X \times_S Y/S)$$

(2.42) is an isomorphism is a consequence of the fact that $\zeta^0_{X,Y}$ (2.46) (or, equivalently, $\zeta_{X,Y}$ (2.44)) is an isomorphism.

(b) If $S$ is locally noetherian and regular and $f: X \to S$ and $g: Y \to S$ are smooth and surjective, then hypothesis (ii) of the proposition holds. See [GA3, Remark 4.10(a)]. Further, since in this case $f \times_S g$ is also smooth and surjective, hypothesis (i) also holds by [EGA, $\text{IV}_4$, Corollary 17.16.3(ii)]. The identity (or, more precisely, canonical isomorphism) (2.49) holds in many cases of interest. See [GA2, Examples 5.9 and 5.12]. Finally, hypothesis (v) holds in the following cases (see [ADT, Remark II.2.2(a), p. 165]):

(1) $S$ is the spectrum of a global field.
(2) $S$ is a proper nonempty open subscheme of the spectrum of the ring of integers of a number field.
(3) $S$ is a nonempty open affine subscheme of a smooth, complete and irreducible curve over a finite field.

Recall that, if $k'/k$ is a field extension, a geometrically integral $k$-scheme $X$ is called *$k'$-rational* if the function field of $X \times_k \operatorname{Spec} k'$ is a purely transcendental extension of $k'$.

**Corollary 2.11.** *Let $S$ be a locally noetherian normal scheme and let $X$ and $G$ be faithfully flat $S$-schemes locally of finite type, where $G$ is an $S$-group scheme. Let $F: (\mathrm{Sch}/S) \to$ **Ab** denote either* $\mathrm{NPic}_S$ (2.19) *or* $\mathrm{Br}_{a,S}$ (2.28). *Assume that*

(i) *the structural morphism $X \to S$ has an étale quasi-section,*
(ii) *for every integer $i \geq 1$ and every étale and surjective morphism $T \to S$, $X_T$, $G_T^i$ and $X_T \times_T G_T^i$ are locally factorial,*
(iii) *for every point $s \in S$ of codimension $\leq 1$, the fibers $X_s$ and $G_s$ are geometrically integral, and*
(iv) *for every maximal point $\eta$ of $S$, $G_\eta$ is $k(\eta)^{\mathrm{s}}$-rational.*

*Then the maps $F(G^i) \oplus F(G) \to F(G^{i+1})$ and $F(X) \oplus F(G^i) \to F(X \times_S G^i)$* (2.40), (2.42) *are isomorphisms of abelian groups for every $i \geq 1$.*

*Proof.* By (ii) and (iii), $X_\eta$ is normal, geometrically integral and locally of finite type over $k(\eta)$. Further, by (iv), $G_\eta^i$ is $k(\eta)^{\mathrm{s}}$-rational for every integer $i \geq 1$. Thus, by [GA2, Example 5.9(a)], $\operatorname{Pic}(X_\eta^{\mathrm{s}} \times_{k(\eta)^{\mathrm{s}}} (G_\eta^i)^{\mathrm{s}}) = \operatorname{Pic} X_\eta^{\mathrm{s}} \oplus \operatorname{Pic}((G_\eta^i)^{\mathrm{s}})$, i.e., (2.49) holds. Similarly, $\operatorname{Pic}((G_\eta^i)^{\mathrm{s}} \times_{k(\eta)^{\mathrm{s}}} G_\eta^{\mathrm{s}}) = \operatorname{Pic}((G_\eta^i)^{\mathrm{s}}) \oplus \operatorname{Pic}(G_\eta^{\mathrm{s}})$. Therefore hypotheses (i)-(iv) and (v″) of the proposition hold for $X, G^i$ and $G^i, G$. The corollary for $F = \mathrm{NPic}_S$ (respectively, $F = \mathrm{Br}_{a,S}$) now follows from part (a) (respectively, (c)) of the proposition. □

**Remark 2.12.** There are many examples where hypothesis (iv) of the proposition holds, e.g., when $G$ is a reductive $S$-group scheme. See Proposition 4.1(i).

## 3. The units-Picard-Brauer sequence of a torsor

Let $S$ be a scheme, let $G$ and $Y$ be flat $S$-schemes locally of finite presentation, where $G$ is an $S$-group scheme, and let $G_Y = G \times_S Y$. A basic problem is to compute the Picard group of a $G_Y$-torsor $X$ over $Y$ in terms of the Picard groups of $Y$ and $G$. When $S = \operatorname{Spec} k$, where $k$ is a field, this problem was discussed by Sansuc in [San81, pp. 43-45], who used a simplicial method to obtain a units-Picard-Brauer exact sequence that relates the groups mentioned above. In this Section we generalize Sansuc's method to deduce, under appropriate conditions, a similar exact sequence over any locally noetherian normal scheme $S$.

### 3.1 A simplicial lemma

Let $\mathscr{C}$ be a full subcategory of $S_{\mathrm{fl}}$ which is stable under products and contains $1_S$. Further, let $F: \mathscr{C} \to$ **Ab** be an abelian presheaf on $\mathscr{C}$ such that $F(1_S) = 0$. If $X \to S$ is an object of $\mathscr{C}$ (respectively, if $\xi: X \to Y$ is a morphism in $\mathscr{C}$), we will write $F(X)$ (respectively, $F(\xi)$) for the abelian group $F(X \to S)$ (respectively, morphism of abelian groups $F(Y) \to F(X)$). The canonical $\mathscr{C}$-morphisms $p_X: X \times_S Y \to X$ and $p_Y: X \times_S Y \to Y$ define a morphism in **Ab**

$$\psi_{X,Y} = \psi_{F,X,Y}: F(X) \oplus F(Y) \to F(X \times_S Y), (a, b) \mapsto F(p_X)(a) + F(p_Y)(b). \tag{3.1}$$

If $Z \to S$ is a third object of $\mathscr{C}$ and we identify $(X \times_S Y) \times_S Z$ and $X \times_S (Y \times_S Z)$ (and write $X \times_S Y \times_S Z$ for the latter), then the following diagram commutes:

$$\begin{array}{ccc} F(X) \oplus F(Y) \oplus F(Z) & \xrightarrow{1_{F(X)} \oplus \psi_{Y,Z}} & F(X) \oplus F(Y \times_S Z) \\ \downarrow{\scriptstyle \psi_{X,Y} \oplus 1_{F(Z)}} & & \downarrow{\scriptstyle \psi_{X, Y \times_S Z}} \\ F(X \times_S Y) \oplus F(Z) & \xrightarrow{\psi_{X \times_S Y, Z}} & F(X \times_S Y \times_S Z). \end{array} \tag{3.2}$$

Now let $h: Y \to S$ be an object of $\mathscr{C}$, write $\mathscr{C}_{/Y}$ for the category of $Y$-objects of $\mathscr{C}$ and let $\xi: X \to Y$ be a faithfully flat morphism locally of finite presentation. Thus $\xi$ is an object of $\mathscr{C}_{/Y}$ which has a section locally for the fppf topology on $Y$ [EGA, IV$_4$, Corollary 17.16.2]. Now, for every $i \geq 0$, let $X^i_{/Y}$ be defined recursively by $X^0_{/Y} = Y$ and $X^{i+1}_{/Y} = X \times_Y X^i_{/Y}$. It is well-known that $\{X^{i+1}_{/Y}\}_{i \geq 0}$ is a simplicial $Y$-scheme with degeneracy and face maps

$$X^{i+1}_{/Y} \to X^{i+2}_{/Y}, (x_0, \dots, x_i) \mapsto (x_0, \dots, x_j, x_j, \dots, x_i), \quad 0 \leq j \leq i, \tag{3.3}$$

and[2]

$$\partial^j_{i+1}: X^{i+2}_{/Y} \to X^{i+1}_{/Y}, (x_0, \dots, x_{i+1}) \mapsto (x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_{i+1}), \quad 0 \leq j \leq i+1. \tag{3.4}$$

Let

$$h^*F: \mathscr{C}_{/Y} \to \mathbf{Ab}, (Z \to Y) \mapsto F(Z \to Y \xrightarrow{h} S),$$

be the restriction of $F$ to $\mathscr{C}_{/Y}$. Since $h^*F$ is an abelian presheaf on $\mathscr{C}_{/Y}$, we may consider the Čech cohomology groups of $h^*F$ relative to the fppf covering $\xi: X \to Y$, i.e., the cohomology groups $\check{H}^i(X/Y, h^*F)$ of the complex of abelian groups $\{F(X^{i+1}_{/Y}), \partial^{i+1}\}_{i \geq 0}$, where $\partial^{i+1} = \sum_{j=0}^{i+1} (-1)^j F(\partial^j_{i+1}): F(X^{i+1}_{/Y}) \to F(X^{i+2}_{/Y})$[3].

Next let $\pi: G \to S$ be a group object of $\mathscr{C}$ with unit section $\varepsilon: S \to G$. We will need the following fact: since $F(S) = 0$, both $F(\pi)$ and $F(\varepsilon)$ are the zero morphism of $\mathbf{Ab}$. Now assume that the $Y$-scheme $X$ is a right (fppf) $G_Y$-torsor over $Y$ with action $\varsigma': X \times_Y G_Y \to X$. We will write

$$\varsigma: X \times_S G \to X, (x, g) \mapsto xg, \tag{3.5}$$

for the composite $S$-morphism $X \times_S G \xrightarrow{\sim} X \times_Y G_Y \xrightarrow{\varsigma'} X$. Now set $\beta_0 = 1_X$ and note that, for every $i \geq 0$, the morphism of $S$-schemes

$$\beta_i: X \times_S G^i \xrightarrow{\sim} X^{i+1}_{/Y}, (x, g_1, \dots, g_i) \mapsto (x, xg_1, \dots, xg_1 \cdots g_i), \tag{3.6}$$

is an isomorphism. The following holds:

$$\partial^0_1 \circ \beta_1 = \varsigma \tag{3.7}$$

and

$$\partial^1_1 \circ \beta_1 = p_X, \tag{3.8}$$

where $\partial^0_1$ and $\partial^1_1$ are given by (3.4) and $\varsigma$ is given by (3.5).

**Lemma 3.1.** *The map $\{\beta_i\}_{i \geq 0}: \{X \times_S G^i\}_{i \geq 0} \to \{X^{i+1}_{/Y}\}_{i \geq 0}$ is an isomorphism of simplicial $S$-schemes.*

*Proof.* The simplicial $S$-scheme structure on $\{X \times_S G^i\}_{i \geq 0}$ is defined as follows. For every $j$ such that $0 \leq j \leq i$, let

$$v^j_i = (1_{G^j}, \varepsilon, 1_{G^{i-j}})_S: G^i \to G^{i+1}, (g_1, \dots, g_i) \mapsto (g_1, \dots, g_j, 1, g_{j+1}, \dots, g_i). \tag{3.9}$$

[2]In several formulas below, ordered tuples of the form $y_k, \dots, y_r$ with $k > r$ must be omitted for the formulas to make sense.

[3]Here we regard $X^{i+1}_{/Y}$ and $\partial^j_{i+1}$ as $S$-schemes and $S$-morphisms, respectively.

Then the degeneracy maps of $\{X \times_S G^i\}_{i \geq 0}$ are the maps

$$s_i^j = (1_X, v_i^j)_S \colon X \times_S G^i \to X \times_S G^{i+1}, (x, g_1, \ldots, g_i) \mapsto (x, g_1, \ldots, g_j, 1, g_{j+1}, \ldots, g_i). \tag{3.10}$$

Now, for $i \geq 1$ and $1 \leq j \leq i+1$, consider

$$w_{i+1}^j \colon G^{i+1} \to G^i, (g_1, \ldots, g_{i+1}) \mapsto (g_1, \ldots, g_{j-1}, g_j g_{j+1}, g_{j+2} \ldots, g_{i+1}). \tag{3.11}$$

We extend the above definition to the pair $(i, j) = (0, 1)$ by setting

$$w_1^1 = \pi : G \to S. \tag{3.12}$$

Then the face maps $d_{i+1}^j \colon X \times_S G^{i+1} \to X \times_S G^i$ of $\{X \times_S G^i\}_{i \geq 0}$ are given by $d_{i+1}^j = (1_X, w_{i+1}^j)_S$ if $1 \leq j \leq i+1$ and

$$d_{i+1}^0(x, g_1, \ldots, g_{i+1}) = (x g_1, g_2, \ldots, g_{i+1}). \tag{3.13}$$

Note that, if $i \geq 1$ and $1 \leq j \leq i+1$, then

$$d_{i+1}^j(x, g_1, \ldots, g_{i+1}) = (x, g_1, \ldots, g_{j-1}, g_j g_{j+1}, g_{j+2} \ldots, g_{i+1}). \tag{3.14}$$

It is not difficult to check that the maps $\beta_i$ (3.6) commute with the operators (3.3), (3.4), (3.10) and (3.13), in the sense of [May, Definition 9.1, p. 83], which yields the lemma. □

We now assume that the map $\psi_{X,G} \colon F(X) \oplus F(G) \to F(X \times_S G)$ (3.1) is an isomorphism of abelian groups and consider the composition

$$\varphi = \varphi_{F,X,G} \colon F(X) \xrightarrow{F(\varsigma)} F(X \times_S G) \xrightarrow{\psi_{X,G}^{-1}} F(X) \oplus F(G) \xrightarrow{p_{F(G)}} F(G), \tag{3.15}$$

where $\varsigma$ is the $S$-morphism (3.5) and $p_{F(G)}$ is the canonical projection.

**Lemma 3.2.** *If $\psi_{X,G} \colon F(X) \oplus F(G) \xrightarrow{\sim} F(X \times_S G)$ is an isomorphism, then there exists a cosimplicial abelian group structure on $\{F(X) \oplus F(G^i)\}_{i \geq 0}$ so that the map*

$$\{\psi_{X,G^i}\}_{i \geq 0} \colon \{F(X) \oplus F(G^i)\}_{i \geq 0} \to \{F(X \times_S G^i)\}_{i \geq 0} \tag{3.16}$$

*is a morphism of cosimplicial abelian groups.*

*Proof.* The codegeneracy maps of $\{F(X) \oplus F(G^i)\}_{i \geq 0}$ are

$$1_{F(X)} \oplus F(v_i^j) \colon F(X) \oplus F(G^{i+1}) \to F(X) \oplus F(G^i),$$

where $0 \leq j \leq i$ and the maps $v_i^j \colon G^i \to G^{i+1}$ are given by (3.9). The coface maps are

$$(p_{F(X)}, \alpha_j^{i+1}) \colon F(X) \oplus F(G^i) \to F(X) \oplus F(G^{i+1}), (a, b) \mapsto (a, \alpha_j^{i+1}(a, b)),$$

where $0 \leq j \leq i+1$, $p_{F(X)} \colon F(X) \oplus F(G^i) \to F(X)$ is the canonical projection and the maps $\alpha_j^{i+1} \colon F(X) \oplus F(G^i) \to F(G^{i+1})$ are defined by

$$\alpha_j^{i+1}(a, b) = \begin{cases} F(p_G^1)(\varphi(a)) + F(p_{G^i})(b) & \text{if } j = 0 \\ F(w_{i+1}^j)(b) & \text{if } 1 \leq j \leq i+1, \end{cases} \tag{3.17}$$

where $a \in F(X), b \in F(G^i)$, $p_G^1 \colon G^{i+1} \to G$ is the first projection, $\varphi$ is the map (3.15), $p_{G^i} \colon G^{i+1} \to G^i$ is the projection onto the last $i$ factors (when $i \geq 1$) and the maps $w_{i+1}^j \colon G^{i+1} \to G^i$ are given by (3.11) and (3.12).

To check that (3.16) is, indeed, an isomorphism of cosimplicial abelian groups, we need to check that the following diagrams commute:

$$\begin{array}{ccc} F(X)\oplus F(G^i) & \xrightarrow{\psi_{X,G^i}} & F(X\times_S G^i) \\ {\scriptstyle (p_{F(X)},\,\alpha_j^{i+1})}\downarrow & & \downarrow{\scriptstyle F(d_{i+1}^j)} \\ F(X)\oplus F(G^{i+1}) & \xrightarrow{\psi_{X,G^{i+1}}} & F(X\times_S G^{i+1}), \end{array} \tag{3.18}$$

where $0\le j\le i+1$, and

$$\begin{array}{ccc} F(X)\oplus F(G^{i+1}) & \xrightarrow{\psi_{X,G^{i+1}}} & F(X\times_S G^{i+1}) \\ {\scriptstyle 1_{F(X)}\oplus F(v_i^j)}\downarrow & & \downarrow{\scriptstyle F(s_i^j)} \\ F(X)\oplus F(G^{i}) & \xrightarrow{\psi_{X,G^{i}}} & F(X\times_S G^{i}), \end{array}$$

where $0\le j\le i$. Except when $j=0$ in (3.18), the commutativity of the preceding diagrams follows without difficulty from the definitions (3.1), (3.9)-(3.12), (3.14) and (3.17). The commutativity of (3.18) when $j=0$ follows from the definitions (3.13), (3.17), the commutativity of diagram (3.2) for $(X,Y,Z)=(X,G,G^i)$ and the following equality of maps $F(X)\to F(X\times_S G)$:

$$\psi_{X,G}\circ(i_{F(X)}+i_{F(G)}\circ\varphi)=F(\varsigma), \tag{3.19}$$

where $i_{F(X)}\colon F(X)\to F(X)\oplus F(G)$ and $i_{F(G)}\colon F(G)\to F(X)\oplus F(G)$ are the canonical embeddings. The identity (3.19) follows, in turn, (by the definition of $\varphi$) from the equality

$$p_{F(X)}\circ\psi_{X,G}^{-1}\circ F(\varsigma)=1_{F(X)},$$

which is [GA2, formula (25), p. 480]. □

Now, for every integer $i\ge 0$, let

$$\gamma_G^{(i)}\colon F(G)^i\to F(G^i)$$

be defined recursively by $\gamma_G^{(0)}=0$ (where $F(G)^0=\{0\}$ by definition), $\gamma_G^{(1)}=1_{F(G)}$ and, for $i\ge 1$,

$$\gamma_G^{(i+1)}(b_1,\dots,b_{i+1})=\psi_{G^i,G}(\gamma_G^{(i)}(b_1,\dots,b_i),b_{i+1}). \tag{3.20}$$

By (3.1), we have

$$\gamma_G^{(i)}(b_1,\dots,b_i)=\sum_{k=1}^{i}F(p_G^k)(b_k), \tag{3.21}$$

where $p_G^k\colon G^i\to G$ is the $k$-th projection morphism. Further, the maps $\gamma_G^{(i)}$ satisfy the following inversion formula (3.22)[4]: for every integer $k$ such that $1\le k\le i$, let $\theta^k\colon G\to G^i$, $g\mapsto(1,\dots,g_{(k)},\dots,1)$, where the subscript indicates position, i.e., $\theta^k$ is the unique $S$-morphism such that $p_G^k\circ\theta^k=1_G$ and $p_G^j\circ\theta^k=\varepsilon\circ\pi$ for $j\ne k$. Then, for every $k$ such that $1\le k\le i$, we have

$$b_k=F(\theta^k)\circ\gamma_G^{(i)}(b_1,\dots,b_i). \tag{3.22}$$

**Lemma 3.3.** *Assume that, for every $i\ge 1$, the maps $\psi_{G^i,G}\colon F(G^i)\oplus F(G)\to F(G^{i+1})$ are isomorphisms of abelian groups. Then there exists a cosimplicial abelian group structure on $\{F(X)\oplus F(G)^i\}_{i\ge 0}$ so that the map*

$$\{1_{F(X)}\oplus\gamma_G^{(i)}\}_{i\ge 0}\colon\{F(X)\oplus F(G)^i\}_{i\ge 0}\xrightarrow{\sim}\{F(X)\oplus F(G^i)\}_{i\ge 0} \tag{3.23}$$

*is an isomorphism of cosimplicial abelian groups.*

[4] This inversion formula is the key to obtaining without difficulty the formulas (3.29) below.

*Proof.* By (3.20), the maps $\gamma_G^{(i)}$ are isomorphisms of abelian groups for every $i \geq 1$. Now, for every $i \geq 0$, we define maps $\delta_j^{i+1}\colon F(X)\oplus F(G)^i \to F(X)\oplus F(G)^{i+1}$ (where $0 \leq j \leq i+1$) and $\sigma_j^i\colon F(G)^{i+1} \to F(G)^i$ (where $0 \leq j \leq i$) by the commutativity of the diagrams

$$\begin{array}{ccc} F(X)\oplus F(G)^i & \xrightarrow[\sim]{1_{F(X)}\oplus\gamma_G^{(i)}} & F(X)\oplus F(G^i) \\ \Big\downarrow{\scriptstyle \delta_j^{i+1}} & & \Big\downarrow{\scriptstyle (p_{F(X)},\,\alpha_j^{i+1})} \\ F(X)\oplus F(G)^{i+1} & \xrightarrow[\sim]{1_{F(X)}\oplus\gamma_G^{(i+1)}} & F(X)\oplus F(G^{i+1}), \end{array} \tag{3.24}$$

where $0 \leq j \leq i+1$, and

$$\begin{array}{ccc} F(G)^{i+1} & \xrightarrow[\sim]{\gamma_G^{(i+1)}} & F(G^{i+1}) \\ \Big\downarrow{\scriptstyle \sigma_j^i} & & \Big\downarrow{\scriptstyle F(\upsilon_i^j)} \\ F(G)^i & \xrightarrow[\sim]{\gamma_G^{(i)}} & F(G^i), \end{array}$$

where $0 \leq j \leq i$. Then $\{F(X)\oplus F(G)^i\}_{i\geq 0}$ is a cosimplicial abelian group with codegeneracy maps $1_{F(X)}\oplus\sigma_j^i$ and coface maps $\delta_j^{i+1}$. Further, it is immediate (by construction) that (3.23) is an isomorphism of cosimplicial abelian groups. □

The next statement is the simplicial lemma alluded to in the title of this subsection. It generalizes [San81, Lemma 6.12], which is the case $S = \operatorname{Spec} k$, where $k$ is a field.

**Lemma 3.4.** *Let $F\colon \mathscr{C} \to \mathbf{Ab}$ with $F(1_S) = 0$, $G \to S$, $h\colon Y \to S$ and $\xi\colon X \to Y$ be as above. Assume that the maps $\psi_{G^i,G}\colon F(G^i)\oplus F(G) \to F(G^{i+1})$ and $\psi_{X,G^i}\colon F(X)\oplus F(G^i) \to F(X\times_S G^i)$ (3.1) are isomorphisms of abelian groups for every $i \geq 1$. Then $\check{H}^i(X/Y, h^*F) = 0$ for every $i \geq 2$ and there exists a canonical exact sequence of abelian groups*

$$0 \to \check{H}^0(X/Y, h^*F) \to F(X) \xrightarrow{\varphi} F(G) \to \check{H}^1(X/Y, h^*F) \to 0, \tag{3.25}$$

*where $\varphi = \varphi_{F,X,G}$ is the map* (3.15).

**Remark 3.5.** Let

$$\vartheta = p_G \circ \beta_1^{-1}\colon X\times_Y X \to G, \tag{3.26}$$

where $\beta_1$ is the isomorphism (3.6) for $i = 1$. Then $\vartheta$ is an $S$-morphism such that, for every $S$-scheme $T$, $\vartheta(T)$ maps $(x_0, x_1) \in X(T)\times_{Y(T)} X(T)$ to the unique element $g \in G(T)$ such that $x_1 = x_0 g$. The proof of the lemma will show that $F(\vartheta)\colon F(G) \to F(X\times_Y X)$ factors as

$$F(G) \xrightarrow{F(\vartheta)'} \operatorname{Ker}\partial^2 \subseteq F(X\times_Y X).$$

It will also show that the maps $\check{H}^0(X/Y, h^*F) \to F(X)$ and $F(G) \to \check{H}^1(X/Y, h^*F)$ in (3.25) are, respectively, the inclusion and the composition

$$F(G) \xrightarrow{F(\vartheta)'} \operatorname{Ker}\partial^2 \twoheadrightarrow \operatorname{Ker}\partial^2/\operatorname{Im}\partial^1 = \check{H}^1(X/Y, h^*F).$$

*Proof.* By Lemmas 3.1, 3.2 and 3.3,

$$\left\{F(\beta_i^{-1})\circ\psi_{X,G^i}\circ\left(1_{F(X)}\oplus\gamma_G^{(i)}\right)\right\}_{i\geq 0}\colon \left\{F(X)\oplus F(G)^i\right\}_{i\geq 0} \xrightarrow{\sim} \left\{F(X_{/Y}^{i+1})\right\}_{i\geq 0}$$

is an isomorphism of cosimplicial abelian groups. The latter map induces isomorphisms of the associated cochain complexes of abelian groups

$$\left\{F(X)\oplus F(G)^i, \delta^i\right\}_{i\geq 0} \xrightarrow{\sim} \left\{F(X_{/Y}^{i+1}), \partial^{i+1}\right\}_{i\geq 0} \tag{3.27}$$

and thus of the corresponding cohomology groups

$$H^i(F(X)\oplus F(G)^\bullet) \xrightarrow{\sim} \check{H}^i(X/Y, h^*F) \qquad (i \geq 0), \tag{3.28}$$

where $\delta^i = \sum_{j=0}^{i+1}(-1)^j\delta_j^{i+1}\colon F(X)\oplus F(G)^i \to F(X)\oplus F(G)^{i+1}$ and the maps $\delta_j^{i+1}$ are given by (3.24) via (3.17) and (3.20). Note that, since $\beta_0 = 1_X$, $\psi_{X,G^0} = 1_{F(X)}$ and $\gamma_G^{(0)} = 0$, the map (3.28) for $i = 0$ is an *equality* $\check{H}^0(X/Y, h^*F) = \operatorname{Ker}\delta^0$.

We will now compute the maps $\delta^i$ explicitly. Using definitions (3.11), (3.12), (3.17), formulas (3.21), (3.22) and the commutativity of diagram (3.24), the maps $\delta_j^{i+1}$ (where $0 \leq j \leq i+1$) are given by $\delta_j^{i+1}(a, b_1, \ldots, b_i) = (a, c_1^{(i,j)}, \ldots, c_{i+1}^{(i,j)})$, where the elements $c_k^{(i,j)} \in F(G)$ for $1 \leq k \leq i+1$ are defined by

$$c_k^{(i,j)} = \begin{cases} \varphi(a) & \text{if } k = 1 \text{ and } j = 0 \\ b_{k-1} & \text{if } 2 \leq k \leq i+1 \text{ and } 0 \leq j \leq k-1 \\ b_k & \text{if } 1 \leq k \leq i \text{ and } k \leq j \leq i+1 \\ 0 & \text{if } (i,j) = (0,1) \text{ or } k = j = i+1 \geq 2. \end{cases} \tag{3.29}$$

The preceding formulas yield[5]:

$$\begin{aligned} \delta^0(a) &= (0, \varphi(a)) \\ \delta^1(a, b_1) &= (a, \varphi(a), 0) \\ \delta^{2r}(a, b_1, \ldots, b_{2r}) &= (0, \varphi(a) - b_1, 0, b_2 - b_3, 0, \ldots, b_{2r-2} - b_{2r-1}, 0, b_{2r}) \\ \delta^{2r+1}(a, b_1, \ldots, b_{2r+1}) &= (a, \varphi(a), b_2, b_2, \ldots, b_{2r}, b_{2r}, 0), \end{aligned}$$

where $r \geq 1$. It is now clear that $\check{H}^0(X/Y, h^*F) = \operatorname{Ker}\delta^0 = \operatorname{Ker}\varphi$. Next we note that the first component of the map (3.27), i.e.,

$$F(\beta_1^{-1}) \circ \psi_{X,G}\colon F(X) \oplus F(G) \xrightarrow{\sim} F(X \times_Y X),$$

induces (via restriction of domain) a map $\operatorname{Ker}\delta^1 \to \operatorname{Ker}\partial^2$. The composition of the latter map and the canonical isomorphism $F(G) \xrightarrow{\sim} \operatorname{Ker}\delta^1 \subseteq F(X) \oplus F(G), b \mapsto (0, b)$, is a map $F(\vartheta)'\colon F(G) \to \operatorname{Ker}\partial^2$ such that the composition

$$F(G) \xrightarrow{F(\vartheta)'} \operatorname{Ker}\partial^2 \hookrightarrow F(X \times_Y X)$$

is the map $F(\vartheta)\colon F(G) \to F(X \times_Y X)$ (3.26). Further, $F(\vartheta)'$ induces an isomorphism of abelian groups $F(G)/\operatorname{Im}\varphi \xrightarrow{\sim} \check{H}^1(X/Y, h^*F)$, namely the composition

$$F(G)/\operatorname{Im}\varphi \xrightarrow{\sim} \operatorname{Ker}\delta^1/\operatorname{Im}\delta^0 \xrightarrow{\sim} \operatorname{Ker}\partial^2/\operatorname{Im}\partial^1 = \check{H}^1(X/Y, h^*F),$$

where the second isomorphism is the map (3.28) for $i = 1$. Thus we can define the map $F(G) \twoheadrightarrow \check{H}^1(X/Y, h^*F)$ appearing in (3.25) as the composition

$$F(G) \xrightarrow{F(\vartheta)'} \operatorname{Ker}\partial^2 \twoheadrightarrow \operatorname{Ker}\partial^2/\operatorname{Im}\partial^1 = \check{H}^1(X/Y, h^*F).$$

It remains only to check that $\check{H}^i(X/Y, h^*F) = 0$ for $i \geq 2$. Following [San81, p. 45], we define the map $\lambda_i\colon F(X) \oplus F(G)^i \to F(X) \oplus F(G)^{i-1}$ by $\lambda_i(a, b_1, \ldots, b_i) = (a, -b_1, b_3, \ldots, b_i)$, where $i \geq 2$. Then $\delta^{i-1} \circ \lambda_i + \lambda_{i+1} \circ \delta^i = 1_{F(X)\oplus F(G)^i}$ for every $i \geq 2$, whence $\operatorname{Ker}\delta^i = \operatorname{Im}\delta^{i-1}$ for all $i \geq 2$, i.e., $\check{H}^i(X/Y, h^*F) = 0$ for every $i \geq 2$ by (3.28). The proof is now complete. □

[5]These formulas generalize those stated (without proof) in [San81, p. 45].

### 3.2 A complement

Let $S$ be a locally noetherian normal scheme, let $\mathscr{C} = S_{\mathrm{fl}}$ and assume that the objects $X \to S$ and $G \to S$ considered above satisfy the conditions of Corollary 2.11. Then, by (2.29), the functor $F = \mathrm{Br}_{\mathrm{a},S} \colon S_{\mathrm{fl}} \to \mathbf{Ab}$ (2.28) satisfies the hypotheses (and therefore the conclusions) of Lemma 3.4. In particular, there exists a canonical map (3.15)

$$\varphi = \varphi_{\mathrm{Br}_{\mathrm{a},S},X,G} = p_{\mathrm{Br_a}(G/S)} \circ \psi_{X,G}^{-1} \circ \mathrm{Br_a}\,\varsigma \colon \mathrm{Br_a}(X/S) \to \mathrm{Br_a}(G/S). \tag{3.30}$$

On the other hand, the functor $F = \mathrm{Br}_{1,S} \colon S_{\mathrm{fl}} \to \mathbf{Ab}$ does *not* satisfy the conditions of Lemma 3.4 since $F(1_S) = \mathrm{Br}\,S$ is nonzero in general. However, we may still define an analog of the map (3.30) for $\mathrm{Br}_{1,S}$:

**Proposition 3.6.** *There exists a morphism of abelian groups*

$$\varphi' = \varphi'_{X,G} \colon \mathrm{Br_1}(X/S) \to \mathrm{Br}_{\varepsilon}(G/S) \tag{3.31}$$

*such that the following diagram of abelian groups commutes*

$$\begin{array}{ccc}
\mathrm{Br_1}(X/S) & \xrightarrow{\varphi'} & \mathrm{Br}_{\varepsilon}(G/S) \\
{\scriptstyle c(X)}\downarrow & & \downarrow{\scriptstyle c(G),\varepsilon} \\
\mathrm{Br_a}(X/S) & \xrightarrow{\varphi} & \mathrm{Br_a}(G/S),
\end{array}$$

*where $\varphi$ is the map* (3.30), *$c_{(X)}$ is the canonical projection* (2.30) *and $c_{(G),\varepsilon}$ is given by* (2.33).

*Proof.* By the proof of Corollary 2.11 and Remark 2.10, the map (2.44)

$$\zeta_{X,G} \colon \mathrm{Br_1}(X/S) \oplus \mathrm{Br}_{\varepsilon}(G/S) \xrightarrow{\sim} \mathrm{Br_1}(X \times_S G/S) \tag{3.32}$$

is an isomorphism of abelian groups. Thus we may define

$$\varphi' = p_{\mathrm{Br}_{\varepsilon}(G/S)} \circ \zeta_{X,G}^{-1} \circ \mathrm{Br_1}\,\varsigma \colon \mathrm{Br_1}(X/S) \to \mathrm{Br}_{\varepsilon}(G/S). \tag{3.33}$$

The diagram of the proposition is the outer circuit of the following diagram with commuting squares

$$\begin{array}{ccccccc}
\mathrm{Br_1}(X/S) & \xrightarrow{\mathrm{Br_1}\varsigma} & \mathrm{Br_1}(X \times_S G/S) & \xrightarrow{\zeta_{X,G}^{-1}} & \mathrm{Br_1}(X/S) \oplus \mathrm{Br}_{\varepsilon}(G/S) & \xrightarrow{p_{\mathrm{Br}_{\varepsilon}(G/S)}} & \mathrm{Br}_{\varepsilon}(G/S) \\
{\scriptstyle c(X)}\downarrow & & {\scriptstyle c(X\times_S G)}\downarrow & & {\scriptstyle (c(X),\, c(G),\varepsilon)}\downarrow & & \downarrow{\scriptstyle c(G),\varepsilon} \\
\mathrm{Br_a}(X/S) & \xrightarrow{\mathrm{Br_a}\varsigma} & \mathrm{Br_a}(X \times_S G/S) & \xrightarrow{\psi_{X,G}^{-1}} & \mathrm{Br_a}(X/S) \oplus \mathrm{Br_a}(G/S) & \xrightarrow{p_{\mathrm{Br_a}(G/S)}} & \mathrm{Br_a}(G/S).
\end{array}$$

□

**Lemma 3.7.** *If $\zeta_{X,G}$ is the isomorphism* (3.32), *then*

$$p_{\mathrm{Br_1}(X/S)} \circ \zeta_{X,G}^{-1} \circ \mathrm{Br_1}\,\varsigma = 1_{\mathrm{Br_1}(X/S)}.$$

*Proof.* Recall $s_0^0 = (1_X, \varepsilon)_S$ (3.10). Since $p_G \circ s_0^0 = \varepsilon \circ h \circ \xi$ and $\mathrm{Br_1}(\varepsilon) \circ p_{\mathrm{Br}_{\varepsilon}(G/S)} = 0$ by definition of $\mathrm{Br}_{\varepsilon}(G/S)$ (2.31), we have

$$(\mathrm{Br_1}\,s_0^0) \circ \zeta_{X,G} = \mathrm{Br_1}(p_X \circ s_0^0) \circ p_{\mathrm{Br_1}(X/S)} + \mathrm{Br_1}(p_G \circ s_0^0) \circ p_{\mathrm{Br}_{\varepsilon}(G/S)} = p_{\mathrm{Br_1}(X/S)},$$

whence $p_{\mathrm{Br_1}(X/S)} \circ \zeta_{X,G}^{-1} \circ \mathrm{Br_1}\,\varsigma = (\mathrm{Br_1}\,s_0^0) \circ (\mathrm{Br_1}\,\varsigma) = \mathrm{Br_1}(1_X) = 1_{\mathrm{Br_1}(X/S)}$, as claimed. □

**Lemma 3.8.** *Let $1 \to H \xrightarrow{i} G \to K \to 1$ be an exact sequence of groups in $\mathscr{C}$, so that the $K$-scheme $G$ is an $H_K$-torsor over $K$, and let $\varepsilon_H$ and $\varepsilon_G$ denote the unit sections of $H$ and $G$, respectively. Assume that $G \to S$ and $H \to S$ satisfy the conditions of Corollary* 2.11, *so that the map* (3.31)

$$\varphi'_{G,H} \colon \mathrm{Br_1}(G/S) \to \mathrm{Br}_{\varepsilon_H}(H/S) \tag{3.34}$$

*is defined. Then the restriction of $\varphi'_{G,H}$ to $\mathrm{Br}_{\varepsilon_G}(G/S) \subset \mathrm{Br_1}(G/S)$ is the map $\mathrm{Br}_{\varepsilon_H,\varepsilon_G}\, i$* (2.35).

*Proof.* The action of $H$ on $G$ is given by $\varsigma\colon G\times_S H\to G,\ (g,h)\mapsto gi(h)$. We claim that the map $\varphi=\varphi_{\mathrm{Br}_{\mathrm{a},S},G,H}=p_{\mathrm{Br}_{\mathrm{a}}(H/S)}\circ\psi_{G,H}^{-1}\circ\mathrm{Br}_{\mathrm{a}}\varsigma\colon \mathrm{Br}_{\mathrm{a}}(G/S)\to\mathrm{Br}_{\mathrm{a}}(H/S)$ (3.30) is the map $\mathrm{Br}_{\mathrm{a}}i$. In effect, applying the argument at the beginning of the proof of [GA2, Lemma 2.7, p. 480] to the unit section $\varepsilon_G$ of $G$ (rather than to the unit section of $H$) and the functor $F=\mathrm{Br}_{\mathrm{a},S}$, we obtain the formula $\mathrm{Br}_{\mathrm{a},S}(\varepsilon_G\times_S 1_H)=p_{\mathrm{Br}_{\mathrm{a}}(H/S)}\circ\psi_{G,H}^{-1}$, whence $\varphi=p_{\mathrm{Br}_{\mathrm{a}}(H/S)}\circ\psi_{G,H}^{-1}\circ\mathrm{Br}_{\mathrm{a}}\varsigma=\mathrm{Br}_{\mathrm{a}}(\varepsilon_G\times_S 1_H)\circ\mathrm{Br}_{\mathrm{a}}\varsigma=\mathrm{Br}_{\mathrm{a}}i$, as claimed. Now consider the diagram

$$\begin{array}{ccccc}
 & \mathrm{Br}_1(G/S) & \xrightarrow{\ \varphi'\ } & \mathrm{Br}_{\varepsilon_H}(H/S) & \\
 & \uparrow & & \| & \\
c_{(G)}\ \Big( & \mathrm{Br}_{\varepsilon_G}(G/S) & \xrightarrow{\ \mathrm{Br}_{\varepsilon_H,\varepsilon_G}i\ } & \mathrm{Br}_{\varepsilon_H}(H/S) & \Big)\ c_{(H),\varepsilon_H} \\
 & \simeq\downarrow c_{(G),\varepsilon_G} & & c_{(H),\varepsilon_H}\downarrow\simeq & \\
 & \mathrm{Br}_{\mathrm{a}}(G/S) & \xrightarrow{\ \varphi=\mathrm{Br}_{\mathrm{a}}i\ } & \mathrm{Br}_{\mathrm{a}}(H/S), &
\end{array}$$

where $\varphi'=\varphi'_{G,H}=p_{\mathrm{Br}_{\varepsilon_H}(H/S)}\circ\zeta_{G,H}^{-1}\circ\mathrm{Br}_1\varsigma$ (3.33). By definition of $c_{(G),\varepsilon_G}$ (2.33), the left-hand semicircle commutes. Further, the bottom square commutes by the commutativity of diagram (2.37) and the outer diagram commutes by Lemma 3.6. Since the map $c_{(H),\varepsilon_H}$ is an isomorphism, we conclude that the top square above commutes, i.e., the lemma holds. □

## 3.3 The units-Picard-Brauer sequence

Let $S$ be a locally noetherian scheme and let $\mathscr{C}$ be the full subcategory of $S_{\mathrm{fl}}$ whose objects are the schematically dominant morphisms. Then $\mathscr{C}$ contains $1_S$ and is stable under products [EGA, IV$_3$, Theorem 11.10.5(ii)]. Now let $\pi: G\to S$ be a flat $S$-group scheme locally of finite type and let $h\colon Y\to S$ be a faithfully flat morphism locally of finite type. Note that both $\pi$ and $h$ are objects of $\mathscr{C}$. Now let $\xi\colon X\to Y$ be a right (fppf) $G_Y$-torsor over $Y$ and consider the restrictions of the abelian presheaves (2.16) and (2.19) to $\mathscr{C}$:

$$\mathcal{U}_S\colon \mathscr{C}\to\mathbf{Ab},\ (Z\to S)\mapsto \mathcal{U}_S(Z),$$

and

$$\mathrm{NPic}_S\colon \mathscr{C}\to\mathbf{Ab},\ (Z\to S)\mapsto \mathrm{NPic}(Z/S).$$

If $F$ denotes either of the above functors, then $F(1_S)=0$. Further, Proposition 2.8 and Corollary 2.11 show that the maps (3.1) $\psi_{G^i,G}\colon F(G^i)\oplus F(G)\to F(G^{i+1})$ and $\psi_{X,G^i}\colon F(X)\oplus F(G^i)\to F(X\times_S G^i)$ are isomorphisms of abelian groups for every $i\geq 0$. Thus Lemma 3.4 yields

$$\check{H}^i(X/Y,h^*\mathcal{U}_S)=\check{H}^i(X/Y,h^*\mathrm{NPic}_S)=0\quad (i\geq 2). \tag{3.35}$$

Further, by Lemmas 2.5 and 3.4, there exist canonical exact sequences of abelian groups

$$0\to\check{H}^0(X/Y,h^*\mathcal{U}_S)\to\mathcal{U}_S(X)\to G^*(S)\to\check{H}^1(X/Y,h^*\mathcal{U}_S)\to 0 \tag{3.36}$$

and

$$0\to\check{H}^0(X/Y,h^*\mathrm{NPic}_S)\to\mathrm{NPic}(X/S)\to\mathrm{NPic}(G/S)\to\check{H}^1(X/Y,h^*\mathrm{NPic}_S)\to 0. \tag{3.37}$$

Next we consider the abelian presheaf

$$Q_S\colon\mathscr{C}\to\mathbf{Ab},\ \big(Z\xrightarrow{g}S\big)\mapsto \mathrm{Im}\Big[\mathrm{Pic}\,S\xrightarrow{\mathrm{Pic}\,g}\mathrm{Pic}\,Z\Big]. \tag{3.38}$$

**Lemma 3.9.** *We have*

(i) $\check{H}^0(X/Y,h^*Q_S)=\mathrm{Im}\,\mathrm{Pic}(h\circ\xi)$, *and*
(ii) $\check{H}^i(X/Y,h^*Q_S)=0$ *for* $i\geq 1$.

*Proof.* Recall the $Y$-morphism $\partial^j_{i+1}\colon X^{i+2}_{/Y}\to X^{i+1}_{/Y}$ (3.4), where $i\geq 0$ and $0\leq j\leq i+1$. Then $(h^*Q_S)(\partial^j_{i+1})=Q_S(\partial^j_{i+1})$ is the restriction of $\operatorname{Pic}\partial^j_{i+1}\colon \operatorname{Pic}X^{i+1}_{/Y}\to\operatorname{Pic}X^{i+2}_{/Y}$ to $\operatorname{Im}\operatorname{Pic}(h\circ p^{[i]}_Y)$, where $p^{[i]}_Y\colon X^{i+1}_{/Y}\to Y$ is the structural morphism of $X^{i+1}_{/Y}$. Since $h\circ p^{[i]}_Y\circ\partial^j_{i+1}=h\circ p^{[i]}_Y\circ\partial^k_{i+1}$ for every pair of integers $j,k$ such that $0\leq j,k\leq i+1$, we have $Q_S(\partial^j_{i+1})=Q_S(\partial^k_{i+1})$ for all $j,k$ as above. Further, since $\partial^j_{i+1}$ has a section, namely (3.3), the map $\operatorname{Pic}\partial^j_{i+1}$ is injective for every $j$. Thus the complex $\{Q_S(X^{i+1}_{/Y}),\partial^{i+1}\}_{i\geq 0}$, where $\partial^{i+1}=\sum_{j=0}^{i+1}(-1)^jQ_S(\partial^j_{i+1})$, is quasi-isomorphic to the complex whose only term is the group $\operatorname{Im}\operatorname{Pic}(h\circ\xi)$ placed in degree 0, where $\xi=p^{[0]}_Y\colon X\to Y$. The lemma is now clear. □

Next, for every integer $j\geq 0$, we consider the abelian presheaf

$$\mathscr{H}^j(\mathbb{G}_{m,Y})\colon \mathscr{C}_{/Y}\to\mathbf{Ab},\ (Z\to Y)\mapsto H^j(Z_{\text{ét}},\mathbb{G}_{m,Z}). \tag{3.39}$$

**Lemma 3.10.** *The following holds*

(i) $\check{H}^0(X/Y,h^*\mathcal{U}_S)=\mathcal{U}_S(Y)$,
(ii) $\check{H}^1(X/Y,h^*\mathcal{U}_S)=\check{H}^1(X/Y,\mathscr{H}^0(\mathbb{G}_{m,Y}))$,
(iii) *there exists a canonical isomorphism of abelian groups*

$$\frac{\check{H}^0(X/Y,\mathscr{H}^1(\mathbb{G}_{m,Y}))}{\operatorname{Im}\operatorname{Pic}(h\circ\xi)}\xrightarrow{\sim}\check{H}^0(X/Y,h^*\mathrm{NPic}_S),$$

*where* $\check{H}^0(X/Y,\mathscr{H}^1(\mathbb{G}_{m,Y}))=\operatorname{Ker}[\operatorname{Pic}\partial^0_1-\operatorname{Pic}\partial^1_1\colon\operatorname{Pic}X\to\operatorname{Pic}(X\times_Y X)]$ *is a subgroup of* $\operatorname{Pic}X$.
(iv) $\check{H}^1(X/Y,h^*\mathrm{NPic}_S)=\check{H}^1(X/Y,\mathscr{H}^1(\mathbb{G}_{m,Y}))$, *and*
(v) $\check{H}^i(X/Y,\mathscr{H}^0(\mathbb{G}_{m,Y}))=\check{H}^i(X/Y,\mathscr{H}^1(\mathbb{G}_{m,Y}))=0$ *for* $i\geq 2$.

*Proof.* Let $P_{\mathbb{G}_{m,S}(S)}$ be the constant presheaf on $\mathscr{C}_{/Y}$ with value $\mathbb{G}_{m,S}(S)$. Since every object $Z\to Y$ of $\mathscr{C}_{/Y}$ is schematically dominant over $S$, the canonical map $\mathbb{G}_{m,S}(S)\to\mathbb{G}_{m,S}(Z)$ is injective. Consequently, there exists a canonical exact sequence of abelian presheaves on $\mathscr{C}_{/Y}$

$$1\to P_{\mathbb{G}_{m,S}(S)}\to\mathscr{H}^0(\mathbb{G}_{m,Y})\to h^*\mathcal{U}_S\to 1.$$

Since $\check{H}^i(X/Y,P_{\mathbb{G}_{m,S}(S)})=0$ for $i\geq 1$, the preceding sequence immediately yields (i) and (ii) and also shows that $\check{H}^i(X/Y,\mathscr{H}^0(\mathbb{G}_{m,Y}))=\check{H}^i(X/Y,h^*\mathcal{U}_S)=0$ for $i\geq 2$ by (3.35), which is the first half of (v). Next, there exists a canonical exact sequence of abelian presheaves on $\mathscr{C}_{/Y}$

$$1\to h^*Q_S\to\mathscr{H}^1(\mathbb{G}_{m,Y})\to h^*\mathrm{NPic}_S\to 1,$$

where $Q_S$ is the presheaf (3.38). By Lemma 3.9, the latter sequence induces the isomorphism in (iii) as well as isomorphisms

$$\check{H}^i(X/Y,h^*\mathrm{NPic}_S)=\check{H}^i(X/Y,\mathscr{H}^1(\mathbb{G}_{m,Y}))\quad(i\geq 1),$$

which yields (iv) and the second half of (v) by (3.35). The proof is now complete. □

The following proposition generalizes [San81, (6.10.1), p. 43]:

**Proposition 3.11.** (The units-Picard-Brauer sequence of a torsor) *Let $S$ be a locally noetherian normal scheme, $G$ a flat $S$-group scheme locally of finite type and $X\to Y$ a $G_Y$-torsor over $Y$, where $Y\to S$ is faithfully flat and locally of finite type. Assume that*

(i) *the structural morphism $X\to S$ has an étale quasi-section,*
(ii) *for every integer $i\geq 1$ and every étale and surjective morphism $T\to S$, $X_T$, $G^i_T$ and $X_T\times_T G^i_T$ are locally factorial,*

(iii) *for every point $s \in S$ of codimension $\leq 1$, the fibers $X_s$ and $G_s$ are geometrically integral, and*
(iv) *for every maximal point $\eta$ of $S$, $G_\eta$ is $k(\eta)^{\mathrm{s}}$-rational.*

*Then there exists a canonical exact sequence of abelian groups*

$$\begin{aligned} 0 \to \mathcal{U}_S(Y) \to \mathcal{U}_S(X) \to G^*(S) \to \operatorname{Pic} Y &\to \operatorname{Pic} X \to \operatorname{NPic}(G/S) \\ &\to \operatorname{Br} Y \to \operatorname{Br} X, \end{aligned}$$

*where $\mathcal{U}_S$ is the functor* (2.15) *and* $\operatorname{NPic}(G/S)$ *is the group* (2.18).

*Proof.* Using Lemma 3.10, (i)-(iv), in (3.36) and (3.37), we obtain exact sequences

$$0 \to \mathcal{U}_S(Y) \to \mathcal{U}_S(X) \to G^*(S) \to \check{H}^1(X/Y, \mathscr{H}^0(\mathbb{G}_{m,Y})) \to 0 \tag{3.40}$$

and

$$0 \to \operatorname{Pic} X/\check{H}^0(X/Y, \mathscr{H}^1(\mathbb{G}_{m,Y})) \to \operatorname{NPic}(G/S) \to \check{H}^1(X/Y, \mathscr{H}^1(\mathbb{G}_{m,Y})) \to 0. \tag{3.41}$$

We now endow $\mathscr{C}_{/Y}$ with the fppf topology and consider the spectral sequence for Čech cohomology associated to the fppf covering $\xi\colon X \to Y$ and the abelian sheaf $\mathbb{G}_{m,Y}$ on $\mathscr{C}_{/Y}$:

$$\check{H}^i(X/Y, \mathscr{H}^j(\mathbb{G}_{m,Y})) \implies H^{i+j}(Y_{\mathrm{fl}}, \mathbb{G}_{m,Y}) = H^{i+j}(Y_{\text{ét}}, \mathbb{G}_{m,Y}).$$

By Lemma 3.10(v) and [CE, Case $E^k$ for $k = 0$, p. 329], the above spectral sequence induces exact sequences of abelian groups

$$0 \to \check{H}^1(X/Y, \mathscr{H}^0(\mathbb{G}_{m,Y})) \to \operatorname{Pic} Y \to \check{H}^0(X/Y, \mathscr{H}^1(\mathbb{G}_{m,Y})) \to 0 \tag{3.42}$$

and

$$0 \to \check{H}^1(X/Y, \mathscr{H}^1(\mathbb{G}_{m,Y})) \to \operatorname{Br} Y \to \check{H}^0(X/Y, \mathscr{H}^2(\mathbb{G}_{m,Y})) \to 0, \tag{3.43}$$

where

$$\check{H}^0(X/Y, \mathscr{H}^2(\mathbb{G}_{m,Y})) = \operatorname{Ker}[\operatorname{Br} \partial_1^0 - \operatorname{Br} \partial_1^1 \colon \operatorname{Br} X \to \operatorname{Br}(X \times_Y X)].$$

Note that, since $\beta_1\colon X \times_S G \to X \times_Y X$ (3.6) is an isomorphism, we have

$$\check{H}^0(X/Y, \mathscr{H}^2(\mathbb{G}_{m,Y})) = \operatorname{Ker}[\operatorname{Br}(\partial_1^0 \circ \beta_1) - \operatorname{Br}(\partial_1^1 \circ \beta_1)] = \operatorname{Ker}[\operatorname{Br} \varsigma - \operatorname{Br} p_X]$$

by (3.7) and (3.8). Thus, if

$$\phi = \operatorname{Br} \varsigma - \operatorname{Br} p_X \colon \operatorname{Br} X \to \operatorname{Br}(X \times_S G), \tag{3.44}$$

then (3.40)-(3.43) yield exact sequences

$$0 \to \mathcal{U}_S(Y) \to \mathcal{U}_S(X) \to G^*(S) \to \operatorname{Pic} Y \to \check{H}^0(X/Y, \mathscr{H}^1(\mathbb{G}_{m,Y})) \to 0 \tag{3.45}$$

and

$$0 \to \operatorname{Pic} X/\check{H}^0(X/Y, \mathscr{H}^1(\mathbb{G}_{m,Y})) \to \operatorname{NPic}(G/S) \to \operatorname{Br} Y \to \operatorname{Br} X \xrightarrow{\phi} \operatorname{Br}(X \times_S G). \tag{3.46}$$

The sequences (3.45) and (3.46) can now be assembled to yield the exact sequence of abelian groups

$$\begin{aligned} 0 \to \mathcal{U}_S(Y) \to \mathcal{U}_S(X) \to G^*(S) &\to \operatorname{Pic} Y \to \operatorname{Pic} X \to \operatorname{NPic}(G/S) \\ &\to \operatorname{Br} Y \to \operatorname{Br} X \xrightarrow{\phi} \operatorname{Br}(X \times_S G), \end{aligned} \tag{3.47}$$

where $\phi$ is the map (3.44). The proposition follows. □

**Remarks 3.12.**

(a) The sequence (3.47) (and therefore also the sequence of the proposition) is functorial in the following sense. Let $G'$ be a flat $S$-group scheme locally of finite type and let $X' \to Y'$ be a (right) $G'_{Y'}$-torsor over $Y'$ such that the $S$-schemes $G', X', Y'$ satisfy all the conditions of the proposition. Assume, furthermore, that there exist a morphism of $S$-group schemes $G \to G'$ and morphisms of $S$-schemes $X \to X'$ and $Y \to Y'$ such that the following diagram, whose horizontal arrows are the corresponding group actions, commutes

$$\begin{array}{ccc} X \times_Y G_Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ X' \times_{Y'} G'_{Y'} & \longrightarrow & X'. \end{array}$$

Then the following diagram, whose top and bottom rows are, respectively, the exact sequences (3.47) associated to the triples $(X', Y', G')$ and $(X, Y, G)$, commutes:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \operatorname{Pic} Y' & \longrightarrow & \operatorname{Pic} X' & \longrightarrow & \operatorname{NPic}(G'/S) & \longrightarrow & \operatorname{Br} Y' & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \operatorname{Pic} Y & \longrightarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{NPic}(G/S) & \longrightarrow & \operatorname{Br} Y & \longrightarrow & \cdots . \end{array}$$

We do not carry out here the (lengthy) verification of this fact. We do, however, include below (after Remark 3.15) a partial verification of the indicated functoriality in a particular case that will be relevant in the next section.

(b) The homomorphism of abelian groups $d\colon G^*(S) \to \operatorname{Pic} Y$ in (3.47) is defined as follows: if $\chi \in G^*(S)$, i.e., $\chi : G \to \mathbb{G}_{m,S}$ is a morphism of $S$-group schemes, then

$$d(\chi) = \chi_Y^{(1)}([X]) = [X \wedge^{G_Y, \chi_Y} \mathbb{G}_{m,Y,r}],$$

where $\chi_Y^{(1)}\colon H^1(Y_{\text{fl}}, G_Y) \to H^1(Y_{\text{fl}}, \mathbb{G}_{m,Y}) = \operatorname{Pic} Y$ is the map (2.3) induced by $\chi_Y$ and $[X]$ is the class of the $G_Y$-torsor $X \to Y$ in $H^1(Y_{\text{fl}}, G_Y)$.

(c) In the setting of the proposition, assume in addition that $S$ is noetherian and (for simplicity) irreducible with function field $K$. Assume also that $G$ is of finite type over $S$ and that, for every point $s$ of $S$ of codimension 1, $Y_s$ is integral. By (a), [GA2, Corollary 5.3], [Ray, Proposition VII.1.3(4), p. 104] and the proposition, there exists a canonical exact and commutative diagram of abelian groups

$$\begin{array}{ccccccccccc} & & \operatorname{Pic} S & = & \operatorname{Pic} S & & & & & & \\ & & \downarrow & & \downarrow & & & & & & \\ G^*(S) & \longrightarrow & \operatorname{Pic} Y & \xrightarrow{\operatorname{Pic}\xi} & \operatorname{Pic} X & \longrightarrow & \operatorname{NPic}(G/S) & \longrightarrow & \operatorname{Br} Y & \xrightarrow{\operatorname{Br}\xi} & \operatorname{Br} X \\ \downarrow{\simeq} & & \downarrow & & \downarrow & & \downarrow{\simeq} & & \downarrow & & \downarrow \\ G^*(K) & \longrightarrow & \operatorname{Pic} Y_K & \xrightarrow{\operatorname{Pic}\xi_K} & \operatorname{Pic} X_K & \longrightarrow & \operatorname{Pic} G_K & \longrightarrow & \operatorname{Br} Y_K & \xrightarrow{\operatorname{Br}\xi_K} & \operatorname{Br} X_K . \end{array}$$

It follows from the diagram that the canonical map $\operatorname{Ker}\operatorname{Br}\xi \to \operatorname{Ker}\operatorname{Br}\xi_K$ is an isomorphism. Further, if the canonical map $\mathcal{U}_S(X) \to \mathcal{U}_K(X_K)$ is surjective, then $\operatorname{Ker}\operatorname{Pic}\xi \to \operatorname{Ker}\operatorname{Pic}\xi_K$ is an isomorphism as well. Thus, in this case, a substantial part of the sequence of the proposition is *essentially equivalent* to the corresponding part of the sequence over the field $K$. See also Remark 4.22.

Recall that, if $G$ is an $S$-group scheme, then $G^*$ is the sheaf $\underline{\operatorname{Hom}}_{S\text{-gr}}(G, \mathbb{G}_{m,S})$ (2.11). Assertion (ii) in the following statement generalizes [San81, (6.10.3), p. 43][6].

[6] Since it seems that the proof of the indicated result in [San81, p. 45, lines 21-26] is incorrect, we provide a modified argument in the setting of this paper.

**Corollary 3.13.** *Let the notation and hypotheses be those of the proposition.*

(i) *There exists a canonical exact sequence of étale sheaves on* $S$

$$\begin{aligned}1 \to \mathrm{U}_{Y/S} \to \mathrm{U}_{X/S} \to G^* \to \mathrm{Pic}_{Y/S} \to \mathrm{Pic}_{X/S} \to \mathrm{Pic}_{G/S}\\ \to \mathrm{Br}_{Y/S} \to \mathrm{Br}_{X/S} \to \mathrm{Br}_{X\times_S G/S}\end{aligned}$$

(ii) *If the morphism* $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{G/S}$ *in* (i) *is surjective, then there exists a canonical exact sequence of abelian groups*

$$\begin{aligned}0 \to \mathcal{U}_S(Y) \to \mathcal{U}_S(X) \to G^*(S) \;\to\; &\mathrm{Pic}\, Y \to \mathrm{Pic}\, X \to \mathrm{NPic}(G/S)\\ \to\; &\mathrm{Br}_1(Y/S) \to \mathrm{Br}_1(X/S) \xrightarrow{\varphi'} \mathrm{Br}_\varepsilon(G/S),\end{aligned}$$

*where* $\varepsilon$ *is the unit section of* $G \to S$ *and* $\varphi'$ *is the map* (3.33).

*Proof.* If $T \to S$ is an etale and surjective morphism of schemes, then $T$, $G_T$, $X_T$ and $Y_T$ satisfy all the hypotheses of the proposition. Thus there exists an exact sequence of abelian groups (3.47)

$$\begin{aligned}0 \to \mathcal{U}_{Y/S}(T) \to \mathcal{U}_{X/S}(T) \to G^*(T) \;\to\; &\mathrm{Pic}\, Y_T \to \mathrm{Pic}\, X_T \to \mathrm{NPic}(G_T/T)\\ \to\; &\mathrm{Br}\, Y_T \to \mathrm{Br} X_T \to \mathrm{Br}(X_T \times_T G_T).\end{aligned} \tag{3.48}$$

The above sequence is the sequence of $T$-sections of a complex of abelian presheaves on $S_{\text{ét}}$ such that the corresponding complex of associated sheaves on $S_{\text{ét}}$ is exact, by the exactness of (3.48) for every $T \to S$ as above. Assertion (i) now follows, noting that $\mathrm{U}_{Y/S}$ and $\mathrm{U}_{X/S}$ are the etale sheaves on $S$ associated to $\mathcal{U}_{Y/S}$ and $\mathcal{U}_{X/S}$ (respectively) by Remark 2.4, and $\mathrm{Pic}_{G/S}$ is the etale sheaf on $S$ associated to the presheaf $T \mapsto \mathrm{NPic}(G_T/T)$ by [Klei, Definition 9.2.2, p. 252]. If $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{G/S}$ is surjective, then (i) yields the exactness of the right-hand column in the following commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathrm{Br}_1(Y/S) & \longrightarrow & \mathrm{Br}\, Y & \longrightarrow & \mathrm{Br}_{Y/S}(S)\\
 & & \downarrow & & \downarrow & & \downarrow\\
1 & \longrightarrow & \mathrm{Br}_1(X/S) & \longrightarrow & \mathrm{Br} X & \longrightarrow & \mathrm{Br}_{X/S}(S)\\
 & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi} & & \downarrow\\
1 & \longrightarrow & \mathrm{Br}_1(X \times_S G/S) & \longrightarrow & \mathrm{Br}(X \times_S G) & \longrightarrow & \mathrm{Br}_{X\times_S G/S}(S),
\end{array}$$

where $\phi$ is the map (3.44) and $\phi_1 = \mathrm{Br}_1 \varsigma - \mathrm{Br}_1 p_X$ is induced by $\phi$. By the exactness of (3.47), the middle column in the above diagram is also exact. It now follows that the left-hand column is exact and $\mathrm{Ker}[\mathrm{Br}\, Y \to \mathrm{Br} X] = \mathrm{Ker}[\mathrm{Br}_1(Y/S) \to \mathrm{Br}_1(X/S)]$. Thus (3.47) yields the exact sequence

$$\begin{aligned}0 \to \mathcal{U}_S(Y) \to \mathcal{U}_S(X) \to G^*(S) \;\to\; &\mathrm{Pic}\, Y \to \mathrm{Pic}\, X \to \mathrm{NPic}(G/S)\\ \to\; &\mathrm{Br}_1(Y/S) \to \mathrm{Br}_1(X/S) \xrightarrow{\phi_1} \mathrm{Br}_1(X \times_S G/S).\end{aligned}$$

Now, since we work under the hypotheses of Proposition 3.11, which are the same as those of Corollary 2.11, the proof of Corollary 2.11 and Remark 2.10(a) together show that the map $\zeta_{X,G}\colon \mathrm{Br}_1(X/S) \oplus \mathrm{Br}_\varepsilon(G/S) \to \mathrm{Br}_1(X \times_S G/S)$ (2.44) is an isomorphism of abelian groups. Thus the kernel of $\phi_1$ is the same as the kernel of the composite map

$$\mathrm{Br}_1(X/S) \xrightarrow{\phi_1} \mathrm{Br}_1(X \times_S G/S) \xrightarrow{\zeta_{X,G}^{-1}} \mathrm{Br}_1(X/S) \oplus \mathrm{Br}_\varepsilon(G/S). \tag{3.49}$$

Now, by (2.47), (2.48) and Lemma 3.7, we have

$$\begin{aligned}p_{\mathrm{Br}_1(X/S)} \circ \zeta_{X,G}^{-1} \circ \phi_1 &= p_{\mathrm{Br}_1(X/S)} \circ \zeta_{X,G}^{-1} \circ \mathrm{Br}_1 \varsigma - p_{\mathrm{Br}_1(X/S)} \circ \zeta_{X,G}^{-1} \circ \mathrm{Br}_1 p_X\\ &= 1_{\mathrm{Br}_1(X/S)} - 1_{\mathrm{Br}_1(X/S)} = 0\end{aligned}$$

and

$$
\begin{aligned}
p_{\mathrm{Br}_\varepsilon(G/S)} \circ \zeta_{X,G}^{-1} \circ \phi_1 &= p_{\mathrm{Br}_\varepsilon(G/S)} \circ \zeta_{X,G}^{-1} \circ \mathrm{Br}_1 \varsigma - p_{\mathrm{Br}_\varepsilon(G/S)} \circ \zeta_{X,G}^{-1} \circ \mathrm{Br}_1 p_X \\
&= \varphi' - 0 = \varphi',
\end{aligned}
$$

where $\varphi' \colon \mathrm{Br}_1(X/S) \to \mathrm{Br}_\varepsilon(G/S)$ is the map (3.33). Thus the kernel of the composition (3.49) is the kernel of $\varphi'$, which completes the proof. □

**Corollary 3.14.** *Let $1 \to H \to G \to F \to 1$ be an exact sequence of smooth $S$-group schemes with connected fibers at all points of $S$ of codimension $\leq 1$, where $S$ is a locally noetherian regular scheme. Assume that, for every maximal point $\eta$ of $S$, $H_\eta$ is $k(\eta)^{\mathrm{s}}$-rational. Then the given sequence induces*

(i) *an exact sequence of abelian groups*

$$
\begin{aligned}
0 \to F^*(S) \to G^*(S) \to H^*(S) \to \mathrm{Pic}\, F &\to \mathrm{Pic}\, G \to \mathrm{NPic}(H/S) \\
&\to \mathrm{Br}\, F \to \mathrm{Br}\, G,
\end{aligned}
$$

(ii) *an exact sequence of étale sheaves on $S$*

$$
\begin{aligned}
0 \to F^* \to G^* \to H^* &\to \mathrm{Pic}_{\,F/S} \to \mathrm{Pic}_{G/S} \to \mathrm{Pic}_{\,H/S} \\
&\to \mathrm{Br}_{\,F/S} \to \mathrm{Br}_{G/S} \to \mathrm{Br}_{G \times_S H/S}
\end{aligned}
$$

*and,*

(iii) *if the morphism $\mathrm{Pic}_{G/S} \to \mathrm{Pic}_{\,H/S}$ in* (ii) *is surjective, an exact sequence of abelian groups*

$$
\begin{aligned}
0 \to F^*(S) \to G^*(S) \to H^*(S) &\to \mathrm{Pic}\, F \to \mathrm{Pic}\, G \to \mathrm{NPic}(G/S) \\
&\to \mathrm{Br}_1(F/S) \to \mathrm{Br}_1(G/S) \xrightarrow{\varphi'} \mathrm{Br}_\varepsilon(H/S),
\end{aligned}
$$

*where $\varepsilon$ denotes the unit section of $H \to S$ and $\varphi'$ is the map* (3.34).

*Proof.* Assertion (i) follows by applying Proposition 3.11 to the $H_F$-torsor $G \to F$, which is justified since all the conditions of that proposition hold true by Remark 2.10(b). Assertion (ii) follows from Corollary 3.13(i) and Lemma 2.3. Assertion (iii) follows from Corollary 3.13(ii) and Lemma 2.5. □

**Remark 3.15.** Regarding assertion (i) of the preceding corollary, Raynaud constructed in [Ray, Proposition VII.1.5, pp. 106-107] a canonical complex of abelian groups $G^*(S) \to H^*(S) \to \mathrm{Pic}\, F \to \mathrm{Pic}\, G$, where $S$ is any scheme, $G$ is an $S$-group scheme and $H$ is a subgroup scheme of $G$ such that the quotient fpqc sheaf $G/H$ is represented by an $S$-scheme $F$. If the maximal fibers of $H$ are not smooth, then the preceding complex may not be exact.

The maps $H^*(S) \to \mathrm{Pic}\, F$ and $H^* \to \mathrm{Pic}_{\,F/S}$ in the sequences of Corollary 3.14, (i) and (ii), can be defined for ***any*** exact sequence of $S$-group schemes

$$
1 \to H \to G \to F \to 1. \tag{3.50}
$$

The first map $d \colon H^*(S) \to \mathrm{Pic}\, F$ is given by

$$
d(\chi) = \chi_F^{(1)}([G]) = [G \wedge^{H_F, \chi_F} \mathbb{G}_{m,F,r}], \tag{3.51}
$$

where $\chi_F^{(1)} \colon H^1(F_{\mathrm{fl}}, H_F) \to H^1(F_{\mathrm{fl}}, \mathbb{G}_{m,F}) = \mathrm{Pic}\, F$ is the map (2.3) induced by $\chi_F$ and $[G]$ is the class of the $H_F$-torsor $G \to F$ in $H^1(F_{\mathrm{fl}}, H_F)$. See Remark 3.12(b). The map (3.51) is compatible with pullbacks[7], i.e., if $g \colon F' \to F$ is a morphism of $S$-group schemes, then the exact sequence $1 \to H \to G_{F'} \to F' \to 1$ induced by (3.50) defines a map $d' \colon H^*(S) \to \mathrm{Pic}\, F'$, namely $d'(\chi) = \chi_{F'}^{(1)}([G_{F'}]) = [G_{F'} \wedge^{H_{F'}, \chi_{F'}} \mathbb{G}_{m,F',r}]$, such that the following diagram commutes

$$
\begin{array}{ccc}
H^*(S) & \xrightarrow{\ d\ } & \mathrm{Pic}\, F \\
& \searrow^{d'} & \downarrow{\scriptstyle \mathrm{Pic}\, g} \\
& & \mathrm{Pic}\, F'.
\end{array} \tag{3.52}
$$

[7]This fact is a particular case of the functoriality assertion in Remark 3.12(a).

See (2.4) and (2.14). Now, for every morphism of schemes $T \to S$, there exist maps $H^*(T) \to \operatorname{Pic} F_T$ defined similarly to (3.51) which induce a morphism of abelian presheaves on $(\mathrm{Sch}/S)$. This morphism of presheaves induces, in turn, a morphism of the associated etale sheaves on $S$

$$\boldsymbol{d}\colon H^* \to \operatorname{Pic}_{F/S}, \tag{3.53}$$

which is the map in the sequence of Corollary 3.14(ii). If $S$ is strictly local, then $\boldsymbol{d}(S)$ agrees with the map $d$ (3.51) (see Remark 2.6 for the equality $\operatorname{Pic}_{F/S}(S) = \operatorname{Pic} F$). Further, there exist commutative diagrams analogous to (3.52) when $S$ is replaced by any $T$ as above. We conclude that there exists a canonical commutative diagram of etale sheaves on $S$:

$$\begin{array}{ccc} H^* & \xrightarrow{\ \boldsymbol{d}\ } & \operatorname{Pic}_{F/S} \\ & {}_{\boldsymbol{d}'}\searrow & \Big\downarrow {\scriptstyle \operatorname{Pic}_S(g)} \\ & & \operatorname{Pic}_{F'/S}. \end{array} \tag{3.54}$$

## 4. Reductive group schemes

In this section we establish the main theorem of the paper (Theorem 4.20 below), which generalizes the main theorem of [BvH].

Let $S$ be a (non-empty) scheme. Henceforth, all $S$-group schemes are tacitly assumed to be of finite type over $S$.

If $M$ is an $S$-group scheme of multiplicative type, then $M^* = \underline{\mathrm{Hom}}_{S\text{-gr}}(M, \mathbb{G}_{m,S})$ (2.11) and $M_* = \underline{\mathrm{Hom}}_{S\text{-gr}}(\mathbb{G}_{m,S}, M)$ are represented by (finitely generated) twisted constant $S$-group schemes [SGA3$_{\text{new}}$, **X**, Corollary 4.5 and Theorem 5.6]. If $f\colon M \to N$ is a morphism of $S$-group schemes, we will write $f^{(*)}\colon N^* \to M^*$ and $f_{(*)}\colon M_* \to N_*$ for the canonical morphisms induced by $f$.

An $S$-group scheme $G$ is called *reductive* (respectively, *semisimple*, *simply connected*) if $G$ is affine and smooth over $S$ and its geometric fibers are *connected* reductive (respectively, semisimple, simply connected) algebraic groups. By convention, the trivial $S$-group scheme is simply connected. If $G$ is a reductive $S$-group scheme, we will write $\pi_G\colon G \to S$ and $\varepsilon_G\colon S \to G$ for the structural morphism and unit section of $G$, respectively. Further, $\mathrm{rad}(G)$ will denote the radical of $G$, i.e., the identity component of the center of $G$. Now recall that the derived group $G^{\mathrm{der}}$ of $G$ is a normal and semisimple $S$-subgroup scheme of $G$ such that $G^{\mathrm{tor}} = G/G^{\mathrm{der}}$ is the largest quotient of $G$ that is an $S$-torus. If $G$ is a semisimple $S$-group scheme, there exists a simply-connected $S$-group scheme $\widetilde{G}$ and a central isogeny $\varphi\colon \widetilde{G} \to G$. The $S$-group scheme $\widetilde{G}$ is called the *simply-connected central cover* of $G$ and the group $\mu = \mu_G = \operatorname{Ker}\varphi$ is called the *fundamental group* of $G$. See [GA1, p. 1161] for more details and relevant references. If $G$ is an arbitrary reductive $S$-group scheme, $\widetilde{G}$ and $\mu$ will denote, respectively, the simply connected central cover and fundamental group of $G^{\mathrm{der}}$ i.e., $\widetilde{G} = \widetilde{G^{\mathrm{der}}}$ and $\mu = \mu_G = \mu_{G^{\mathrm{der}}}$. Then there exists a canonical central extension of flat $S$-group schemes of finite type

$$1 \to \mu \xrightarrow{i} \widetilde{G} \to G^{\mathrm{der}} \to 1, \tag{4.1}$$

where $\mu = \operatorname{Ker}[\widetilde{G} \to G^{\mathrm{der}}] = S \times_{G^{\mathrm{der}}} \widetilde{G}$ and $i = \varepsilon_{G^{\mathrm{der}}} \times_{G^{\mathrm{der}}} \widetilde{G}$. The $S$-group scheme $\mu$ is a (possibly non-smooth) finite subgroup scheme of $\widetilde{G}$ of multiplicative type. Consequently, $\mu^*$ is a finite and etale $S$-group scheme. We note that $G^{\mathrm{der}}$ is simply connected if, and only if, $\mu_G = 0$. We also note that there exists a canonical exact sequence of reductive $S$-group schemes

$$1 \to G^{\mathrm{der}} \xrightarrow{\iota} G \xrightarrow{q} G^{\mathrm{tor}} \to 1. \tag{4.2}$$

The composition $\widetilde{G} \twoheadrightarrow G^{\mathrm{der}} \overset{\iota}{\hookrightarrow} G$ and the product in $G$ define a faithfully flat morphism of $S$-group schemes $\mathrm{rad}(G) \times_S \widetilde{G} \to G$ which fits into a central extension of flat $S$-group schemes of finite type

$$1 \to \mu' \xrightarrow{i'} \mathrm{rad}(G) \times_S \widetilde{G} \to G \to 1, \tag{4.3}$$

where $\mu' = \operatorname{Ker}[\mathrm{rad}(G) \times_S \widetilde{G} \to G] = S \times_G \big(\mathrm{rad}(G) \times_S \widetilde{G}\big)$ is a finite $S$-group scheme of multiplicative type and $i' = \varepsilon_G \times_G \big(\mathrm{rad}(G) \times_S \widetilde{G}\big)$. Since $\mathrm{rad}(G) \times_G G^{\mathrm{der}}$ is the trivial $S$-group scheme, we may make the identification

$$i' \times_G G^{\mathrm{der}} = i. \tag{4.4}$$

Further, there exists a canonical exact sequence of $S$-group schemes of multiplicative type

$$1 \to \mu \to \mu' \to \mathrm{rad}(G) \to G^{\mathrm{tor}} \to 1. \tag{4.5}$$

See [GA1, proof of Proposition 3.2]. When reference to $G$ is necessary, we will write

$$q_{(G)}\colon G \to G^{\mathrm{tor}} \tag{4.6}$$

for the morphism $q$ in (4.2). Note that, since $(G^{\mathrm{der}})^* = 0$, (4.2) induces an isomorphism of etale twisted constant $S$-group schemes

$$q_{(G)}^{(*)}\colon (G^{\mathrm{tor}})^* \xrightarrow{\sim} G^*. \tag{4.7}$$

Further, the exact sequence (4.1) induces a morphism of abelian groups (3.51)

$$e\colon \mu^*(S) \to \operatorname{Pic} G^{\mathrm{der}},\ \chi \mapsto \left[\widetilde{G} \wedge^{\mu_{G^{\mathrm{der}}},\, \chi_{G^{\mathrm{der}}}} \mathbb{G}_{m,\, G^{\mathrm{der}},\, r}\right], \tag{4.8}$$

and a morphism of etale sheaves on $S$ (3.53)

$$\boldsymbol{e}\colon \mu^* \to \mathrm{Pic}_{G^{\mathrm{der}}/S}. \tag{4.9}$$

If $S$ is strictly local, then $\boldsymbol{e}(S) = e$.

For lack of adequate references, we now present proofs of the following "well-known" facts.

**Proposition 4.1.** *Let $k$ be a field with fixed separable algebraic closure $k^{s}$ and let $G$ be a (connected) reductive algebraic $k$-group scheme. Then*

(i) *$G$ is a $k^{s}$-rational variety, and*
(ii) *$\operatorname{Pic} G = 0$ if $G$ is simply connected.*

*Proof.* [See [Nfdc23-1] and [Nfdc23-2]] To prove (i), we may assume that $k = k^{s}$. Let $T$ be a (split) maximal $k$-torus in $G$, let $B \supset T$ be the Borel $k$-subgroup of $G$ such that the set of $T$-roots on $\mathrm{Lie}(B)$ is a chosen positive system of $T$-roots for $G$ and let $B' \supset T$ be the Borel $k$-subgroup of $G$ opposite to $B$. If $U \subset B$ and $U' \subset B'$ are the $k$-split $k$-unipotent radicals, then the multiplication morphism $U' \times_k T \times_k U \to G$ is an open immersion with $k$-rational source, which yields (i). See [CGP, Propositions 2.1.8,(2),(3), p. 53, 2.1.10, p. 58 and 2.2.9, p. 67]. Now let $k$ be any field and let $G$ be a simply connected $k$-group scheme. By [CT08, Corollary 5.7][8], $\operatorname{Pic} G$ is canonically isomorphic to the group of central extensions of $G$ by $\mathbb{G}_{m,k}$. Since any such extension is trivial by [GA1, Proposition 2.4], the proof is complete. □

**Lemma 4.2.** *Let $S$ be a scheme and let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of reductive $S$-group schemes. Then the given sequence induces an exact sequence of $S$-group schemes of multiplicative type*

$$1 \to \mu_1 \to \mu_2 \to \mu_3 \to G_1^{\mathrm{tor}} \to G_2^{\mathrm{tor}} \to G_3^{\mathrm{tor}} \to 1,$$

*where $\mu_i = \mu_{G_i}$.*

*Proof.* By [GA1, Proposition 2.10], there exists an exact and commutative diagram of reductive $S$-group schemes

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{G}_1 & \longrightarrow & \widetilde{G}_2 & \longrightarrow & \widetilde{G}_3 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1.
\end{array}$$

Now, although the category of reductive $S$-group schemes is not abelian, the proof of the snake lemma given in [Bey, Lemma 2.3, p. 307] can be adapted so that it applies to the above diagram (for example, the decomposition [Bey, (2.2), p. 306] is valid if we set $X = G_2 \times_{G_3} \widetilde{G}_3$ there). The sequence of the lemma then follows from the above diagram as in [Bey, Lemma 2.3, p. 307]. □

[8]The proof of this result depends on (i) but not on the assertion that we aim to prove, i.e., (ii).

**Lemma 4.3.** *Let $S$ be a locally noetherian normal scheme and let $G$ be a smooth $S$-group scheme with connected fibers at every point of $S$ of codimension $\leq 1$. Then there exists a canonically split exact sequence of abelian groups*

$$0 \to \operatorname{Pic} S \to \operatorname{Pic} G \to \prod \operatorname{Pic} G_\eta \to 0,$$

*where the product runs over the set of maximal points $\eta$ of $S$.*

*Proof.* See [GA2, Corollary 5.3] and note that the unit section of $G$ defines a retraction of the canonical map $\operatorname{Pic} S \to \operatorname{Pic} G$ which splits the above sequence [GA2, Remark 3.1(c)]. □

**Proposition 4.4.** *Let $S$ be a noetherian strictly local regular scheme and let $G$ be a reductive $S$-group scheme such that $G^{\text{der}}$ is simply connected. Then* $\operatorname{Pic} G = 0$.

*Proof.* Since $G^{\text{tor}}$ splits over a finite etale cover of $S$ [SGA3$_{\text{new}}$, X, Corollary 4.6(i)], we have $G^{\text{tor}} \simeq \mathbb{G}_{m,S}^n$ for some $n \geq 0$ by [EGA, IV$_4$, Proposition 18.8.1(b)]. Now, since $\operatorname{Pic} S = 0$, Lemma 4.3 shows that $\operatorname{Pic} G^{\text{tor}} \simeq \operatorname{Pic} \mathbb{G}_{m,S}^n \simeq \operatorname{Pic} \mathbb{G}_{m,K}^n$, where $K$ is the function field of $S$. Thus, since the ring of regular functions on $\mathbb{G}_{m,K}^n$ is a UFD and therefore $\operatorname{Pic} \mathbb{G}_{m,K}^n = 0$, we have $\operatorname{Pic} G^{\text{tor}} = 0$. Now Corollary 3.14(i) applied to the exact sequence (4.2) shows that the canonical map $\operatorname{Pic} G \to \operatorname{Pic} G_K^{\text{der}}$ is injective. Since $\operatorname{Pic} G_K^{\text{der}} = 0$ by Proposition 4.1(ii), the proposition follows. □

**Definition 4.5.** *A* $t$-resolution *of a reductive $S$-group scheme $G$ is a central extension of reductive $S$-group schemes $1 \to T \to H \to G \to 1$, where $T$ is an $S$-torus and $H^{\text{der}}$ is simply connected. A morphism of $t$-resolutions of $G$ is a morphism of central extensions of $G$ [BGA, Definition 2.4].*

**Lemma 4.6.** *There exists a $t$-resolution of $G$*

$$1 \to T \xrightarrow{j} H \xrightarrow{p} G \to 1 \qquad (\mathscr{R}) \tag{4.10}$$

*which fits into an exact and commutative diagram of $S$-group schemes*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu' & \xrightarrow{\;i'\;} & \operatorname{rad}(G) \times_S \widetilde{G} & \longrightarrow & G & \longrightarrow & 1 \\ & & {\scriptstyle l'}\big\downarrow & & \big\downarrow & & \big\| & & \\ 1 & \longrightarrow & T & \xrightarrow{\;j\;} & H & \xrightarrow{\;p\;} & G & \longrightarrow & 1, \end{array} \tag{4.11}$$

*where the top row is the sequence* (4.3).

*Proof.* [Cf. [BGA, proof of Proposition 2.2]] Choose an $S$-torus $T$ and a closed immersion $l' \colon \mu' \hookrightarrow T$ and let

$$H = \big(\operatorname{rad}(G) \times_S \widetilde{G}\big) \wedge^{i', \mu', l'} T \tag{4.12}$$

be the pushout of $i'$ and $l'$. Then $H^{\text{der}}$ is isomorphic to $\widetilde{G}$ [GA1, proof of Proposition 3.2], (4.11) is a particular case of diagram (2.6) and (4.10) is indeed a $t$-resolution of $G$. □

**Lemma 4.7.** *The $t$-resolution* (4.10) *of $G$ induces a $t$-resolution of $G^{\text{der}}$*

$$1 \to T \xrightarrow{j_1} H_1 \xrightarrow{p_1} G^{\text{der}} \to 1, \tag{4.13}$$

*where $H_1 = H \times_G G^{\text{der}}$. Further, there exists a closed immersion $l \colon \mu \hookrightarrow T$ and a canonical isomorphism of right $T_{G^{\text{der}}}$-torsors over $G^{\text{der}}$*

$$H_1 \simeq \widetilde{G} \wedge^{\mu_{G^{\text{der}}}, l_{G^{\text{der}}}} T_{G^{\text{der}}, r}. \tag{4.14}$$

*Proof.* The exact sequence (4.13) is the pullback of (4.10) along $\iota \colon G^{\text{der}} \hookrightarrow G$. Thus there exists a canonical exact and commutative diagram of reductive $S$-group schemes

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T & \xrightarrow{j_1} & H_1 & \xrightarrow{p_1} & G^{\text{der}} & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle \iota_H} & & \downarrow{\scriptstyle \iota} & & \\
1 & \longrightarrow & T & \xrightarrow{j} & H & \xrightarrow{p} & G & \longrightarrow & 1.
\end{array}$$

The above diagram induces an exact sequence of reductive $S$-group schemes

$$1 \to H_1 \xrightarrow{\iota_H} H \to G^{\text{tor}} \to 1. \tag{4.15}$$

Set $R = H^{\text{tor}}$ and $R_1 = H_1^{\text{tor}}$. Then Lemma 4.2 applied to (4.15) yields $\mu_{H_1} = \mu_H = 0$, i.e., $H_1^{\text{der}}$ is simply connected and therefore (4.13) is a $t$-resolution of $G^{\text{der}}$, and an exact sequence of $S$-tori

$$1 \to R_1 \xrightarrow{r} R \to G^{\text{tor}} \to 1. \tag{4.16}$$

Now let $l \colon \mu \hookrightarrow T$ be the composition of closed immersions $\mu \hookrightarrow \mu' \overset{l'}{\hookrightarrow} T$, where the first map comes from (4.5) and the second morphism is the left-hand vertical map in diagram (4.11). Via the identification (4.4), the pullback of diagram (4.11) along $\iota \colon G^{\text{der}} \hookrightarrow G$ is (isomorphic to) the pushout diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu & \xrightarrow{i} & \widetilde{G} & \longrightarrow & G^{\text{der}} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle l} & & \downarrow & & \| & & \\
1 & \longrightarrow & T & \xrightarrow{j_1} & H_1 & \xrightarrow{p_1} & G^{\text{der}} & \longrightarrow & 1,
\end{array}$$

i.e., $H_1 \simeq \widetilde{G} \wedge^{i, \mu, l} T$. See (2.4). Now (2.7) yields the isomorphism (4.14). □

If $q_{(H_1)}$ and $q_{(H)}$ are the maps (4.6) associated to $H_1$ and $H$, respectively, then there exist exact and commutative diagrams of $S$-group schemes

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mu & \xrightarrow{i} & \widetilde{G} & \longrightarrow & G^{\text{der}} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle l} & & \downarrow & & \| & & \\
1 & \longrightarrow & T & \xrightarrow{j_1} & H_1 & \xrightarrow{p_1} & G^{\text{der}} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle q_{(H_1)}} & & & & \\
 & & R_1 & = & R_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{4.17}$$

and

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mu & \xrightarrow{i} & \widetilde{G} & \longrightarrow & G^{\mathrm{der}} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle l} & & \downarrow & & \downarrow{\scriptstyle \iota} & & \\
1 & \longrightarrow & T & \xrightarrow{j} & H & \xrightarrow{p} & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \rho_1} & \searrow^{\rho} & \downarrow{\scriptstyle q_{(H)}} & & \downarrow & & \\
1 & \longrightarrow & R_1 & \xrightarrow{r} & R & \longrightarrow & G^{\mathrm{tor}} & \longrightarrow & 1, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{4.18}$$

where the bottom row in (4.18) is the sequence (4.16) and

$$\rho = r \circ \rho_1 = q_{(H)} \circ j \colon T \to R. \tag{4.19}$$

Note that the left-hand column in (4.17) induces an exact sequence of etale twisted constant $S$-group schemes

$$1 \to R_1^* \xrightarrow{\rho_1^{(*)}} T^* \xrightarrow{l^{(*)}} \mu^* \to 1. \tag{4.20}$$

Further, by the commutativity of (4.17), we have

$$\rho_1^{(*)}(S) = j_1^{(*)}(S) \circ q_{(H_1)}^{(*)}(S), \tag{4.21}$$

where $q_{(H_1)}^{(*)} \colon R_1^* \xrightarrow{\sim} H_1^*$ is the isomorphism (4.7) associated to $H_1$.

Next we observe that the $t$-resolution (4.10) of $G$ and the associated $t$-resolution (4.13) of $G^{\mathrm{der}}$ induce, respectively, morphisms of abelian groups (3.51)

$$d \colon T^*(S) \to \operatorname{Pic} G,\ \chi \mapsto \left[H \wedge^{T_G, \chi_G} \mathbb{G}_{m,G,r}\right] \tag{4.22}$$

and

$$d' \colon T^*(S) \to \operatorname{Pic} G^{\mathrm{der}},\ \chi \mapsto \left[H_1 \wedge^{T_{G^{\mathrm{der}}}, \chi_{G^{\mathrm{der}}}} \mathbb{G}_{m,G^{\mathrm{der}},r}\right] \tag{4.23}$$

such that the following diagram commutes (3.52)

$$\begin{array}{ccc}
T^*(S) & \xrightarrow{d} & \operatorname{Pic} G \\
 & \searrow^{d'} & \downarrow{\scriptstyle \operatorname{Pic} \iota} \\
 & & \operatorname{Pic} G^{\mathrm{der}}.
\end{array} \tag{4.24}$$

Further, there exists a canonical morphism of etale sheaves on $S$ (3.53)

$$\boldsymbol{d} \colon T^* \to \mathrm{Pic}_{G/S} \tag{4.25}$$

such that $\boldsymbol{d}(S) = d$ if $S$ is strictly local and the diagram of etale sheaves on $S$ (3.54)

$$\begin{array}{ccc}
T^* & \xrightarrow{\boldsymbol{d}} & \mathrm{Pic}_{G/S} \\
 & \searrow^{\boldsymbol{d}'} & \downarrow{\scriptstyle \mathrm{Pic}_S(\iota)} \\
 & & \mathrm{Pic}_{G^{\mathrm{der}}/S}
\end{array} \tag{4.26}$$

commutes.

**Proposition 4.8.** *Let $S$ be a locally noetherian regular scheme and let $G$ be a reductive $S$-group scheme. Then there exists a canonical isomorphism of étale sheaves on $S$*

$$\mathrm{Pic}_{G/S} \simeq \mu^*,$$

*where $\mu$ is the fundamental group of $G$.*

*Proof.* We will show that the canonical maps $\mathrm{Pic}_S(\iota)\colon \mathrm{Pic}_{G/S} \to \mathrm{Pic}_{G^{\mathrm{der}}/S}$ (induced by $\iota\colon G^{\mathrm{der}} \hookrightarrow G$) and $\boldsymbol{e}\colon \mu^* \to \mathrm{Pic}_{G^{\mathrm{der}}/S}$ (4.9) are isomorphisms of etale sheaves on $S$. By [SP, Tag 07QL, Lemma 15.42.10] and standard considerations [T, Theorem 5.6(i), p. 118, Lemma 6.2.3, p. 124, and Theorem 6.4.1, p. 128], we may assume that $S$ is noetherian, strictly local and regular, in which case the proof reduces to checking that the induced maps $e = \boldsymbol{e}(S)\colon \mu^*(S) \to \mathrm{Pic}\, G^{\mathrm{der}}$ (4.8) and $\mathrm{Pic}\,\iota\colon \mathrm{Pic}\, G \to \mathrm{Pic}\, G^{\mathrm{der}}$ are isomorphisms of abelian groups. Let $H$ be given by (4.12) and let $H_1 = H \times_G G^{\mathrm{der}}$. Since $H_1^{\mathrm{der}}$ is simply connected, we have $\mathrm{Pic}\, H_1 = 0$ by Proposition 4.4. Thus Corollary 3.14(i) applied to (4.13) yields the bottom row of the following diagram of abelian groups with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_1^*(S) & \xrightarrow{\rho_1^{(*)}(S)} & T^*(S) & \xrightarrow{l^{(*)}(S)} & \mu^*(S) & \longrightarrow & 0 \\
 & & {\scriptstyle q_{(H_1)}^{(*)}(S)}\downarrow\simeq & & \| & & \downarrow{\scriptstyle e} & & \\
0 & \longrightarrow & H_1^*(S) & \xrightarrow{j_1^{(*)}(S)} & T^*(S) & \xrightarrow{d'} & \mathrm{Pic}\, G^{\mathrm{der}} & \longrightarrow & 0,
\end{array} \tag{4.27}$$

where the top row is induced by the exact sequence (4.20) using [T, II, Lemma 6.2.3, p. 124], the left-hand square commutes by (4.21) and $d'$ is the map (4.23). We will show that the right-hand square in (4.27) commutes, which will show that the right-hand vertical map $e$ is an isomorphism. Let $\chi : T \to \mathbb{G}_{m,S}$ be a morphism of $S$-group schemes. By (4.14) and [Gi, III, 1.3.1.3, p. 115, and 1.3.5, p. 116], there exist isomorphisms of right $\mathbb{G}_{m,G^{\mathrm{der}}}$-torsors over $G^{\mathrm{der}}$

$$\begin{aligned}
H_1 \wedge^{T_{G^{\mathrm{der}}},\chi_{G^{\mathrm{der}}}} \mathbb{G}_{m,G^{\mathrm{der}},r} &\simeq (\widetilde{G} \wedge^{\mu_{G^{\mathrm{der}}}, l_{G^{\mathrm{der}}}} T_{G^{\mathrm{der}},r}) \wedge^{T_{G^{\mathrm{der}}},\chi_{G^{\mathrm{der}}}} \mathbb{G}_{m,G^{\mathrm{der}},r} \\
&\simeq \widetilde{G} \wedge^{\mu_{G^{\mathrm{der}}},(\chi\circ l)_{G^{\mathrm{der}}}} \mathbb{G}_{m,G^{\mathrm{der}},r}.
\end{aligned}$$

Thus, by definitions (4.8) and (4.23), we have

$$d'(\chi) = [H_1 \wedge^{T_{G^{\mathrm{der}}},\chi_{G^{\mathrm{der}}}} \mathbb{G}_{m,G^{\mathrm{der}},r}] = [\widetilde{G} \wedge^{\mu_{G^{\mathrm{der}}},(\chi\circ l)_{G^{\mathrm{der}}}} \mathbb{G}_{m,G^{\mathrm{der}},r}] = (e \circ l^{(*)}(S))(\chi),$$

whence $d' = e \circ l^{(*)}(S)$, as claimed. Thus $e$ is an isomorphism. Now, by Proposition 4.4 and Corollary 3.14(i) applied to the sequence (4.2), the map $\mathrm{Pic}\,\iota\colon \mathrm{Pic}\, G \to \mathrm{Pic}\, G^{\mathrm{der}}$ is injective. On the other hand, the commutative diagram (4.24) and the surjectivity of $d'$ (4.27) show that $\mathrm{Pic}\,\iota$ is surjective as well, which completes the proof. □

**Remarks 4.9.**

(a) In the case $S = \mathrm{Spec}\, k$, where $k$ is a separably closed field, the preceding argument yields a new proof of the "well-known" facts $\mu^*(k) = \mathrm{Pic}\, G^{\mathrm{der}} = \mathrm{Pic}\, G$ [San81, Lemma 6.9(iii), p. 41, and (6.11.4), p. 43].

(b) Under the hypotheses of the proposition, the above proof and the commutativity of diagram (4.26) show that the following diagram of etale sheaves on $S$ commutes:

$$\begin{array}{ccc}
T^* & \xrightarrow{\boldsymbol{d}} & \mathrm{Pic}_{G/S} \\
{\scriptstyle l^{(*)}}\downarrow & & \simeq\downarrow{\scriptstyle \mathrm{Pic}_S(\iota)} \\
\mu^* & \xrightarrow[\simeq]{\boldsymbol{e}} & \mathrm{Pic}_{G^{\mathrm{der}}/S},
\end{array} \tag{4.28}$$

where $\boldsymbol{d}$ and $\boldsymbol{e}$ are the maps (4.25) and (4.9), respectively.

**Corollary 4.10.** *Let $S$ be a locally noetherian regular scheme and let $G$ be a reductive $S$-group scheme such that $G^{\mathrm{der}}$ is simply connected. Then*

$$\mathrm{Pic}_{G/S} = 0.$$

*Consequently, there exists a canonical isomorphism in $D^b(S_{\text{ét}})$*

$$\mathrm{UPic}_{G/S} \xrightarrow{\sim} \mathrm{U}_{G/S}[1].$$

*Proof.* The first assertion of the corollary is immediate from the proposition since $\mu = 0$. The triangle (2.22) yields an isomorphism $\mathrm{U}_{G/S}[1] \xrightarrow{\sim} \mathrm{UPic}_{G/S}$ whose inverse is the isomorphism of the corollary. □

We now apply Lemma 4.2 to the middle row of diagram (4.18), i.e., the given $t$-resolution $(\mathscr{R})$ of $G$ (4.10), and obtain an exact sequence of $S$-group schemes of multiplicative type

$$1 \to \mu \xrightarrow{l} T \xrightarrow{\rho} R \to G^{\mathrm{tor}} \to 1, \tag{4.29}$$

where $\rho$ is the map (4.19). The latter sequence induces an exact sequence of etale twisted constant $S$-group schemes

$$1 \to (G^{\mathrm{tor}})^* \to R^* \xrightarrow{\rho^{(*)}} T^* \xrightarrow{l^{(*)}} \mu^* \to 1. \tag{4.30}$$

Consider the following object of $C^b(S_{\text{ét}})$:

$$\pi_1^D(\mathscr{R}) = C^\bullet(\rho^{(*)}) = (R^* \xrightarrow{\rho^{(*)}} T^*),$$

where $R^*$ and $T^*$ are placed in degrees $-1$ and 0, respectively.

**Lemma 4.11.** *Let $G$ be a reductive $S$-group scheme. Then a morphism $\phi\colon \mathscr{R}_1 \to \mathscr{R}_2$ of $t$-resolutions of $G$ induces a quasi-isomorphism $\pi_1^D(\mathscr{R}_2) \xrightarrow{\sim} \pi_1^D(\mathscr{R}_1)$ in $C^b(S_{\text{ét}})$ such that, for $i = -1$ and 0, the induced isomorphisms of étale sheaves on $S$*

$$H^i(\pi_1^D(\mathscr{R}_2)) \xrightarrow{\sim} H^i(\pi_1^D(\mathscr{R}_1))$$

*are independent of the choice of $\phi$.*

*Proof.* Let $\mathscr{R}_i\colon 1 \to T_i \to H_i \to G \to 1$, where $i = 1$ and 2, be the given $t$-resolutions of $G$. By (4.29), the morphism of complexes $(\phi_{(T)}, \phi_{(R)})\colon (T_1 \to R_1) \to (T_2 \to R_2)$ is a quasi-isomorphism. Further, if $\psi\colon \mathscr{R}_1 \to \mathscr{R}_2$ is another morphism of $t$-resolutions, then the morphisms $\phi_{(H)}, \psi_{(H)}\colon H_1 \to H_2$ differ by a morphism of $S$-group schemes $\alpha\colon H_1 \to T_2$ that factors through $R_1$ [BGA, proof of Lemma 2.7]. Thus, since $\alpha$ is trivial on $\mathrm{Ker}(T_1 \to R_1)$, we conclude that the two isomorphisms $\mathrm{Ker}(T_1 \to R_1) \xrightarrow{\sim} \mathrm{Ker}(T_2 \to R_2)$ (respectively, $\mathrm{Coker}(T_1 \to R_1) \xrightarrow{\sim} \mathrm{Coker}(T_2 \to R_2)$) induced by $\phi_{(H)}$ and $\psi_{(H)}$ are equal. The lemma now follows from the above by duality, i.e., by applying to the preceding considerations the exact functor $M \to M^*$ on the category of $S$-group schemes $M$ (of finite type and) of multiplicative type. □

**Lemma 4.12.** *Let $G$ be a reductive $S$-group scheme and let $\mathscr{R}_1$ and $\mathscr{R}_2$ be two $t$-resolutions of $G$. Then $\pi_1^D(\mathscr{R}_2)$ and $\pi_1^D(\mathscr{R}_1)$ are canonically isomorphic in $D^b(S_{\text{ét}})$.*

*Proof.* The proof is similar to the proof of [BGA, Proposition 2.10], using the previous lemma in place of [BGA, Lemma 2.7]. □

**Definition 4.13.** *Let $G$ be a reductive $S$-group scheme. Using the preceding lemma, we shall henceforth identify the objects $\pi_1^D(\mathscr{R}) \in D^b(S_{\text{ét}})$ as $\mathscr{R}$ ranges over the family of all $t$-resolutions of $G$. Their common value will be denoted by $\pi_1^D(G)$ and called the dual algebraic fundamental complex of $G$. Thus $\pi_1^D(G) = \pi_1^D(\mathscr{R}) \in D^b(S_{\text{ét}})$ for any $t$-resolution $\mathscr{R}$ of $G$.*

**Remark 4.14.** As noted in the Introduction, over a field (of characteristic zero), Borovoi and van Hamel defined $\pi_1^D(G)$ in terms of a maximal torus in $G$. Their definition can be extended over any base scheme $S$ using the etale-local existence of maximal tori in reductive $S$-group schemes. This definition and Definition 4.13 can then be shown to be equivalent using the fact that a maximal torus in a reductive $S$-group scheme $G$ canonically defines a $t$-resolution of $G$. See [BGA, Lemma 3.9 and the following discussion].

A morphism of reductive $S$-group schemes $\varphi\colon G' \to G$ induces a morphism $\pi_1^D(\varphi)\colon \pi_1^D(G) \to \pi_1^D(G')$ in $D^b(S_{\text{ét}})$. Thus we obtain a contravariant functor $\pi_1^D$ from the category of reductive $S$-group schemes to $D^b(S_{\text{ét}})$. Next we will show that $\pi_1^D$ is *exact*, i.e., it transforms short exact sequences of reductive $S$-group schemes into distinguished triangles in $D^b(S_{\text{ét}})$. To this end, we first prove

**Lemma 4.15.** *Let $S$ be a scheme and let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of reductive $S$-group schemes. Then there exists an exact and commutative diagram of reductive $S$-group schemes*

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T_1 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T_2 & \longrightarrow & H_2 & \longrightarrow & G_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T_3 & \longrightarrow & H_3 & \longrightarrow & G_3 & \longrightarrow & 1, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array} \tag{4.31}$$

*where the top, middle and bottom rows are $t$-resolutions of $G_1$, $G_2$ and $G_3$, respectively.*

*Proof.* Let $1 \to T_3 \to H_3 \to G_3 \to 1$ be a $t$-resolution of $G_3$ and let

$$1 \to T_1 \to H \overset{p}{\to} G_2 \to 1 \tag{4.32}$$

be a $t$-resolution of $G_2$. Set $T_2 = T_1 \times_S T_3$ and $H_2 = H_3 \times_{G_3} H$, where $H \to G_3$ is the composition $H \to G_2 \to G_3$. By [GA1, Proposition 2.8 and its proof], there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_1 & \longrightarrow & H & \overset{p}{\longrightarrow} & G_2 & \longrightarrow & 1 \\ & & \uparrow & & \uparrow{\scriptstyle q} & & \| & & \\ 1 & \longrightarrow & T_2 & \overset{j_2}{\longrightarrow} & H_2 & \overset{p_2}{\longrightarrow} & G_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T_3 & \longrightarrow & H_3 & \longrightarrow & G_3 & \longrightarrow & 1, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array} \tag{4.33}$$

where the middle row is a $t$-resolution of $G_2$. Note that, by the definition of $H_2$, there exists a canonical exact and commutative diagram

$$\begin{array}{ccccccccc} & & & & H_2 \times_{H_3} S & \overset{\tilde{p}_0}{\underset{\sim}{\longrightarrow}} & H \times_{G_3} S & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T_3 & \longrightarrow & H_2 & \overset{\tilde{p}}{\longrightarrow} & H & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T_3 & \longrightarrow & H_3 & \longrightarrow & G_3 & \longrightarrow & 1. \end{array} \tag{4.34}$$

Now set $H_1 = H \times_{G_2} G_1$. Then the pullback of (4.32) along $G_1 \to G_2$ is an exact sequence of reductive $S$-groups schemes

$$1 \to T_1 \overset{j_1}{\to} H_1 \overset{p_1}{\to} G_1 \to 1.$$

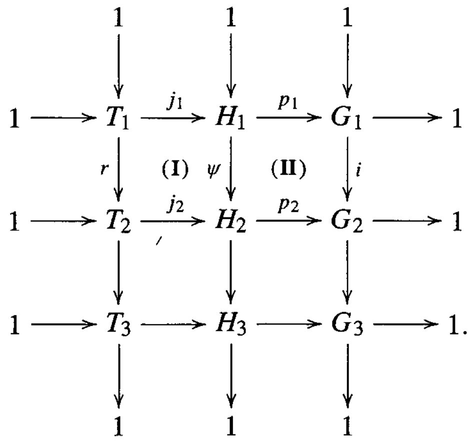
Next let $\nu \colon H_1 \overset{\sim}{\to} H \times_{G_3} S$ be the composition of canonical isomorphisms

$$H_1 = H \times_{G_2} G_1 \overset{\sim}{\to} H \times_{G_2} (G_2 \times_{G_3} S) \overset{\sim}{\to} H \times_{G_3} S \tag{4.35}$$

and consider the composition

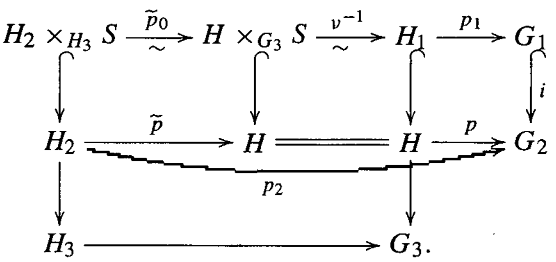
$$\psi\colon H_1 \xrightarrow[\sim]{\nu} H\times_{G_3} S \xrightarrow[\sim]{\widetilde{p}_0^{\,-1}} H_2\times_{H_3} S \hookrightarrow H_2. \tag{4.36}$$

Then $1 \to H_1 \xrightarrow{\psi} H_2 \to H_3 \to 1$ is an exact sequence of reductive $S$-group schemes. Thus the rows and columns of the following diagram of reductive $S$-group schemes are exact:




(4.37)

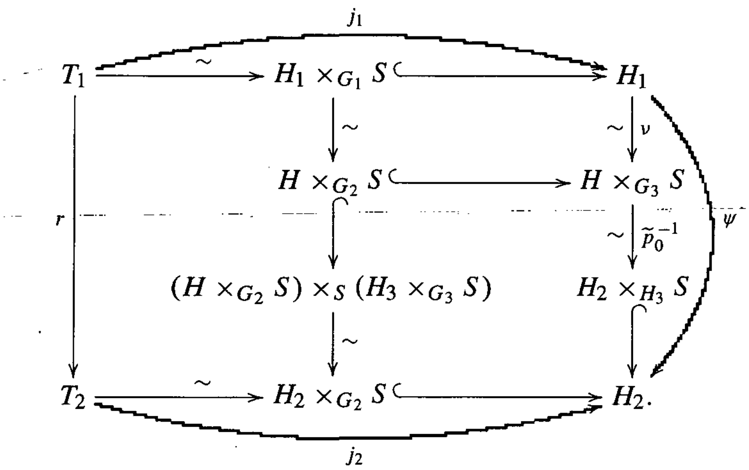
An application of [GA1, Corollary 2.11] to the middle column above shows at once that $H_1^{\rm der}$ is simply connected. Thus the top, middle and bottom rows of diagram (4.37) are $t$-resolutions of $G_1$, $G_2$ and $G_3$, respectively. Further, by the commutativity of (4.33), the lower half of diagram (4.37) commutes. Thus it remains only to check that the squares labeled (I) and (II) above commute. By the definitions of $H_1$ and $\nu$ (4.35) and the commutativity of diagrams (4.33) and (4.34), there exists an exact and commutative diagram of reductive $S$-group schemes




Thus, by the definition of $\psi$ (4.36), we have

$$p_2\circ\psi = p_2\,|_{H_2\times_{H_3}S}\circ\widetilde{p}_0^{\,-1}\circ\nu = (i\circ p_1\circ\nu^{-1}\circ\widetilde{p}_0)\circ\widetilde{p}_0^{\,-1}\circ\nu = i\circ p_1,$$

i.e., the square labeled (II) in diagram (4.37) commutes. The commutativity of the square labeled (I) in (4.37) follows from the commutativity of the following diagram:



□

**Proposition 4.16.** *Let $S$ be a scheme and let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of reductive $S$-group schemes. Then the given sequence induces a distinguished triangle in $D^b(S_{\text{ét}})$*

$$\pi_1^D(G_3) \to \pi_1^D(G_2) \to \pi_1^D(G_1) \to \pi_1^D(G_3)[1].$$

*Proof.* By Lemma 4.15, there exist $t$-resolutions $(\mathscr{R}_i)\colon 1 \to T_i \to H_i \to G_i \to 1$ of $G_i$, where $i = 1, 2$ or $3$, and an exact and commutative diagram of reductive $S$-group schemes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_1 & \longrightarrow & T_2 & \longrightarrow & T_3 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & 1. \end{array}$$

The preceding diagram induces an exact and commutative diagram of $S$-tori

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_1 & \longrightarrow & T_2 & \longrightarrow & T_3 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_3} & & \\ 1 & \longrightarrow & R_1 & \longrightarrow & R_2 & \longrightarrow & R_3 & \longrightarrow & 1, \end{array} \tag{4.38}$$

where $R_i = H_i^{\text{tor}}$, the bottom sequence is obtained by applying Lemma 4.2 to the sequence $1 \to H_1 \to H_2 \to H_3 \to 1$ and the maps $\rho_i$ are the compositions $T_i \to H_i \to R_i$ (4.19). Now (4.38) induces an exact and commutative diagram of etale twisted constant $S$-group schemes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R_3^* & \longrightarrow & R_2^* & \longrightarrow & R_1^* & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho_3^{(*)}} & & \downarrow{\scriptstyle \rho_2^{(*)}} & & \downarrow{\scriptstyle \rho_1^{(*)}} & & \\ 1 & \longrightarrow & T_3^* & \longrightarrow & T_2^* & \longrightarrow & T_1^* & \longrightarrow & 1 \end{array}$$

which induces, in turn, an exact sequence in $C^b(S_{\text{ét}})$

$$1 \to \pi_1^D(\mathscr{R}_3) \to \pi_1^D(\mathscr{R}_2) \to \pi_1^D(\mathscr{R}_1) \to 1.$$

The latter sequence induces a distinguished triangle in $D^b(S_{\text{ét}})$

$$\pi_1^D(\mathscr{R}_3) \to \pi_1^D(\mathscr{R}_2) \to \pi_1^D(\mathscr{R}_1) \to \pi_1^D(\mathscr{R}_3)[1],$$

which yields the proposition. □

**Remark 4.17.** Lemma 4.15 also yields a new proof of the exactness of the covariant functor $\pi_1(G) = \pi_1(\mathscr{R}) = \operatorname{Coker}[T_* \to R_*]$ on the category of reductive $S$-group schemes [BGA, Theorem 3.8]. Indeed, diagram (4.38) induces an exact and commutative diagram of etale twisted constant $S$-group schemes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (T_1)_* & \longrightarrow & (T_2)_* & \longrightarrow & (T_3)_* & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho_{1,(*)}} & & \downarrow{\scriptstyle \rho_{2,(*)}} & & \downarrow{\scriptstyle \rho_{3,(*)}} & & \\ 1 & \longrightarrow & (R_1)_* & \longrightarrow & (R_2)_* & \longrightarrow & (R_3)_* & \longrightarrow & 1 \end{array}$$

which immediately yields an exact sequence of etale twisted constant $S$-group schemes

$$1 \to \pi_1(G_1) \to \pi_1(G_2) \to \pi_1(G_3) \to 1.$$

**Proposition 4.18.** *Let $S$ be a locally noetherian regular scheme, let $G$ be a reductive $S$-group scheme and let $(\mathscr{R})\colon 1 \to T \xrightarrow{j} H \xrightarrow{p} G \to 1$ be a $t$-resolution of $G$. Then*

(i) *the map* $\mathrm{U}_S(p)\colon \mathrm{U}_{G/S} \to \mathrm{U}_{H/S}$ *induces an isomorphism of étale sheaves on* $S$

$$\mathrm{U}_{G/S} \xrightarrow{\sim} \operatorname{Ker}\mathrm{U}_S(j),$$

*and*

(ii) *there exists a canonical quasi-isomorphism in* $C^b(S_{\text{ét}})$

$$C^\bullet(\mathrm{U}_S(j)) \xrightarrow{\sim} \pi_1^D(\mathscr{R}).$$

*Proof.* Since $H^{\text{der}}$ is simply connected, Corollary 4.10 shows that $\operatorname{Pic}_{H/S} = 0$. Thus Corollary 3.14(ii) applied to the given $t$-resolution $(\mathscr{R})$ yields a canonical exact sequence of etale sheaves on $S$

$$1 \to G^* \xrightarrow{p^{(*)}} H^* \xrightarrow{j^{(*)}} T^* \xrightarrow{d} \operatorname{Pic}_{G/S} \to 1, \tag{4.39}$$

where $d$ is the map (4.25). On the other hand, since $\mathrm{U}_S(\varepsilon_G)$ is the zero morphism and $p \circ j = \varepsilon_G \circ \pi_T$, we have $\mathrm{U}_S(j) \circ \mathrm{U}_S(p) = 0$. Thus the sequence

$$1 \to \mathrm{U}_{G/S} \xrightarrow{\mathrm{U}_S(p)} \mathrm{U}_{H/S} \xrightarrow{\mathrm{U}_S(j)} \mathrm{U}_{T/S} \to \operatorname{Coker}\mathrm{U}_S(j) \to 1 \tag{4.40}$$

is a complex of etale sheaves on $S$. Now, since $p$ is faithfully flat and therefore schematically dominant, Lemma 2.2 shows that $\mathrm{U}_S(p)$ is injective. Thus (4.40) can fail to be exact only at $\mathrm{U}_{H/S}$. Consider the diagram

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & \mathrm{U}_{G/S} & \xrightarrow{\mathrm{U}_S(p)} & \mathrm{U}_{H/S} & \xrightarrow{\mathrm{U}_S(j)} & \mathrm{U}_{T/S} & \longrightarrow & \operatorname{Coker}\mathrm{U}_S(j) & \longrightarrow & 1 \\
 & & {\scriptstyle\omega_G}\downarrow\simeq & \text{(I)} & {\scriptstyle\omega_H}\downarrow\simeq & \text{(II)} & {\scriptstyle\omega_T}\downarrow\simeq & \text{(III)} & \vdots\simeq & & \\
1 & \longrightarrow & G^* & \xrightarrow{p^{(*)}} & H^* & \xrightarrow{j^{(*)}} & T^* & \xrightarrow{d} & \operatorname{Pic}_{G/S} & \longrightarrow & 1 \\
 & & {\scriptstyle\left(q_{(G)}^{(*)}\right)^{-1}}\downarrow\simeq & & {\scriptstyle\left(q_{(H)}^{(*)}\right)^{-1}}\downarrow\simeq & & \| & & {\scriptstyle e^{-1}\circ\operatorname{Pic}_S(\iota)}\downarrow\simeq & & \\
1 & \longrightarrow & (G^{\text{tor}})^* & \longrightarrow & R^* & \xrightarrow{\rho^{(*)}} & T^* & \xrightarrow{\iota^{(*)}} & \mu^* & \longrightarrow & 1.
\end{array} \tag{4.41}$$

The top row above is the complex (4.40), and the middle and bottom rows are the exact sequences (4.39) and (4.30), respectively. The continuous vertical arrows in the upper half of the diagram are the maps (2.13). In the lower half of the diagram, the left-hand and middle vertical arrows are the inverses of the maps (4.7) associated to $G$ and $H$, respectively. Further, the map $e^{-1} \circ \operatorname{Pic}_S(\iota)$ is an isomorphism by the proof of Proposition 4.8. Using the definitions of the maps (2.13) (see [GA2, Lemma 4.8]) and the identities $p \circ \varepsilon_H = \varepsilon_G$ and $j \circ \varepsilon_T = \varepsilon_H$, it is not difficult to check that the squares labeled **(I)** and **(II)** in (4.41) commute. Thus the top row of diagram (4.41), i.e., the complex (4.40), is exact, whence (i) follows. Further, there exists a unique way to define the discontinuous vertical arrow in diagram (4.41) so that the resulting map is an isomorphism of etale sheaves on $S$ and the square labeled **(III)** in (4.41) commutes. Since the bottom squares in diagram (4.41) also commute by the commutativity of diagrams (4.18) and (4.28), the entire diagram (4.41) commutes, whence the map $C^\bullet(\mathrm{U}_S(j)) \to C^\bullet\big(\rho^{(*)}\big) = \pi_1^D(\mathscr{R})$ with components $\big(q_{(H)}^{(*)}\big)^{-1} \circ \omega_H\colon \mathrm{U}_{H/S} \to R^*$ and $\omega_T\colon \mathrm{U}_{T/S} \to T^*$ (in degrees $-1$ and 0, respectively) is a morphism of complexes. The diagram shows that the map just defined is, in fact, a quasi-isomorphism in $C^b(S_{\text{ét}})$, which completes the proof. □

**Lemma 4.19.** *Let* $S$ *be a locally noetherian regular scheme and let* $G$ *be a reductive* $S$*-group scheme. Every* $t$*-resolution* $(\mathscr{R})$ *of* $G$ *induces an isomorphism in* $D^b(S_{\text{ét}})$

$$\operatorname{UPic}_{G/S} \xrightarrow{\sim} \pi_1^D(\mathscr{R}).$$

*Proof.* Let $1 \to T \xrightarrow{j} H \xrightarrow{p} G \to 1$ be the given $t$-resolution of $G$. Then

$$\operatorname{UPic}_S(j) \circ \operatorname{UPic}_S(p) = \operatorname{UPic}_S(p \circ j) = \operatorname{UPic}_S(\varepsilon_G \circ \pi_T) = 0, \tag{4.42}$$

since $\mathrm{UPic}_S(\varepsilon_G)\colon \mathrm{UPic}_{G/S} \to \mathrm{UPic}_{S/S} = 0$ is the zero morphism. Now, since $D^b(S_{\text{ét}})$ is a triangulated category, there exists a distinguished triangle in $D^b(S_{\text{ét}})$ containing the morphism $\mathrm{UPic}_S(p)\colon \mathrm{UPic}_{G/S} \to \mathrm{UPic}_{H/S}$, i.e., for some object $Z$ of $D^b(S_{\text{ét}})$, the top row of the following diagram is a distinguished triangle in $D^b(S_{\text{ét}})$:

$$\begin{array}{ccccccc}
\mathrm{UPic}_{G/S} & \xrightarrow{\mathrm{UPic}_S(p)} & \mathrm{UPic}_{H/S} & \xrightarrow{u} & Z & \longrightarrow & \mathrm{UPic}_{G/S}[1] \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow\simeq & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\
C^\bullet(\mathrm{U}_S(j)) & \xrightarrow{v} & \mathrm{U}_{H/S}[1] & \xrightarrow{\mathrm{U}_S(j)[1]} & \mathrm{U}_{T/S}[1] & \longrightarrow & C^\bullet(\mathrm{U}_S(j))[1].
\end{array} \tag{4.43}$$

The bottom row of the above diagram is a distinguished triangle of the form (2.1) and the map labeled $g$ is the isomorphism of Corollary 4.10. Note that, since $H^r(\mathrm{UPic}_{H/S}) = 0$ for $r \neq -1$ and $H^r(\mathrm{UPic}_{G/S}) = 0$ for $r \neq -1, 0$, we have $H^r(Z) = 0$ for $r \neq -2, -1$. Now, by (4.42) and the commutativity of diagram (2.25) applied to the $S$-morphism $j$, we have $\mathrm{U}_S(j)[1] \circ g \circ \mathrm{UPic}_S(p) = 0$. Thus, by [BBD, Proposition 1.1.9, p. 23], there exist morphisms $f\colon \mathrm{UPic}_{G/S} \to C^\bullet(\mathrm{U}_S(j))$ and $h\colon Z \to \mathrm{U}_{T/S}[1]$ in diagram (4.43) such that $(f, g, h)$ is a morphism of distinguished triangles. Now, since $H^{-1}(v)$ is the inclusion $\mathrm{Ker}\,\mathrm{U}_S(j) \hookrightarrow \mathrm{U}_{H/S}$, the commutativity of the left-hand square in (4.43) shows that the map $H^{-1}(f)\colon H^{-1}(\mathrm{UPic}_{G/S}) \to H^{-1}(C^\bullet(\mathrm{U}_S(j)))$ is the isomorphism $\mathrm{U}_{G/S} \xrightarrow{\sim} \mathrm{Ker}\,\mathrm{U}_S(j)$ of Proposition (4.18)(i). Further, $H^{-2}(Z) = 0$, whence $H^r(Z) = 0$ for all $r \neq -1$. We now consider the diagram

$$\begin{array}{ccccccc}
C^\bullet(\mathrm{U}_S(j)) & \xrightarrow{v} & \mathrm{U}_{H/S}[1] & \xrightarrow{\mathrm{U}_S(j)[1]} & \mathrm{U}_{T/S}[1] & \longrightarrow & C^\bullet(\mathrm{U}_S(j))[1] \\
{\scriptstyle \tilde{f}}\downarrow & & {\scriptstyle g^{-1}}\downarrow\simeq & & {\scriptstyle \tilde{h}}\downarrow & & \downarrow{\scriptstyle \tilde{f}[1]} \\
\mathrm{UPic}_{G/S} & \xrightarrow{\mathrm{UPic}_S(p)} & \mathrm{UPic}_{H/S} & \xrightarrow{u} & Z & \longrightarrow & \mathrm{UPic}_{G/S}[1].
\end{array} \tag{4.44}$$

We claim that the composition $u \circ g^{-1} \circ v : C^\bullet(\mathrm{U}_S(j)) \to Z$ is the zero morphism. Indeed, since $v \circ f = g \circ \mathrm{UPic}_S(p)$ and $H^{-1}(f)$ is an isomorphism, we have

$$H^{-1}(u) \circ H^{-1}(g^{-1}) \circ H^{-1}(v) = H^{-1}(u) \circ H^{-1}(\mathrm{UPic}_S(p)) \circ H^{-1}(f)^{-1} = 0$$

since $u \circ \mathrm{UPic}_S(p) = 0$ [Ver, Corollary 1.2.2, p. 97]. We conclude, as above, that there exist morphisms $\tilde{f}$ and $\tilde{h}$ in diagram (4.44) such that $(\tilde{f}, g^{-1}, \tilde{h})$ is a morphism of distinguished triangles. Now the concatenation of diagrams (4.44) and (4.43) (in that order) is a diagram of the form

$$\begin{array}{ccccccc}
C^\bullet(\mathrm{U}_S(j)) & \xrightarrow{v} & \mathrm{U}_{H/S}[1] & \xrightarrow{\mathrm{U}_S(j)[1]} & \mathrm{U}_{T/S}[1] & \longrightarrow & C^\bullet(\mathrm{U}_S(j))[1] \\
{\scriptstyle \alpha}\downarrow & & \| & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \alpha[1]} \\
C^\bullet(\mathrm{U}_S(j)) & \xrightarrow{v} & \mathrm{U}_{H/S}[1] & \xrightarrow{\mathrm{U}_S(j)[1]} & \mathrm{U}_{T/S}[1] & \longrightarrow & C^\bullet(\mathrm{U}_S(j))[1]
\end{array} \tag{4.45}$$

(namely for $\alpha = f \circ \tilde{f}$ and $\beta = h \circ \tilde{h}$). Since $H^0(C^\bullet(\mathrm{U}_S(j))) = \mathrm{Coker}\,\mathrm{U}_S(j) \simeq \mu^*$ by the proof of Proposition 4.18 (see diagram (4.41)) and $\mathrm{U}_{T/S} \simeq T^*$ by Lemma 2.3, we have

$$\mathrm{Hom}_{D^b(S_{\text{ét}})}(C^\bullet(\mathrm{U}_S(j)), \mathrm{U}_{T/S}) = \mathrm{Hom}(\mu^*, T^*) = 0.$$

Thus [BBD, Proposition 1.1.9, p. 23] yields the existence of *unique* morphisms $\alpha$ and $\beta$ in diagram (4.45) such that $(\alpha, 1_{\mathrm{U}_{H/S}[1]}, \beta)$ is a morphism of distinguished triangles, i.e., $\alpha = 1_{C^\bullet(\mathrm{U}_S(j))}$ and $\beta = 1_{\mathrm{U}_{T/S}[1]}$. We conclude that $f \circ \tilde{f} = 1_{C^\bullet(\mathrm{U}_S(j))}$, whence $H^0(f)\colon H^0(\mathrm{UPic}_{G/S}) \to H^0(C^\bullet(\mathrm{U}_S(j)))$ is a surjective morphism of etale sheaves on $S$. Since $H^0(\mathrm{UPic}_{G/S}) = \mathrm{Pic}_{G/S}$ and $H^0(C^\bullet(\mathrm{U}_S(j))) = \mathrm{Coker}\,\mathrm{U}_S(j)$ are both isomorphic to the etale sheaf $\mu^*$, which has finite stalks, a counting argument now shows that $H^0(f)$ is, in fact, an isomorphism of etale sheaves on $S$. We conclude that $f\colon \mathrm{UPic}_{G/S} \to C^\bullet(\mathrm{U}_S(j))$ is an isomorphism in $D^b(S_{\text{ét}})$. The composition of the preceding isomorphism and the canonical isomorphism $C^\bullet(\mathrm{U}_S(j)) \xrightarrow{\sim} \pi_1^D(\mathscr{R})$ of Proposition 4.18(ii) is the isomorphism of the lemma. □

We may now prove the main theorem of the paper.

**Theorem 4.20.** *Let $S$ be a locally noetherian regular scheme and let $G$ be a reductive $S$-group scheme. Then there exists an isomorphism in $D^b(S_{\text{ét}})$*

$$\mathrm{UPic}_{G/S} \xrightarrow{\sim} \pi_1^D(G)$$

*which is functorial in $G$.*

*Proof.* By definition of $\pi_1^D(G)$ (see Definition 4.13), it suffices to check that the isomorphism $\mathrm{UPic}_{G/S} \xrightarrow{\sim} \pi_1^D(\mathscr{R})$ of Lemma 4.19 induced by a given $t$-resolution $(\mathscr{R})$ of $G$ is functorial in $G$. To this end, let $\varphi\colon G' \to G$ be a morphism of reductive $S$-group schemes. By [BGA, Lemma 3.3], there exists a $t$-resolution of $\varphi$, i.e., an exact and commutative diagram of reductive $S$-group schemes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T' & \xrightarrow{j'} & H' & \xrightarrow{p'} & G' & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varphi} & & \\ 1 & \longrightarrow & T & \xrightarrow{j} & H & \xrightarrow{p} & G & \longrightarrow & 1 \end{array} \quad \begin{array}{c}(\mathscr{R}')\\ \\ (\mathscr{R}),\end{array} \tag{4.46}$$

where the top and bottom rows are $t$-resolutions of $G'$ and $G$, respectively. The left-hand square in the above diagram induces a morphism of complexes $\theta\colon C^\bullet(\mathrm{U}_S(j)) \to C^\bullet(\mathrm{U}_S(j'))$ whose components are $\mathrm{U}_S(\delta)$ and $\mathrm{U}_S(\gamma)$ in degrees $-1$ and 0, respectively. We also note that, since the map $f_{(G)} = f_{(G,\mathscr{R})}\colon \mathrm{UPic}_{G/S} \to C^\bullet(\mathrm{U}_S(j))$ in diagram (4.43) is an isomorphism, the map $h\colon Z \to \mathrm{U}_{T/S}[1]$ in that diagram is an isomorphism as well [Ver, Corollary 1.2.3, p. 97]. Now let $f_{(G')} = f_{(G',\mathscr{R}')}\colon \mathrm{UPic}_{G'/S} \xrightarrow{\sim} C^\bullet(\mathrm{U}_S(j'))$ and consider the following diagram whose rows are distinguished triangles in $D^b(S_{\text{ét}})$:

$$\theta\left(\begin{array}{ccccccc} C^\bullet(\mathrm{U}_S(j)) & \xrightarrow{v} & \mathrm{U}_{H/S}[1] & \xrightarrow{\mathrm{U}_S(j)[1]} & \mathrm{U}_{T/S}[1] & \longrightarrow & C^\bullet(\mathrm{U}_S(j))[1] \\ {\scriptstyle f_{(G)}^{-1}}\downarrow\simeq & & {\scriptstyle g_{(H)}^{-1}}\downarrow\simeq & & {\scriptstyle h^{-1}}\downarrow\simeq & & {\scriptstyle f_{(G)}^{-1}[1]}\downarrow\simeq \\ \mathrm{UPic}_{G/S} & \xrightarrow{\mathrm{UPic}_S(p)} & \mathrm{UPic}_{H/S} & \xrightarrow{u} & Z & \longrightarrow & \mathrm{UPic}_{G/S}[1] \\ {\scriptstyle \mathrm{UPic}_S(\varphi)}\downarrow & (\mathrm{I}) & {\scriptstyle \mathrm{UPic}_S(\delta)}\downarrow & & {\scriptstyle l}\downarrow & m & {\scriptstyle \mathrm{UPic}_S(\varphi)[1]}\downarrow \\ \mathrm{UPic}_{G'/S} & \xrightarrow{\mathrm{UPic}_S(p')} & \mathrm{UPic}_{H'/S} & \xrightarrow{u'} & Z' & \xrightarrow{l} & \mathrm{UPic}_{G'/S}[1] \\ {\scriptstyle f_{(G')}}\downarrow\simeq & & {\scriptstyle g_{(H')}}\downarrow\simeq & & {\scriptstyle h'}\downarrow\simeq & & {\scriptstyle f_{(G')}[1]}\downarrow\simeq \\ C^\bullet(\mathrm{U}_S(j')) & \xrightarrow{v'} & \mathrm{U}_{H'/S}[1] & \xrightarrow{\mathrm{U}_S(j')[1]} & \mathrm{U}_{T'/S}[1] & \longrightarrow & C^\bullet(\mathrm{U}_S(j'))[1]. \end{array}\right)\theta[1] \tag{4.47}$$

Let

$$\lambda_1 = f_{(G')} \circ \mathrm{UPic}_S(\varphi) \circ f_{(G)}^{-1}$$
$$\lambda_2 = g_{(H')} \circ \mathrm{UPic}_S(\delta) \circ g_{(H)}^{-1}$$

be the first and second vertical compositions in the preceding diagram, respectively. By the definitions of $f_{(G)}$ and $f_{(G')}$, $(f_{(G)}^{-1}, g_{(H)}^{-1}, h^{-1})$ and $(f_{(G')}, g_{(H')}, h')$ are morphisms of distinguished triangles. Further, the square labeled (I) above commutes since (4.46) commutes. On the other hand, it follows from the definitions of the maps $g_{(H)}, g_{(H')}, v, v'$ (2.1) and $\theta$ that the following diagram commutes:

$$\begin{array}{ccc} C^\bullet(\mathrm{U}_S(j)) & \xrightarrow{v} & \mathrm{U}_{H/S}[1] \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\lambda_2} \\ C^\bullet(\mathrm{U}_S(j')) & \xrightarrow{v'} & \mathrm{U}_{H'/S}[1]. \end{array}$$

Thus, by [BBD, Proposition 1.1.9, p. 23], there exist morphisms $l\colon Z \to Z'$ and $m\colon \mathrm{U}_{T/S}[1] \to \mathrm{U}_{T'/S}[1]$ in (4.47) such that $(\lambda_1, \lambda_2, h' \circ l \circ h^{-1})$ and $(\theta, \lambda_2, m)$ are morphisms of distinguished triangles with the same second component $\lambda_2$. Now, since

$$\mathrm{Hom}_{D^b(S_{\mathrm{\acute{e}t}})}(C^\bullet(\mathrm{U}_S(j)), \mathrm{U}_{T'/S}) = \mathrm{Hom}(\mu^*, (T')^*) = 0,$$

the uniqueness assertion in [BBD, Proposition 1.1.9, p. 23] yields $\theta = \lambda_1$, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathrm{UPic}_{G/S} & \xrightarrow{f_{(G)}} & C^\bullet(\mathrm{U}_S(j)) \\ \downarrow{\scriptstyle \mathrm{UPic}_S(\varphi)} & & \downarrow{\scriptstyle \theta} \\ \mathrm{UPic}_{G'/S} & \xrightarrow{f_{(G')}} & C^\bullet(\mathrm{U}_S(j')). \end{array}$$

Therefore the following diagram commutes as well

$$\begin{array}{ccccc} \mathrm{UPic}_{G/S} & \xrightarrow{f_{(G)}} & C^\bullet(\mathrm{U}_S(j)) & \xrightarrow{\sim} & \pi_1^D(\mathscr{R}) \\ \downarrow{\scriptstyle \mathrm{UPic}_S(\varphi)} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \pi_1^D(\varphi)} \\ \mathrm{UPic}_{G'/S} & \xrightarrow{f_{(G')}} & C^\bullet(\mathrm{U}_S(j')) & \xrightarrow{\sim} & \pi_1^D(\mathscr{R}'), \end{array}$$

where the unlabeled maps are the canonical isomorphisms of Proposition (4.18)(ii). The required functoriality is now established. □

The sequences in (ii) and (iii) below generalize (respectively) [San81, (6.11.4) and (6.11.2)].

**Corollary 4.21.** *Let $S$ be a locally noetherian regular scheme and let*

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

*be an exact sequence of reductive $S$-group schemes. Then the above sequence induces*

(i) *a distinguished triangle in $D^b(S_{\mathrm{\acute{e}t}})$*

$$\mathrm{UPic}_{G_3/S} \to \mathrm{UPic}_{G_2/S} \to \mathrm{UPic}_{G_1/S} \to \mathrm{UPic}_{G_3/S}[1],$$

(ii) *an exact sequence of étale sheaves on $S$*

$$1 \to G_3^* \to G_2^* \to G_1^* \to \mathrm{Pic}_{G_3/S} \to \mathrm{Pic}_{G_2/S} \to \mathrm{Pic}_{G_1/S} \to 1$$

*and*

(iii) *an exact sequence of abelian groups*

$$\begin{aligned} 1 \to G_3^*(S) \to G_2^*(S) \to G_1^*(S) &\to \mathrm{Pic}\, G_3 \to \mathrm{Pic}\, G_2 \to \mathrm{NPic}(G_1/S) \\ &\to \mathrm{Br}_{\mathrm{a}}(G_3/S) \to \mathrm{Br}_{\mathrm{a}}(G_2/S) \to \mathrm{Br}_{\mathrm{a}}(G_1/S). \end{aligned}$$

*Proof.* Assertion (i) is immediate from the theorem and Proposition 4.16. By (2.23) and (2.24), the distinguished triangle in (i) induces an exact sequence (of etale sheaves on $S$) $1 \to \mathrm{U}_{G_3/S} \to \mathrm{U}_{G_2/S} \to \mathrm{U}_{G_1/S} \to \mathrm{Pic}_{G_3/S} \to \mathrm{Pic}_{G_2/S} \to \mathrm{Pic}_{G_1/S} \to 1$. Assertion (ii) now follows from Lemma 2.3. To prove (iii), let $i\colon G_1 \to G_2$ and $p\colon G_2 \to G_3$ be the given $S$-morphisms and, for $j = 1, 2$ or $3$, let $\pi_j\colon G_j \to S$ and $\varepsilon_j\colon S \to G_j$ denote the structural morphism and unit section of $G_j$, respectively. Now recall from Corollary 3.14(iii) the exact sequence of abelian groups

$$\begin{aligned} 0 \to G_3^*(S) \to G_2^*(S) \to G_1^*(S) &\to \mathrm{Pic}\, G_3 \to \mathrm{Pic}\, G_2 \to \mathrm{NPic}(G_1/S) \\ &\xrightarrow{\alpha} \mathrm{Br}_1(G_3/S) \xrightarrow{\mathrm{Br}_1 p} \mathrm{Br}_1(G_2/S) \xrightarrow{\varphi'} \mathrm{Br}_{\varepsilon_1}(G_1/S), \end{aligned} \tag{4.48}$$

where $\varphi'$ is the map (3.31), and consider the following diagram

$$
\begin{array}{ccccccc}
 & & \mathrm{Br}_{\varepsilon_3}(G_3/S) & \xrightarrow{\mathrm{Br}_{\varepsilon_2,\varepsilon_3}p} & \mathrm{Br}_{\varepsilon_2}(G_2/S) & \xrightarrow{\mathrm{Br}_{\varepsilon_1,\varepsilon_2}i} & \mathrm{Br}_{\varepsilon_1}(G_1/S) \\
 & \overset{\alpha'}{\nearrow} & \downarrow & & \downarrow & & \Vert \\
\mathrm{NPic}(G_1/S) & \xrightarrow{\alpha} & \mathrm{Br}_1(G_3/S) & \xrightarrow{\mathrm{Br}_1 p} & \mathrm{Br}_1(G_2/S) & \xrightarrow{\varphi'} & \mathrm{Br}_{\varepsilon_1}(G_1/S) \\
 & & \downarrow & & \downarrow & & \\
 & & \mathrm{Im}\,\mathrm{Br}_1\pi_3 & \xrightarrow{\sim} & \mathrm{Im}\,\mathrm{Br}_1\pi_2 & & 
\end{array}
$$

The bottom horizontal arrow is the isomorphism (2.34) induced by $\mathrm{Br}_1 p$ and all squares above commute by Remark 3.8 and the definition of $\mathrm{Br}_{\varepsilon_2,\varepsilon_3} p$ (2.35). Further, the middle row is exact by the exactness of (4.48) and the left-hand and middle columns are (split) exact sequences of abelian groups by (2.32). Since the bottom map is injective (in fact, an isomorphism), the commutativity of all three squares in the diagram shows that the top row is exact as well. On the other hand, by Remark 3.12(a) applied to the triples $(G_2, G_3, G_1)$ and $(S, S, S)$, the composition of $\alpha$ and the map $\mathrm{Br}_1\varepsilon_3\colon \mathrm{Br}_1(G_3/S) \to \mathrm{Br}\,S$ factors through $\mathrm{NPic}_S(\varepsilon_1)\colon \mathrm{NPic}(G_1/S) \to \mathrm{NPic}(S/S) = 0$. Thus $\alpha$ factors through the map labeled $\alpha'$ above and (2.36) (applied to the $S$-morphism $G_2 \to G_3$) shows that $\mathrm{Im}\,\alpha' = \mathrm{Ker}\,\mathrm{Br}_{\varepsilon_2,\varepsilon_3} p$. Thus (4.48) induces an exact sequence

$$
\begin{aligned}
0 \to G_3^*(S) \to G_2^*(S) &\to G_1^*(S) \to \mathrm{Pic}\,G_3 \to \mathrm{Pic}\,G_2 \to \mathrm{NPic}(G_1/S) \\
&\overset{\alpha'}{\to} \mathrm{Br}_{\varepsilon_3}(G_3/S) \overset{\mathrm{Br}_{\varepsilon_2,\varepsilon_3}p}{\longrightarrow} \mathrm{Br}_{\varepsilon_2}(G_2/S) \overset{\mathrm{Br}_{\varepsilon_1,\varepsilon_2}i}{\longrightarrow} \mathrm{Br}_{\varepsilon_1}(G_1/S).
\end{aligned}
$$

Finally, by the commutativity of diagram (2.37), the above sequence induces an exact sequence of abelian groups

$$
\begin{aligned}
0 \to G_3^*(S) \to G_2^*(S) \to G_1^*(S) &\to \mathrm{Pic}\,G_3 \to \mathrm{Pic}\,G_2 \to \mathrm{NPic}(G_1/S) \\
&\overset{\alpha''}{\to} \mathrm{Br}_{\mathrm{a}}(G_3/S) \overset{\mathrm{Br}_{\mathrm{a}}p}{\longrightarrow} \mathrm{Br}_{\mathrm{a}}(G_2/S) \overset{\mathrm{Br}_{\mathrm{a}}i}{\longrightarrow} \mathrm{Br}_{\mathrm{a}}(G_1/S),
\end{aligned}
$$

where $\alpha'' = c_{(G_3),\varepsilon_3} \circ \alpha'$ and $c_{(G_3),\varepsilon_3}$ is given by (2.33). □

**Remark 4.22.** In the setting of the corollary assume, in addition, that $S$ is noetherian and irreducible with function field $K$. As in Remark 3.12(c), the canonical maps $G_i^*(S) \to G_i^*(K)$ (where $i = 1, 2$ and $3$) and $\mathrm{NPic}(G_1/S) \to \mathrm{Pic}\,G_{1,K}$ are isomorphisms of abelian groups. Further, by Lemma 4.3, $\mathrm{Pic}\,G_i$ is canonically isomorphic to $\mathrm{Pic}\,S \oplus \mathrm{Pic}\,G_{i,K}$ for $i = 2$ and $3$. Thus the sequence in part (iii) of the corollary is canonically isomorphic to an exact sequence

$$
\begin{aligned}
1 \to G_3^*(K) \to G_2^*(K) &\to G_1^*(K) \to \mathrm{Pic}\,S \oplus \mathrm{Pic}\,G_{3,K} \to \mathrm{Pic}\,S \oplus \mathrm{Pic}\,G_{2,K} \\
&\to \mathrm{Pic}\,G_{1,K} \to \mathrm{Br}_{\mathrm{a}}(G_3/S) \overset{\mathrm{Br}_{\mathrm{a}}p}{\longrightarrow} \mathrm{Br}_{\mathrm{a}}(G_2/S) \to \mathrm{Br}_{\mathrm{a}}(G_1/S).
\end{aligned}
$$

Now, by a functoriality argument similar to that given in Remark 3.12(c), $\mathrm{Ker}\,\mathrm{Br}_{\mathrm{a}}\,p$ is canonically isomorphic to $\mathrm{Ker}\,\mathrm{Br}_{\mathrm{a}}\,p_K$. Thus most of the sequence in part (iii) of the corollary is essentially equivalent to the corresponding subsequence over $K$. A similar fact becomes evident when the sequence in Corollary 3.14(iii) is compared with [San81, (6.10.3), p. 43].

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