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Divisibility of class numbers of certain families of quadratic fields

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Abstract. We construct some families of quadratic fields whose class numbers are divisible by 3. The main tools used are a trinomial introduced by Kishi and a parametrization of Kishi and Miyake of a family of quadratic fields whose class numbers are divisible by 3. At the end we compute class number of these fields for some small values and verify our results.

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1. Introduction

The ideal class group or more precisely the class number of number fields is one of the most fundamental and mysterious objects associated with the field extensions. Starting from Gauss, this area has attracted the attention of many researchers. It is well-known that there exist infinitely many quadratic fields each with class number divisible by a given integer $g \geq 2$. In particular, Nagell [11] proved that there are infinitely many imaginary quadratic fields with class number divisible by a given integer $g \geq 2$. On the other hand, for real quadratic field case, Honda [2], Yamamoto [13], Weinberger [12] and Ichimura [5] independently proved that there are infinitely many real quadratic fields each with class number divisible by a given integer $g \geq 2$. In recent years, the study is concentrating on characterising such fields, i.e. each with class number divisible by a given integer $g \geq 2$. In this direction, Kishi and Miyake [9] gave a parametrization of quadratic fields with class number divisible by 3 and this enabled enlargement of the list of families of quadratic fields with the divisibility properties. Chakraborty and Murty [1], Kishi [6], Hoque and Saikia [3] contributed some members to this list.

In this paper we provide some infinite, simply parametrized new families of quadratic fields with class number divisible by 3. More precisely, we show that under certain conditions on the integers a, b, m, n, p and r, the class numbers of the fields $\mathbb{Q}(\sqrt{3(4m^{3n} - k^2)})$, $\mathbb{Q}(\sqrt{-(m^2n^2 \pm 4n)/3})$, $\mathbb{Q}(\sqrt{-(3^m p^{2n} + r)})$, $\mathbb{Q}(\sqrt{3(a^{3n} - b^{2n})})$, $\mathbb{Q}(\sqrt{3(4a^{3n} - b^{2n})})$, $\mathbb{Q}(\sqrt{-3(4m^3 + 1)})$, $\mathbb{Q}(\sqrt{3(2m^{3n} - 1)})$ and $\mathbb{Q}(\sqrt{1 - 2m^3})$ are divisible by 3. We begin by fixing some notations.

Notations. For a number field K, Δ_K and $\mathcal{O}_K$ denote the discriminant and the ring of integers of K, respectively. We denote by $N_{K/\mathbb{Q}}$ and $T_{K/\mathbb{Q}}$ the norm and trace map of a number field K, respectively. For a non-square integer d, $h(d)$ denotes the class number of $\mathbb{Q}(\sqrt{d})$. For a prime number p and an integer n, $v_p(n)$ denotes the greatest exponent μ of p such that $p^\mu \mid n$. For a polynomial f, $S_{\mathbb{Q}}(f)$ denotes the splitting field of f over $\mathbb{Q}$.

2. Kishi's trinomial and class numbers of quadratic fields

We begin with some lemmas and then recall a criterion for an extension to be unramified. Finally, we construct some quadratic fields each with class number divisible by 3 using Kishi's trinomial.

Let $\alpha \in \mathcal{O}_K$ with $N_{K/\mathbb{Q}}(\alpha) \in \mathbb{Z}^3$ and

$$f_\alpha(X) := X^3 - 3[N_{K/\mathbb{Q}}(\alpha)]^{1/3}X - T_{K/\mathbb{Q}}(\alpha) \tag{2.1}$$

The trinomial $f_\alpha(X)$ was introduced by Kishi in [7]. We recall the following result of Kishi [8].

Lemma 2.1. *Let $K = \mathbb{Q}(\sqrt{d})$. Suppose $\alpha = \frac{a+b\sqrt{d}}{2} \in \mathcal{O}_K$ with $N_{K/\mathbb{Q}}(\alpha) \in \mathbb{Z}^3$. Then $f_\alpha(X)$ is reducible over $\mathbb{Q}$ if and only if α is a cube in K.*

Let $d(\neq 1, -3)$ be a square-free integer and

$$D = \begin{cases} -d/3 & \text{if } d \text{ is a multiple of } 3, \\ -3d & \text{otherwise.} \end{cases}$$

Let $K = \mathbb{Q}(\sqrt{d})$ and $L = \mathbb{Q}(\sqrt{D})$. Also,

$$R_d := \{\alpha \in \mathcal{O}_K : \alpha \text{ is not a cube in } K \text{ and } N_{K/\mathbb{Q}}(\alpha) \text{ is a cube in } \mathbb{Z}\}$$

and

$$R_D := \{\alpha \in \mathcal{O}_L : \alpha \text{ is not a cube in } L \text{ and } N_{L/\mathbb{Q}}(\alpha) \text{ is a cube in } \mathbb{Z}\}.$$

It is clear that the subset R_d (respectively R_D) contains all those units in K which are not cubes in K (respectively in L). Further let,

$$R_d^* := \{\alpha \in R_d : \gcd(N_{K/\mathbb{Q}}(\alpha), T_{K/\mathbb{Q}}(\alpha)) = 1\}$$

and

$$R_D^* := \{\alpha \in R_D : \gcd(N_{L/\mathbb{Q}}(\alpha), T_{L/\mathbb{Q}}(\alpha)) = 1\}.$$

We can now recall a result of Kishi ([7], Proposition 6.5) which is one of the main ingredient for deriving our results.

Lemma 2.2. *Let $\alpha \in R_D^*$ (resp. $\alpha \in R_d^*$). Then $S_{\mathbb{Q}}(f_\alpha)$ is an S_3-field containing $K = \mathbb{Q}(\sqrt{d})$ (resp. $L = \mathbb{Q}(\sqrt{D})$) which is a cyclic cubic extension of K (resp. L) unramified outside 3 and contains a cubic subfield K' with $v_3(\Delta_{K'}) \neq 5$. Conversely, every S_3-field containing K (resp. L) which is unramified outside 3 over K (resp. L) and contains a cubic subfield K' satisfying $v_3(\Delta_{K'}) \neq 5$ is given by $S_{\mathbb{Q}}(f_\alpha)$ with $\alpha \in R_D^*$ (resp. $\alpha \in R_d^*$).*

The following result of Llorente and Nart ([10], Theorem 1) talks about ramification at the prime $p = 3$.

Lemma 2.3. *Suppose that*

$$g(X) := X^3 - aX - b \in \mathbb{Z}[X]$$

is irreducible over $\mathbb{Q}$ and that either $v_3(a) < 2$ or $v_3(b) < 3$ holds. Let θ be a root of $g(X)$. Then 3 is totaly ramified in $\mathbb{Q}(\theta)/\mathbb{Q}$ if and only if one of the following conditions holds:

(LN-1) $1 \leq v_3(b) \leq v_3(a)$,
(LN-2) $3 \mid a$, $a \not\equiv 3 \pmod 9$, $3 \nmid b$ *and* $b^2 \not\equiv a + 1 \pmod 9$,
(LN-3) $a \equiv 3 \pmod 9$, $3 \nmid b$ *and* $b^2 \not\equiv a + 1 \pmod{27}$.

Now we can proceed to our first result.

Theorem 2.1. *Let $m \equiv 0 \pmod 3$ be odd and n be any positive integers. If d_1 is the square-free part of $3(4m^{3n} - k^2)$ with $k \equiv \pm 1 \pmod{18}$ and $\gcd(m, k) = 1$, then $3 | h(d_1)$.*

Proof. Let $D_1 = -d_1/3$, $K_1 = \mathbb{Q}(\sqrt{d_1})$ and $L_1 = \mathbb{Q}(\sqrt{D_1})$. Let $\alpha_1 \in \mathcal{O}_{L_1}$ so that

$$\alpha_1 = \frac{k + \sqrt{k^2 - 4m^{3n}}}{2}.$$

Then $T_{L_1/\mathbb{Q}}(\alpha_1) = k$ and $N_{L_1/\mathbb{Q}}(\alpha_1) = m^{3n}$. Since $\gcd(m, k) = 1$, so that

$$\gcd(T_{L_1/\mathbb{Q}}(\alpha_1), N_{L_1/\mathbb{Q}}(\alpha_1)) = 1.$$

Now with respect to α_1,

$$\begin{aligned} f_{\alpha_1}(X) :&= X^3 - 3[N_{L_1/\mathbb{Q}}(\alpha_1)]^{1/3}X - T_{L_1/\mathbb{Q}}(\alpha_1) \\ &= X^3 - 3m^n X - k \\ &\equiv X^3 + X + 1 \pmod 2. \end{aligned}$$

Thus the polynomial $f_{\alpha_1}(X)$ is irreducible over $\mathbb{Z}_2$ and therefore it is irreducible over $\mathbb{Q}$. Therefore by Lemma 2.1, α_1 is not a cube in L_1 and thus $\alpha_1 \in R_{D_1}$. Since $\gcd(T_{L_1/\mathbb{Q}}(\alpha_1), N_{L_1/\mathbb{Q}}(\alpha_1)) = 1$, so that $\alpha_1 \in R^*_{D_1}$ too. Therefore by Lemma 2.2, $S_{\mathbb{Q}}(f_{\alpha_1})$ is a cyclic cubic extension of K_1 which is unramified outside 3.

Now we are left to show that $S_{\mathbb{Q}}(f_{\alpha_1})$ is unramified over K_1 at 3 too. The polynomial $f_{\alpha_1}(X)$ does not satisfy the condition (LN-1) as $v_3(1) = 0$. Also $f_{\alpha_1}(X)$ does not satisfy the conditions (LN-2) and (LN-3) since $m \equiv 0 \pmod 3$. Therefore by Lemma 2.3, $S_{\mathbb{Q}}(f_{\alpha_1})$ is unramified over K_1 at 3. Thus by Hilbert class field theory the class number of K_1 is divisible by 3. □

Theorem 2.2. *Let $m \equiv 0 \pmod 3$ and n be any odd integers such that $v_3(n) = 1$. Then the class number of $\mathbb{Q}(\sqrt{-(m^2n^2 \pm 4n)/3})$ is divisible by* 3.

Proof. Let d_2 be the square-free part of $-(m^2n^2 \pm 4n)/3$ and $K_2 = \mathbb{Q}(\sqrt{d_2})$. Suppose that $D_2 = -3d_2$ and $L_2 = \mathbb{Q}(\sqrt{D_2})$. Let $\alpha_2 \in \mathcal{O}_{L_2}$ be so that

$$\alpha_2 = \frac{m^2n \pm 2 + m\sqrt{m^2n^2 \pm 4n}}{2}.$$

Then $T_{L_2/\mathbb{Q}}(\alpha) = m^2n \pm 2$ and $N_{L_2/\mathbb{Q}}(\alpha_2) = 1$. Thus again their gcd is 1.

We can now have the cubic polynomial corresponding to such an α_2:

$$\begin{aligned} f_{\alpha_2}(X) :&= X^3 - 3[N_{L_2/\mathbb{Q}}(\alpha_2)]^{1/3}X - T_{L_2/\mathbb{Q}}(\alpha_2) \\ &= X^3 - 3X - (m^2n \pm 2) \\ &\equiv X^3 - X - 1 \pmod 2. \end{aligned}$$

Thus the polynomial $f_{\alpha_2}(X)$ is irreducible over $\mathbb{Z}_2$ and therefore it is irreducible over $\mathbb{Q}$. Therefore by lemma 2.1, α_2 is not a cube in L_2 and thus $\alpha_2 \in R^*_{D_2}$. Therefore by Lemma 2.2, $S_{\mathbb{Q}}(f_{\alpha_2})$ is a cyclic cubic extension of K_2 unramified outside 3.

Now it remains to show that $S_{\mathbb{Q}}(f_{\alpha_2})$ is unramified over K_2 at 3. Since $m \equiv 0 \pmod 3$ and $v_3(n) = 1$, the polynomial $f_{\alpha_2}(X)$ does not satisfy the conditions (LN-1), (LN-2) and (LN-3). Therefore by Lemma 2.3, $S_{\mathbb{Q}}(f_{\alpha_2})$ is unramified over K_2 at 3 too. Thus by Hilbert class field theory the class number of K_2 is divisible by 3. □

We now give an extension of a result proved by Hoque and Saikia [[4], Theorem 3.1].

Theorem 2.3. *Let $m > 1$ and p be odd integers, and n be any positive integer. Let d_3 be the square-free part of $-(3^m p^{2n} + r)$ with $r \in \{-2, 4\}$. Then $3|h(d_3)$.*

Proof. Let $r = 4$ and then $d_3 \equiv 1 \pmod 4$. As before we set $D_3 = -3d_3$ and $L_3 = \mathbb{Q}(\sqrt{D_3})$. Then $L_3 = \mathbb{Q}(\sqrt{3^{m+1}p^{2n} + 12})$. Choose $\alpha_3 \in \mathcal{O}_{L_3}$ by

$$\alpha_3 := \frac{3^m p^{2n} + 2 + 3^{(m-1)/2}p^n\sqrt{3^{m+1}p^{2n} + 12}}{2}.$$

Then $T_{L_3/\mathbb{Q}}(\alpha_3) = 3^m p^{2n} + 2$, $N_{L_3/\mathbb{Q}}(\alpha_3) = 1$ and thus

$$\gcd(T_{L_3/\mathbb{Q}}(\alpha_3), N_{L_3/\mathbb{Q}}(\alpha_3)) = 1.$$

The cubic polynomial corresponding to α_3 is:

$$\begin{aligned} f_{\alpha_3}(X) &:= X^3 - 3[N_{L_3/\mathbb{Q}}(\alpha_3)]^{1/3}X - T_{L_3/\mathbb{Q}}(\alpha_3) \\ &= X^3 - 3X - 3^m p^{2n} - 2 \\ &\equiv X^3 - X - 1 \pmod 2. \end{aligned}$$

Thus the polynomial $f_{\alpha_3}(X)$ is irreducible over $\mathbb{Z}_2$ and therefore it is irreducible over $\mathbb{Q}$. Therefore by Lemma 2.1, α_3 is not a cube in L_3 and hence $\alpha_3 \in R^*_{D_3}$. Thus by Lemma 2.2, $S_{\mathbb{Q}}(f_{\alpha_3})$ is a cyclic cubic extension of K_3 unramified outside 3.

We now claim that $S_{\mathbb{Q}}(f_{\alpha_3})$ is unramified over K_3 at 3 too. The polynomial $f_{\alpha_3}(X)$ does not satisfy the conditions (LN-1), (LN-2) and (LN-3) as $v_3(3^m p^{2n} + 2) = 0$ and $m > 1$. Therefore by Lemma 2.3, we proof the claim. Thus by Hilbert class field theory the class number of K_3 is divisible by 3.

Now let $r = -2$. Then $3 \nmid d_3$ and $d_3 \equiv 3 \pmod 4$. Let us set $D'_3 = -3d_3$ and $L'_3 = \mathbb{Q}(\sqrt{D'_3})$. Then $L'_3 = \mathbb{Q}(\sqrt{3^{m+1}p^{2n} - 6})$ and choose an element $\alpha'_3 \in \mathcal{O}_{L'_3}$ by

$$\alpha'_3 := 3^m p^{2n} - 1 + 3^{(m-1)/2} p^n \sqrt{3^{m+1} p^{2n} - 6}.$$

One can now complete the proof by a similar argument as in the previous case. □

Theorem 2.4. *Let $n > 1$ be an odd integer and a, b two more integers such that:*

(C3.1) $a \equiv 19 \pmod{30}$,
(C3.2) $b \equiv 6 \pmod{15}$, *and is coprime to a,*

then 3 *divides the class number of* $\mathbb{Q}(\sqrt{3(a^{3n} - b^{2n})})$.

Proof. Let d_4 be the square-free part of $3(a^{3n} - b^{2n})$ and $K_4 = \mathbb{Q}(\sqrt{d_4})$. Suppose that $D_4 = -d_4/3$ and $L_4 = \mathbb{Q}(\sqrt{D_4})$.

Let $\alpha_4 \in \mathcal{O}_{L_4}$ so that

$$\alpha_4 = b^n + \sqrt{b^{2n} - a^{3n}}.$$

Then $T_{L_4/\mathbb{Q}}(\alpha_4) = 2b^n$ and $N_{L_4/\mathbb{Q}}(\alpha_4) = a^{3n}$. Then

$$\begin{aligned} f_{\alpha_4}(X) &:= X^3 - 3\big(N_{L_4/\mathbb{Q}}(\alpha_4)\big)^{1/3}X - T_{L_4/\mathbb{Q}}(\alpha_4) \\ &= X^3 - 3a^n X - 2b^n. \end{aligned}$$

By the conditions (C3.1) and (C3.2), we have

$$f_{\alpha_4}(X) \equiv X^3 + 3X - 2 \pmod 5.$$

Clearly the polynomial $f_{\alpha_4}(X)$ is irreducible over $\mathbb{Z}_5$ and hence it is irreducible over $\mathbb{Q}$ too. Thus α_4 is not a cube in L_4 by Lemma 2.1 and hence $\alpha_4 \in R_{D_4}$. The conditions (C3.1) and (C3.2) entail $\gcd(N_{L_4/\mathbb{Q}}(\alpha), T_{L_4/\mathbb{Q}}(\alpha_4)) = 1$ and therefore by Lemma 2.2, $S_{\mathbb{Q}}(f_{\alpha_4})$ is a cyclic cubic extension of K_4 unramified outside 3.

Now we are left to show that $S_{\mathbb{Q}}(f_{\alpha_4})$ is unramified over K_4 at 3 also. The polynomial $f_{\alpha_4}(X)$ does not satisfy the condition (LN-1) since $n > 1$, $a \equiv 1 \pmod 3$ and $b \equiv 0 \pmod 3$. Again $3 \mid 2b^n$ by the condition (C3.2), and thus $f_{\alpha_4}(X)$ satisfies none of the conditions (LN-2) and (LN-3). Therefore by Lemma 2.3, $S_{\mathbb{Q}}(f_{\alpha_4})$ is unramified over K_4 at 3. Thus by Hilbert class field theory the class number of K_4 is divisible by 3. □

Theorem 2.5. *Let $n > 1$ be an integer and a, b two more integers satisfying:*

(C3.3) $\gcd(a, b) = 1$;
(C3.4) $a \equiv 1 \pmod 3$ *and is odd;*
(C3.5) $b \equiv 0 \pmod 3$ *and is odd;*

Then 3 *divides the class number of* $\mathbb{Q}(\sqrt{3(4a^{3n} - b^{2n})})$.

Proof. Let d_5 be the sqaure-free part of $3(4a^{3n} - b^{2n})$ and $K_5 = \mathbb{Q}(\sqrt{d_5})$. Suppose that $D_5 = -d_5/3$ and $L_5 = \mathbb{Q}(\sqrt{D_5})$. Let $\alpha_5 \in \mathcal{O}_{L_5}$ be of the form

$$\alpha_5 = \frac{b^n + \sqrt{b^{2n} - 4a^{3n}}}{2}.$$

Then $T_{L_5/\mathbb{Q}}(\alpha_5) = b^n$ and $N_{L_5/\mathbb{Q}}(\alpha_5) = a^{3n}$ and the corresponding polynomial

$$\begin{aligned} f_{\alpha_5}(X) :&= X^3 - 3[N_{L_5/\mathbb{Q}}(\alpha_5)]^{1/3}X - T_{L_5/\mathbb{Q}}(\alpha_5) \\ &= X^3 - 3a^nX - b^n. \end{aligned}$$

The conditions (C3.4) and (C3.5) would imply $f_{\alpha_5}(X) \equiv X^3 - X - 1 \pmod 2$. Therefore $f_{\alpha_5}(X)$ is irreducible over $\mathbb{Z}_2$ and thus it is irreducible over $\mathbb{Q}$. Thus by Lemma 2.1, α_5 is not a cube in L_5 and hence $\alpha_5 \in R_{D_5}$. By (C3.3), $\gcd(N_{L_5/\mathbb{Q}}(\alpha_5), T_{L_5/\mathbb{Q}}(\alpha_5)) = 1$ and therefore by Lemma 2.2, $S_{\mathbb{Q}}(f_{\alpha_5})$ is a cyclic cubic extension of K_5 unramified outside 3.

It remains to prove that $S_{\mathbb{Q}}(f_{\alpha_5})$ is unramified over K_5 at 3 too. Clearly $v_3(b^n) > v_3(3a^n) = 1$ owing to (C3.4) and (C3.5) and thus $f_{\alpha_5}(X)$ does not satisfy the condition (LN-1). Also $3 \mid b^n$ due to (C3.5) and thus $f_{\alpha_5}(X)$ does not satisfy the conditions (LN-2) and (LN-3). Therefore by Lemma 2.3, $S_{\mathbb{Q}}(f_{\alpha_5})$ is unramified over K_5 at 3. Thus by Hilbert class field theory the class number of K_5 is divisible by 3. □

3. Some more families of quadratic fields

In this section, we shall use a result of Kishi and Miyake [9] for proving the first of the two theorems. Let us first recall Kishi-Miyake parametrization.

Lemma 3.1. *Let u and v be two integers and*

$$f_{u,v}(x) = x^3 - uvx - u^2. \tag{3.1}$$

If

(KM-1) *u and v are relatively prime;*
(KM-2) *$f_{u,v}(x)$ is irreducible over $\mathbb{Q}$;*
(KM-3) *discriminant $D_{f_{u,v}}$ of $f_{u,v}(x)$ is not a perfect square in $\mathbb{Z}$;*
(KM-4) *one of the following conditions hold:*

(a) $3 \nmid v$,
(b) $3 \mid v,\ uv \not\equiv 3 \pmod 9$ *and* $u \equiv v \pm 1 \pmod 9$,
(c) $3 \mid v,\ uv \equiv 3 \pmod 9$ *and* $u \equiv v \pm 1 \pmod{27}$,

then 3 divides the class number of $\mathbb{Q}(\sqrt{D_{f_{u,v}}})$. Conversely, every quadratic number field $\mathbb{Q}(\sqrt{D_{f_{u,v}}})$ with class number divisible by 3 arises in the above way from a suitable choices of integers u and v.

We use this to prove:

Theorem 3.1. *Let m be an odd positive integer.*

(I) *If $m \equiv 0 \pmod 3$, then 3 divides the class number of the field $\mathbb{Q}(\sqrt{-3(4m^3+1)})$.*
(II) *If $m \equiv 4 \pmod{15}$, then 3 divides the class number of the field $\mathbb{Q}(\sqrt{3(2m^{3n}-1)})$ for any odd integer $n \geq 3$.*

Proof. We prove (I) and outline the proof of (II) as in most aspects these are very similar to the proof of (I).

Let us put $u = -1$ and $v = 3m$ in (3.1). Then

$$f_{-1,3m}(X) = X^3 + 3mX - 1$$

and $D_{f_{-1,3m}} = 9d$ with $d = -3(4m^3 + 1)$.

Clearly u and v are relatively prime, and $D_{f_{-1,3m}} \neq \square$ in $\mathbb{Z}$. Now

$$f_{-1,3m}(X) \equiv X^3 - X - 1 \pmod 2$$

as m is odd. Thus $f_{-1,3m}(X)$ is irreducible over $\mathbb{Z}_2$ and therefore it is irreducible over $\mathbb{Q}$. Again $uv = -3m \equiv 0 \pmod 9$ (as $m \equiv 0 \pmod 3$). Furthermore $v - 1 = 3m - 1 \equiv -1 \pmod 9 \equiv u \pmod 9$. Therefore 3 divides class number of $\mathbb{Q}(\sqrt{-3(4m^3+1)})$ by invoking Lemma 3.1.

Finally to prove (II), we put $u = 2$ and $v = 3m^n$ in (3.1). The irreduciblity of $f_{2,3m^n}(X)$ follows from the irreducibility of $f_{2,3m^n}(X)$ over $\mathbb{Z}_5$. The condition (KM-3) holds since the discriminant of $f_{2,3m^n}(X)$ is $144d$ with $d = 3(2m^{3n} - 1) \equiv 3 \pmod 4$. Moreover we can show that the condition (b) of (KM-4) holds. □

We conclude this section by providing another family of quadratic fields whose class number is divisible by 3 by actually producing an element of order 3.

Theorem 3.2. *The class number of* $\mathbb{Q}(\sqrt{1-2m^3})$ *is divisible by* 3 *for any odd integer* $m > 1$.

Proof. Let $d = 1 - 2m^3$ and $K = \mathbb{Q}(\sqrt{d})$. Thus $d \equiv 3 \pmod 4$. Let $\alpha \in \mathcal{O}_K$ be of the form $\alpha = 1 + \sqrt{1-2m^3}$. Then $N_{K/\mathbb{Q}}(\alpha) = 2m^3$.

Suppose that p_j is a prime factor of m. Then the Kronecker symbol $\left(\frac{d}{p_j}\right) = 1$. Thus we can write

$$(p_j) = \mathcal{P}_j \mathcal{P}'_j$$

with distinct prime ideals $\mathcal{P}_j$ and $\mathcal{P}'_j$ in $\mathcal{O}_K$ which are conjugate to each other. Furthermore

$$(2) = \mathcal{P}^2 \quad \text{with} \quad \mathcal{P} = (2, 1+\sqrt{d}).$$

We may express the prime ideal decomposition of (α) as

$$(\alpha) = \mathcal{P} \prod_j \mathcal{P}'^{t_j},$$

because α is not divisible by any rational integers except ± 1. Then $N_{K/\mathbb{Q}}((\alpha)) = 2\prod p_j^{t_j}$ with $p_j = N_{K/\mathbb{Q}}(P_j)$ and thus $3 \mid t_j$. Therefore

$$\left(\mathcal{P} \prod_j \mathcal{P}'^{t_j/3}_j\right)^3 = (2)\mathcal{P} \prod_j \mathcal{P}'^{t_j}_j = (2)(\alpha),$$

which is principal in $\mathcal{O}_K$. If $\langle \mathcal{I} \rangle$ is an ideal class containing $\mathcal{P}\prod_j \mathcal{P}'^{t_j/3}_j$, then the order of $\langle \mathcal{I} \rangle$ is 3 if $(\mathcal{P}\prod_j \mathcal{P}'^{t_j/3}_j)$ is not principal in $\mathcal{O}_K$. To show $(\mathcal{P}\prod_j \mathcal{P}'^{t_j/3}_j)$ is not principal in $\mathcal{O}_K$, it is sufficient to show 2α is not a cube in $\mathcal{O}_K$. Let d' be the square-free part of $1 - 2m^3$ and denote

$$1 - 2m^3 = t^2 d' \quad (t \in \mathbb{Z}). \tag{3.2}$$

If $2\alpha = (a + b\sqrt{d'})^3$ for some $a, b \in \mathbb{Z}$, then we obtain

$$2 = a^3 + 3ab^2 d', \tag{3.3}$$

$$2|t| = 3a^2 b + b^3 d'. \tag{3.4}$$

From (3.3), it holds $a|2$, that is, $a \in \{\pm 1, \pm 2\}$. Taking modulo 3 in (3.3), we see that $a \neq 1, -2$. When $a = -1$, we see from (3.3) and (3.4) that $d' = -1$ and $|t| = 1$. Then by (3.2), we obtain $m = 1$, which contradicts to the assumption $m > 1$. When $a = 2$, we see from (3.3) and (3.4) that $d' = -1$ and $|t| = 11/2$. This is a contradiction. This completes the proof. □

4. Numerical Examples

In this section, we give some numerical examples corroborating our results in §2 and §3. We compute the class numbers of each of the above families of fields for some small values of d and list them in tables below. All the computations in this paper were done using PARI/GP (version 2.7.6).

Table 1. Numerical examples of Theorem 2.1 for $k = 1$ only

m	n	$d = 3(4m^{3n} - 1)$	$h(d)$	m	n	$d = 3(4m^{3n} - 1)$	$h(d)$
3	1	321	3	3	2	8745	12
3	3	2361953	36	3	4	6377289	36
9	1	8745	12	9	2	6377289	36
15	1	40497	3	21	1	111129	6
27	1	2361953	36	33	1	431241	6
39	1	711825	3	45	1	1093497	3
51	1	1591809	3	57	1	2222313	36
63	1	30000561	9	69	1	3942105	6
75	1	5062497	108	81	1	6377289	36
87	1	7902033	3	93	1	9652281	6
99	1	11643585	24	105	1	13891487	18
111	1	16411569	6	117	1	19219353	18
123	1	22330401	12	129	1	25760265	60
135	1	29524497	18	141	1	33638649	60
147	1	38118273	282	153	1	42978921	9
159	1	48236145	6	165	1	53905497	6

Table 2. Numerical examples of Theorem 2.2

m	n	$d = -(m^2n^2 + 4n)/3$	$D = -(m^2n^2 - 4n)/3$	$h(d)$	$h(D)$
3	3	−31	−23	3	3
3	15	−695	−655	24	12
3	21	−1351	−1295	24	36
3	33	−3311	−3223	72	30
3	39	−4615	−4511	36	84
3	51	−7871	−7735	120	48
9	3	−247	−239	6	15
9	15	−6095	−6055	84	36
9	21	−11935	−11879	72	150
9	33	−29447	−29359	132	72
9	39	−41119	−41015	120	180
9	51	−70295	−70159	252	168
15	3	−679	−671	18	30
15	15	−16895	−16855	96	84
15	21	−33103	−33047	60	150
21	3	−1327	−1319	15	45
21	15	−33095	−33055	240	72
21	21	−64855	−64799	120	222
27	3	−2191	−2183	30	42
27	15	−54695	−54655	216	156

Table 3. Numerical examples of Theorem 2.3

m	n	p	$d = -(3^m p^{2n} - 2)$	$D = -(3^m p^{2n} + 4)$	$h(d)$	$h(D)$
3	1	3	−241	−247	12	6
3	2	3	−2185	−2191	24	30
3	1	7	−1321	−1327	24	15
3	2	7	−64825	−64831	24	162
3	3	3	−19681	−19687	84	81
5	1	3	−2185	−2191	24	30
5	2	3	−19681	−19687	84	81
5	1	5	−6073	−6079	24	57
5	2	5	−151873	−151879	120	300
7	1	3	−19681	−19687	84	81
7	1	5	−107161	−107167	216	108

Table 4. Numerical examples of Theorem 2.4

a	b	n	$d = 3(a^{3n} - b^{2n})$	$h(d)$
19	6	3	968062953369	6
19	21	3	967805794974	648
19	36	3	961532746329	24
19	51	3	915274229934	48
19	66	3	720101243289	12
19	81	3	120774483894	24
19	96	3	−1380210275751	1388160
19	111	3	−4643180563146	1951488
19	126	3	−11036449330791	4263624
19	141	3	−22606080831186	3780672
49	306	3	2422323582800979	192

Table 5. Numerical examples of Theorem 2.5

a	b	n	$d = 3(4a^{3n} - b^{2n})$	$h(d)$
1	3	2	−331	3
1	3	3	−725	6
1	3	4	−19671	84
1	9	2	−19671	84
1	9	3	−531437	480
1	15	2	−151863	324
7	3	2	1411545	12
7	15	2	1259913	6
7	27	2	−60845	192
13	9	2	57902025	72
13	15	2	57769833	48
13	21	2	57338265	96

Table 6. Numerical examples of Theorem 3.1 (I)

m	$d = -3(4m^3 + 1)$	$h(d)$	m	$d = -3(4m^3 + 1)$	$h(d)$
3	−327	12	9	−8751	72
15	−40503	96	21	−111135	240
27	−236199	504	33	−431247	360
39	−711831	648	45	−1093503	540
51	−1591815	780	57	−2222319	984
63	−3000567	1152	69	−3942111	2568
75	−5062503	1800	81	−6377295	1296
87	−7902039	2772	93	−9652287	1452
99	−11643591	2160	105	−13891503	2448

Table 7. Numerical examples of Theorem 3.2

m	$d = 1 - 2m^3$	$h(d)$	m	$d = 1 - 2m^3$	$h(d)$
3	−53	6	5	−249	12
7	−685	12	9	−1457	24
11	−2661	48	13	−4393	24
15	−6749	84	17	−9825	12
19	−13717	48	21	−18521	228
23	−24333	72	25	−31249	96
27	−39365	180	29	−48777	96
31	−59581	126	33	−71873	240
35	−85749	336	37	−101305	144
39	−118637	216	41	−137841	264
43	−159013	162	45	−182249	408
47	−207645	288	49	−235297	264
51	−-265301	684	53	−297753	228
55	−332749	318	57	−370385	360
59	−410757	336	61	−453961	420
63	−500093	78	65	−549249	624
67	−601525	66	69	−657017	660
71	−715821	672	73	−778033	324
75	−843749	1128	77	−913065	912

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