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Polynomials associated with the fragments of coset diagrams

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Abstract. The coset diagrams for $PSL(2, \mathbb{Z})$ are composed of fragments, and the fragments are further composed of circuits. Mushtaq has found that, the condition for the existence of a fragment in coset diagram is a polynomial f in $\mathbb{Z}[z]$. Higman has conjectured that, the polynomials related to the fragments are monic and for a fixed degree, there are finite number of such polynomials. In this paper, we consider a family Ω of fragments such that each fragment in Ω contains one vertex fixed by a pair of words $(xy)^{q_1}(xy^{-1})^{q_2}$, $(xy^{-1})^{s_1}(xy)^{s_2}$, where $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$, and prove Higman's conjecture for the polynomials obtained from Ω. At the end, we answer the question; for a fixed degree n, how many polynomials are evolved from Ω.

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1. Introduction

A central theme in the group theory, since the completion of the classification of simple groups in the 1980's, has been the study of groups via their actions. This most commonly takes the form of actions on vector spaces and similar commutative objects (linear representation theory) or on more elementary combinatorial objects.

It is not an exaggeration to say that the modular group $PSL(2, \mathbb{Z})$ is the single most important infinite discrete group, through its myriad connections with number theory, geometry and topology. There is a long and venerable history of studying its actions, particularly on finite sets, that goes back to before the turn of the 20$^{\text{th}}$ century. The modular group $PSL(2, \mathbb{Z})$ [2] has finite presentation

$$\langle x, y : x^2 = y^3 = 1 \rangle$$

where x and y are the linear fractional transformations defined by $z \rightarrow \frac{-1}{z}$ and $z \rightarrow \frac{z-1}{z}$ respectively. If we add a new generator $t : z \rightarrow \frac{1}{z}$ in the modular group $PSL(2, \mathbb{Z})$, we obtain a group

$$PGL(2, \mathbb{Z}) = \langle x, y, t : x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1 \rangle$$

known as extended modular group.

Let p be a prime number and $q = p^n$, where $n \in \mathbb{N}$. Then by the projective line over the finite field F_q, denoted by $PL(F_q)$, we mean $F_q \cup \{\infty\}$.

The group $PGL(2, q)$ has its customary meaning, as the group of all linear fractional transformations $z \rightarrow \frac{az+b}{cz+d}$ such that $a, b, c, d \in F_q$ and $ad - bc \neq 0$, while $PSL(2, q)$ is its subgroup consisting of all those where $ad - bc$ is a quadratic residue in F_q.

The coset diagram for $PGL(2\mathbb{Z})$ was introduced by Higman in 1978. Later, in 1983 Mushtaq [4] laid its foundation. The three cycles of y are denoted by small triangles whose vertices are permuted counter-clockwise by y. If $(v_i)x = v_j$, then we join v_i and v_j by an edge. The vertices fixed by x and y are denoted by heavy dots. Since $(yt)^2 = 1$, so $tyt = y^{-1}$, thus t reverses the orientation of the triangles representing the three cycles of y (as reflection does). Consider the action of $PGL(2, \mathbb{Z})$ on $PL(F_{19})$. We can calculate the permutation representations x, y and t by $(z)x = \frac{-1}{z}$, $(z)y = \frac{z-1}{z}$ and $(z)t = \frac{1}{z}$ respectively. So

$$x : (0\ \infty)(1\ 18)(2\ 9)(3\ 6)(4\ 14)(5\ 15)(7\ 8)(10\ 17)(1112)(13\ 16),$$
$$y : (0\ \infty\ 1)(2\ 10\ 18)(3\ 7\ 9)(4\ 15\ 6)(5\ 16\ 14)(13\ 17\ 11)(8)(12),$$
$$t : (0\ \infty)(2\ 10)(3\ 13)(4\ 5)(6\ 16)(7\ 11)(8\ 12)(9\ 17)(14\ 15)(1)(18).$$

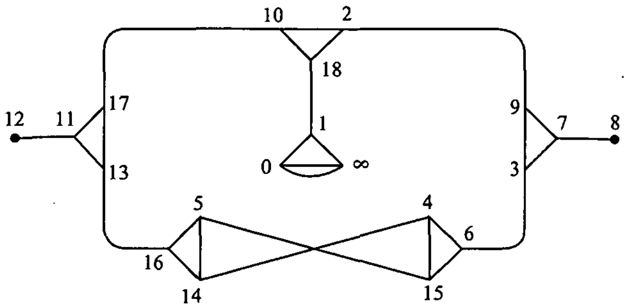

Figure 1.

For more on coset diagrams, we suggest reading of [1], [7], [8] and [9].

Two homomorphisms α and β from $PGL(2, \mathbb{Z})$ to $PGL(2, q)$ are called conjugate if $\beta = \alpha\rho$ for some inner automorphism ρ on $PGL(2, q)$. We call α to be non-degenerate if neither of x, y lies in the kernel of α. In [5] it is proved that there is a one to one correspondence between the conjugacy classes of non-degenerate homomorphisms from $PGL(2, \mathbb{Z})$ to $PGL(2, q)$ and the elements $\theta \neq 0, 3$ of F_q under the correspondence which maps each class to its parameter θ. As in [5], the coset diagram corresponding to the action of $PGL(2, \mathbb{Z})$ on $PL(F_q)$ via a homomorphism α with parameter θ is denoted by $D(\theta, q)$.

2. Occurrence of fragments in $D(\theta, q)$

By a circuit in a coset diagram for modular group, we mean a closed path of triangles and edges. For a sequence of positive integers $n_1, n_2, \ldots, n_{2k}$, the circuit which has a vertex fixed by the word

$$w = (xy)^{n_1}(xy^{-1})^{n_2}\ldots(xy^{-1})^{n_{2k}} \in PSL(2, \mathbb{Z}), \quad \text{where} \quad k \in \mathbb{N},$$

we mean a closed path having n_1 triangles with one vertex inside the circuit and n_2 triangles with one vertex outside the circuit and so on. Since it is a cycle $(n_1, n_2, \ldots, n_{2k})$, so it does not make any difference if n_1 triangles have one vertex outside the circuit and n_2 triangles have one vertex inside the circuit and so on.

For a given sequence of positive integers $n_1, n_2, \ldots, n_{2k}$ the circuit of the type $(n_1, n_2, \ldots, n_{2k'}, n_1, n_2, \ldots, n_{2k'}, \ldots n_1, n_2, \ldots, n_{2k'})$ where k' divides k, is said to have a period of length $2k'$. A circuit which is not of this type is called non-periodic circuit. A circuit is simple circuit, if its each vertex is fixed by a unique word w.

3. Joining of Circuits

Throughout this paper, by joining of a vertex v_i in $(n_1, n_2, \ldots, n_{2k})$ with the vertex v_j in $(m_1, m_2, \ldots, m_{2k})$, we mean, the vertices v_i and v_j melt together and become one node. Let v_i and v_j be fixed by the words w_i and w_j respectively. In order to join these two circuits at v_i and v_j, we choose, without loss of generality $(n_1, n_2, \ldots, n_{2k})$ and apply w_j on v_i in such a way that w_j ends at v_i. In this way, a fragment say γ is created. In other words by γ, we mean a non-simple fragment whose one vertex $v = v_i = v_j$ is fixed by a pair of words w_i, w_j.

Example 1. Let us join the vertex u in $(3, 2)$ with the vertex v in $(4, 3)$ to create a fragment γ. In figures 2 and 3 one can see that the vertices v and u are the fixed points of $(xy)(xy^{-1})^3(xy)^3$ and $(xy)^3(xy^{-1})^2$ respectively.

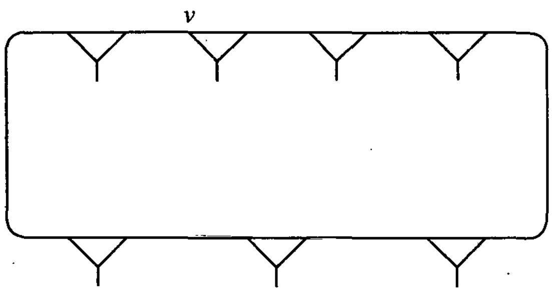

Figure 2.

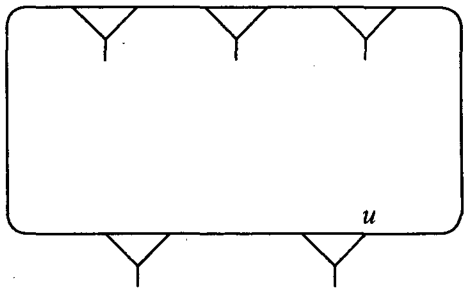

Figure 3.

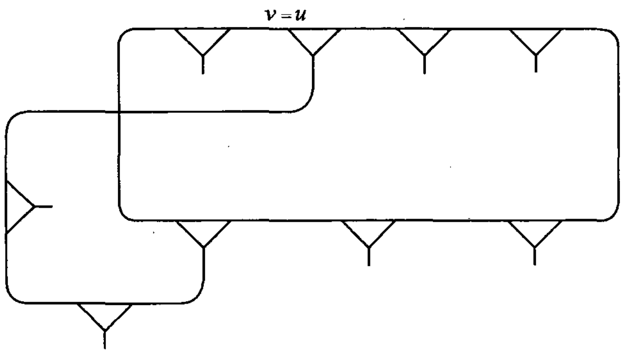

Figure 4.

It can be seen in figure 4 that the vertex $v = u$ in γ is fixed by

$$(xy)^3(xy^{-1})^2, (xy)(xy^{-1})^3(xy)^3.$$

For more on joining of circuits of coset diagrams, we refer to [6].

There is a question, going back to Higman, as to the structure of coset diagram $D(\theta, q)$. Specifically, for a given θ and q, what types of circuits appear in them? Some progress was made by Mushtaq in the late 1980's with the appearance of certain "fragment" in a diagram $D(\theta, q)$ determined by vanishing of a certain polynomial associated to it [3]. In this paper, we also provide some combinatorial data in this direction.

The method to compute a polynomial from a fragment is presented in [3]. Here we narrate this process briefly. Since a fragment is created by joining $(n_1, n_2, \ldots, n_{2k})$ and $(m_1, m_2, \ldots, m_{2k})$ at a common point v, so there is a pair of words $w_i = (xy)^{l_1}(xy^{-1})^{l_2} \ldots (xy^{-1})^{l_{2k_1}}$, $w_j = (xy)^{m_1}(xy^{-1})^{m_2} \ldots (xy^{-1})^{m_{2k_2}}$ such that $(v)w_i = v$ and $(v)w_j = v$. Let X and Y be the matrices corresponding to x and y of $PGL(2, q)$. Then w_i and w_j can be expressed as

$$W_i = (XY)^{l_1}(XY^{-1})^{l_2} \ldots (XY^{-1})^{l_{2k_1}}$$
$$W_j = (XY)^{m_1}(XY^{-1})^{m_2} \ldots (XY^{-1})^{m_{2k_2}}$$

where $k_1, k_2 \in \mathbb{N}$. The matrices X and Y having entries from F_q and satisfying

$$X^2 = Y^3 = \lambda I \tag{3.1}$$

can be represented by

$$X = \begin{pmatrix} a & kc \\ c & -a \end{pmatrix}, \quad Y = \begin{pmatrix} d & kf \\ f & -d-1 \end{pmatrix}.$$

Of course $a, c, d, f, k \in F_q$. We shall write

$$a^2 + kc^2 = -\Delta \neq 0 \tag{3.2}$$

and require that

$$d^2 + d + kf^2 + 1 = 0 \tag{3.3}$$

This gives us the elements, which satisfy the relations 3.1.

Since $\det(Y) = 1$, so $\Delta = -(a^2 + kc^2)$ and $r = a(2d+1) + 2kcf$ are determinant and trace of XY respectively. Also trace $(X) = 0$ and $\det(X) = \Delta$; therefore the characteristic equation of X can be written as

$$X^2 + \Delta I = 0. \tag{3.4}$$

Similarly, since trace$(Y) = -1$ and $\det(Y) = 1$, so

$$Y^2 + Y + I = 0. \tag{3.5}$$

is the characteristic equation of Y. Moreover, trace $(XY) = r$ and $\det(XY) = \Delta$, implying that the characteristic equation of the matrix XY is

$$(XY)^2 - r(XY) + \Delta I = 0. \tag{3.6}$$

On recursion, Equation 3.6 yields

$$(XY)^n = \left\{\binom{n-1}{0} r^{n-1} - \binom{n-2}{1} r^{n-3}\Delta + \dots\right\} XY - \left\{\binom{n-2}{0} r^{n-2}\Delta - \binom{n-3}{1} r^{n-4}\Delta^2 + \dots\right\} I. \tag{3.7}$$

After suitable manipulation, Equations 3.4, 3.5 and 3.6 give the following equations

$$XYX = rX + \Delta I + \Delta Y. \tag{3.8}$$

$$XYY = -X - XY \tag{3.9}$$

$$YXY = rY + X. \tag{3.10}$$

$$YX = rI - X - XY. \tag{3.11}$$

Thus, by making use of Equations 3.4 to 3.11 the matrices the matrices W_i and W_j can be expressed linearly as

$$W_i = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$$

$$W_j = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY$$

where λ_l and μ_l for $l = 0, 1, 2, 3$ are polynomials in r and Δ. Since $(v)w_i = v$ and $(v)w_j = v$, the 2×2 matrices W_i and W_j have an eigenvector in common. This by Lemma 3.1 of [3], we have the algebra generated by W_i and W_j has dimension 3. The algebra contains I, W_i, W_j, W_iW_j and so these are linearly dependent. Using Equations 3.4 to 3.11 the matrix W_iW_j can be expressed as

$$W_iW_j = \nu_0 I + \nu_1 X + \nu_2 Y + \nu_3 XY$$

where v_i, for $i = 0, 1, 2, 3$ can be computed in terms of the λ_i and μ_i, using 3.4 to 3.11. The condition that I, W_i, W_j and W_iW_j are linearly dependent, can be expressed as

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} = 0. \tag{3.12}$$

If we carry out the calculation of v_1, v_2, v_3 in terms of λ_i and μ_i and use in 3.12, we find the following homogeneous equation.

$$\begin{aligned} &-(\lambda_2\mu_3 - \mu_2\lambda_3)^2 - \Delta(\lambda_3\mu_1 - \mu_3\lambda_1)^2 - (\lambda_1\mu_2 - \mu_1\lambda_2)^2 \\ &- r(\lambda_2\mu_3 - \mu_2\lambda_3)(\lambda_3\mu_1 - \mu_3\lambda_1) - (\lambda_2\mu_3 - \mu_2\lambda_3)(\lambda_1\mu_2 - \mu_1\lambda_2) = 0. \end{aligned} \tag{3.13}$$

In [5], θ is defined as $\frac{r^2}{\Delta}$, so we can substitute $\Delta\theta$ for r^2 to get a polynomial in θ.

Higman has conjectured that, the polynomials related to the fragments are monic and for a fixed degree, there are finite number of such polynomials. In this paper, we consider a family Ω of fragments such that each fragment in Ω contains one vertex fixed by $(xy)^{q_1}(xy^{-1})^{q_2}$, $(xy^{-1})^{s_1}(xy)^{s_2}$, where $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$, and prove the Higman's conjecture for the polynomials obtained from Ω.

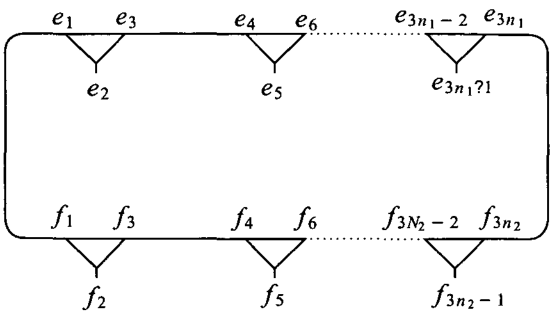

Figure 5.

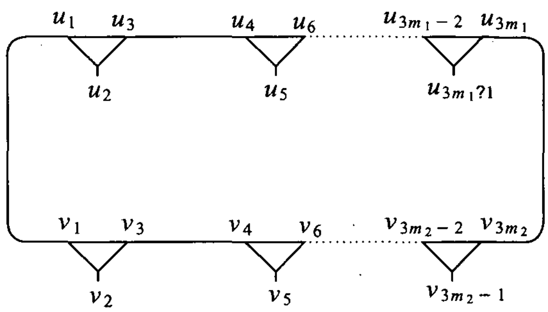

Figure 6.

Consider two circuits (n_1, n_2) and (m_1, m_2) Ω is constructed by joining

$$\begin{array}{llll} e_{3n_1} & \text{with} & u_1 & \text{and} \quad v_1, \\ f_{3n_2} & \text{with} & u_1 & \text{and} \quad v_1, \\ u_{3m_1} & \text{with} & e_1 & \text{and} \quad f_1, \\ v_{3m_2} & \text{with} & e_1 & \text{and} \quad f_1. \end{array}$$

4. Main results

Theorem 1. *Number of triangles in any fragment $\gamma \in \Omega$ is*

$$s_1 + s_2 + q_1 + q_2 - 2.$$

Proof. Let γ be any fragment in Ω. Then its one vertex say v, is a fixed point of the circuits $(xy)^{q_1}(xy^{-1})^{q_2}$ and $(xy^{-1})^{s_1}(xy)^{s_2}$, where $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$. Diagrammatically, it means:

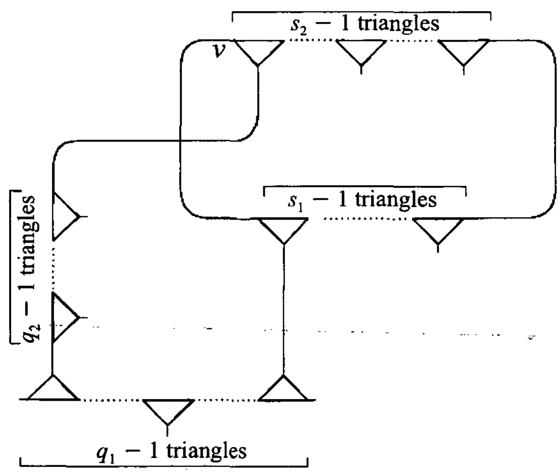

Figure 7.

From figure 7, it is clear that, $\gamma \in \Omega$ has $s_1 + s_2 + q_1 + q_2 - 2$ triangles. □

Total number of triangles in the circuit $(xy)^{q_1}(xy^{-1})^{q_2}$ are $q_1 + q_2$, let $q_1 + q_2 = \tau_1$. Since $(xy)^{q_1}(xy^{-1})^{q_2}$ can be expressed linearly as

$$(XY)^{q_1}(XY^{-1})^{q_2} = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY$$

where μ_i, for $i = 0, 1, 2, 3$ is polynomial in r and Δ, we use $\max(\mu_i)$ for the term containing the highest power of r, in μ_i.

Proposition 1.

(i) *If* $w = xy(x\,y^{-1})^{q_2}$ *where* $q_2 \in \mathbb{Z}^+$. *Then the corresponding matrix can be expressed linearly as*

$$\begin{aligned} XY(XY^{-1})^{q_2} &= (-1)^{q_2}\left\{\binom{q_2-1}{0}r^{q_2} - \binom{q_2-2}{1}r^{q_2-2}\Delta + \ldots\right\}X \\ &+ (-1)^{q_2}\left\{\binom{q_2-1}{0}r^{q_2-1}\Delta - \binom{q_2-2}{1}r^{q_2-3}\Delta^2 + \ldots\right\}Y \\ &+ (-1)^{q_2}\left\{\binom{q_2}{0}r^{q_2} - \binom{q_2-1}{1}r^{q_2-2}\Delta + \ldots\right\}XY. \end{aligned}$$

(ii) *If* $w = (xy)^{q_1}x\,y^{-1}$ *where* $q_1 \in \mathbb{Z}^+$. *Then the corresponding matrix can be expressed linearly as*

$$\begin{aligned} (XY)^{q_1}XY^{-1} &= \left\{-\binom{q_1-1}{0}r^{q_1} + \binom{q_1-2}{1}r^{q_1-2}\Delta - \ldots\right\}X \\ &+ \left\{-\binom{q_1-2}{0}r^{q_1-1}\Delta + \binom{q_1-3}{1}r^{q_1-3}\Delta^2 - \ldots\right\}Y \\ &+ \left\{-\binom{q_1-1}{0}r^{q_1} + \binom{q_1-2}{1}r^{q_1-2}\Delta - \ldots\right\}XY. \end{aligned}$$

The proof is obtained by using mathematical induction.

Proposition 2. *If* $w = (xy)^{q_1}(xy^{-1})^{q_2}$, *where* $q_1, q_2 \in \mathbb{Z}^+\setminus\{1\}$, *then the corresponding matrix can be expressed linearly as* $W = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY$, *such that*

$$\begin{aligned} \max(\mu_0) &= (-1)^{q_2}\xi r^{\tau_1-4}\Delta^2, \quad \textit{where} \quad \xi \in \mathbb{Z}, \\ \max(\mu_1) &= (-1)^{q_2}r^{\tau_1-1}, \\ \max(\mu_2) &= (-1)^{q_2}r^{\tau_1-2}\Delta, \\ \max(\mu_3) &= (-1)^{q_2}r^{\tau_1-1}. \end{aligned}$$

Proof. By Proposition 1,

$$(XY)(XY^{-1})^2 = r^2X + r\Delta Y + (-\Delta + r^2)XY.$$

Now

$$(XY)^2(XY^{-1})^2 = r^2XYX + r\Delta XYY + (-\Delta + r^2)XYXY.$$

By making use of Equations 3.4 to 3.11, we get

$$\begin{aligned} (XY)^2(XY^{-1})^2 &= r^2(\Delta I + rX + \Delta Y) + r\Delta(-X - XY) + (-\Delta + r^2)(-\Delta I + rXY) \\ &= r^2\Delta I + r^3X + r^2\Delta Y - r\Delta X - r\Delta XY + \Delta^2 I - r^2\Delta I - r\Delta XY + r^3XY \\ &= \Delta^2 I + (r^3 - r\Delta)X + r^2\Delta Y + (r^3 - 2r\Delta)XY. \end{aligned}$$

Hence the result is true for $(XY)^2(XY^{-1})^2$.

Let the result be true for $(XY)^k(XY^{-1})^k$, that is

$$(XY)^k(XY^{-1})^k = (-1)^k\{\xi r^{k+k-4}\Delta^2 - \ldots\}I + (-1)^k\{r^{k+k-1} - \ldots\}X + (-1)^k\{r^{k+k-2}\Delta - \ldots\}Y + (-1)^k\{r^{k+k-1} - \ldots\}XY.$$

Now

$$(XY)^{k+1}(XY^{-1})^{k+1} = (-1)^k\{\xi r^{k+k-4}\Delta^2 - \ldots\}XYIXY^{-1} + (-1)^k\{r^{k+k-1} - \ldots\}XYXXY^{-1} + (-1)^k\{r^{k+k-2}\Delta - \ldots\}XYYXY^{-1} + (-1)^k\{r^{k+k-1} - \ldots\}XYXYXY^{-1}.$$

By making use of Equations 3.4 to 3.11, we get

$$\begin{aligned}(XY)^{k+1}(XY^{-1})^{k+1} &= (-1)^k\{\xi r^{k+k-4}\Delta^2 - \ldots\}(-rX - \Delta Y - rXY)\\ &\quad + (-1)^k\{r^{k+k-1} - \ldots\}(-\Delta X) + (-1)^k\{r^{k+k-2}\Delta - \ldots\}(-\Delta I + rX + rXY)\\ &\quad + (-1)^k\{r^{k+k-1} - \ldots\}\{(-r^2 + \Delta)X + -r\Delta Y + (-r^2 + \Delta)XY\}\\ &= (-1)^{k+1}\{r^{(k+1)+(k+1)-4}\Delta^2 - \ldots\}I + (-1)^{k+1}\{r^{(k+1)+(k+1)-1} - \ldots\}X\\ &\quad + (-1)^{k+1}\{r^{(k+1)+(k+1)-2}\Delta - \ldots\}Y + (-1)^{k+1}\{r^{(k+1)+(k+1)-1} - \ldots\}XY.\end{aligned}$$

Hence the result is true for $(XY)^{q_1}(XY^{-1})^{q_2}$ such that $q_1 = q_2 \in \mathbb{Z}^+\setminus\{1\}$ and in this case, we have $\xi = 1$. So, for $k_1 = k_2$

$$(XY)^{k_1}(XY^{-1})^{k_2} = (-1)^{k_2}\{r^{k_1+k_2-4}\Delta^2 - \ldots\}I + (-1)^{k_2}\{r^{k_1+k_2-1} - \ldots\}X + (-1)^{k_2}\{r^{k_1+k_2-2}\Delta - \ldots\}Y + (-1)^{k_2}\{r^{k_1+k_2-1} - \ldots\}XY.$$

Therefore

$$(XY)^{k_1+1}(XY^{-1})^{k_2} = (-1)^{k_2}\{r^{k_1+k_2-4}\Delta^2 - \ldots\}XYI + (-1)^{k_2}\{r^{k_1+k_2-1} - \ldots\}XYX + (-1)^{k_2}\{r^{k_1+k_2-2}\Delta - \ldots\}XYY + (-1)^{k_2}\{r^{k_1+k_2-1} - \ldots\}XYXY.$$

By making use of Equations 3.4 to 3.11, we have

$$\begin{aligned}(XY)^{k_1+1}(XY^{-1})^{k_2} &= (-1)^{k_2}\{r^{k_1+k_2-4}\Delta^2 - \ldots\}XY + (-1)^{k_2}\{r^{k_1+k_2-1} - \ldots\}(\Delta I + rX + \Delta Y)\\ &\quad + (-1)^{k_2}\{r^{k_1+k_2-2}\Delta - \ldots\}(-X - XY) + (-1)^{k_2}\{r^{k_1+k_2-1} - \ldots\}(-\Delta I + rXY)\\ &= (-1)^{k_2}\{\xi r^{k_1+k_2-3}\Delta^2 - \ldots\}I + (-1)^{k_2}\{r^{k_1+k_2} - \ldots\}X\\ &\quad + (-1)^{k_2}\{r^{k_1+k_2-1}\Delta - \ldots\}Y + (-1)^{k_2}\{r^{k_1+k_2} - \ldots\}XY\\ &= (-1)^{k_2}\{\xi r^{(k_1+1)+k_2-4}\Delta^2 - \ldots\}I + (-1)^{k_2}\{r^{(k_1+1)+k_2-1} - \ldots\}X\\ &\quad + (-1)^{k_2}\{r^{(k_1+1)+k_2-2}\Delta - \ldots\}Y + (-1)^{k_2}\{r^{(k_1+1)+k_2-1} - \ldots\}XY.\end{aligned}$$

Hence the result is true for $(XY)^{q_1}(XY^{-1})^{q_2}$ such that $q_1 - q_2 = 1$.

Let the result be true for $(XY)^{q_1}(XY^{-1})^{q_2}$ such that $q_1 - q_2 = n$, that is

$$(XY)^{k_2+n}(XY^{-1})^{k_2} = (-1)^{k_2}\{\xi r^{(k_2+n)+k_2-4}\Delta^2 - \ldots\}I + (-1)^{k_2}\{r^{(k_2+n)+k_2-1} - \ldots\}X + (-1)^{k_2}\{r^{(k_2+n)+k_2-2}\Delta - \ldots\}Y + (-1)^{k_2}\{r^{(k_2+n)+k_2-1} - \ldots\}XY.$$

Now

$$(XY)^{k_2+(n+1)}(XY^{-1})^{k_2} = (-1)^{k_2}\{\xi r^{(k_2+n)+k_2-4}\Delta^2 - \dots\}XYI + (-1)^{k_2}\{r^{(k_2+n)+k_2-1} - \dots\}XYX + (-1)^{k_2}\{r^{(k_2+n)+k_2-2}\Delta - \dots\}XYY + (-1)^{k_2}\{r^{(k_2+n)+k_2-1} - \dots\}XYXY.$$

By making use of Equations 3.4 to 3.11, we have

$$\begin{aligned}(XY)^{k_2+(n+1)}(XY^{-1})^{k_2} &= (-1)^{k_2}\{\xi r^{(k_2+n)+k_2-4}\Delta^2 - \dots\}XY + (-1)^{k_2}\{r^{(k_2+n)+k_2-1} - \dots\}(\Delta I + rX + \Delta Y)\\ &+ (-1)^{k_2}\{r^{(k_2+n)+k_2-2}\Delta - \dots\}(-X - XY)\\ &+ (-1)^{k_2}\{r^{(k_2+n)+k_2-1} - \dots\}(-\Delta I + rXY)\\ &= (-1)^{k_2}\{\xi r^{(k_2+n)+k_2-3}\Delta - \dots\}I + (-1)^{k_2}\{r^{(k_2+n)+k_2} - \dots\}X\\ &+ (-1)^{k_2}\{r^{(k_2+n)+k_2-1}\Delta - \dots\}Y + (-1)^{k_2}\{r^{(k_2+n)+k_2} - \dots\}XY\\ &= (-1)^{k_2}\{\xi r^{(k_2+n+1)+k_2-4}\Delta - \dots\}I + (-1)^{k_2}\{r^{(k_2+n+1)+k_2-1} - \dots\}X\\ &+ (-1)^{k_2}\{r^{(k_2+n+1)+k_2-2}\Delta - \dots\}Y + (-1)^{k_2}\{r^{(k_2+n+1)+k_2-1} - \dots\}XY.\end{aligned}$$

Hence the result is true for $(XY)^{q_1}(XY^{-1})^{q_2}$ such that $q_1 > q_2$. For $k_1 = k_2$

$$(XY)^{k_1}(XY^{-1})^{k_2} = (-1)^{k_2}\{r^{k_1+k_2-4}\Delta^2 - \dots\}I + (-1)^{k_2}\{r^{k_1+k_2-1} - \dots\}X + (-1)^{k_2}\{r^{k_1+k_2-2}\Delta - \dots\}Y + (-1)^{k_2}\{r^{k_1+k_2-1} - \dots\}XY.$$

Therefore

$$(XY)^{k_1}(XY^{-1})^{k_2+1} = (-1)^{k_2}\{r^{k_1+k_2-4}\Delta^2 - \dots\}IXY^{-1} + (-1)^{k_2}\{r^{k_1+k_2-1} - \dots\}XXY^{-1} + (-1)^{k_2}\{r^{k_1+k_2-2}\Delta - \dots\}YXY^{-1} + (-1)^{k_2}\{r^{k_1+k_2-1} - \dots\}XYXY^{-1}.$$

By making use of Equations 3.4 to 3.11, we have

$$\begin{aligned}(XY)^{k_1}(XY^{-1})^{k_2+1} &= (-1)^{k_2}\{r^{k_1+k_2-4}\Delta^2 - \dots\}(-X - XY) + (-1)^{k_2}\{r^{k_1+k_2-1} - \dots\}(\Delta I + \Delta Y)\\ &+ (-1)^{k_2}\{r^{k_1+k_2-2}\Delta - \dots\}(-rI - rY + XY)\\ &+ (-1)^{k_2}\{r^{k_1+k_2-1} - \dots\}(-rX - \Delta Y - rXY)\\ &= (-1)^{k_2}\{\xi r^{k_1+k_2-3}\Delta^2 - \dots\}I + (-1)^{k_2+1}\{r^{k_1+k_2} - \dots\}X\\ &+ (-1)^{k_2+1}\{r^{k_1+k_2-1}\Delta - \dots\}Y + (-1)^{k_2+1}\{r^{k_1+k_2} - \dots\}XY\\ &= (-1)^{k_2}\{\xi r^{k_1+(k_2+1)-4}\Delta^2 - \dots\}I + (-1)^{k_2+1}\{r^{k_1+(k_2+1)-1} - \dots\}X\\ &+ (-1)^{k_2+1}\{r^{k_1+(k_2+1)-2}\Delta - \dots\}Y + (-1)^{k_2+1}\{r^{k_1+(k_2+1)-1} - \dots\}XY.\end{aligned}$$

Hence the result is true for $(XY)^{q_1}(XY^{-1})^{q_2}$ such that $q_1 - q_1 = 1$.

Let the result be true for $(XY)^{q_1}(XY^{-1})^{q_2}$ such that $q_2 - q_1 = n$, that is

$$(XY)^{k_1}(XY^{-1})^{k_1+n} = (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-4}\Delta^2 - \dots\}I + (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-1} - \dots\}X + (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-2}\Delta - \dots\}Y + (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-1} - \dots\}XY.$$

Now

$$\begin{aligned}(XY)^{k_1}(XY^{-1})^{k_1+(n+1)} &= (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-4}\Delta^2 - \ldots\}IXY^{-1}\\ &+ (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-1} - \ldots\}XXY^{-1}\\ &+ (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-2}\Delta - \ldots\}YXY^{-1}\\ &+ (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-1} - \ldots\}XYXY^{-1}.\end{aligned}$$

By making use of Equations 3.4 to 3.11, we have

$$\begin{aligned}(XY)^{k_1}(XY^{-1})^{k_1+(n+1)} &= (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-4}\Delta^2 - \ldots\}(-X - XY)\\ &+ (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-1} - \ldots\}(\Delta I + \Delta Y)\\ &+ (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-2}\Delta - \ldots\}(-rI - rY + XY)\\ &+ (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-1} - \ldots\}(-rX - \Delta Y - rXY)\\ &= (-1)^{(k_1+n)}\{r^{k_1+(k_1+n)-3}\Delta^2 - \ldots\}I\\ &+ (-1)^{(k_1+n)+1}\{r^{k_1+(k_1+n)} - \ldots\}X\\ &+ (-1)^{(k_1+n)+1}\{r^{k_1+(k_1+n)-1}\Delta - \ldots\}Y\\ &+ (-1)^{(k_1+n)+1}\{r^{k_1+(k_1+n)} - \ldots\}XY\\ &= (-1)^{(k_1+n)}\{\xi r^{k_1+\{k_1+(n+1)\}-4}\Delta^2 - \ldots\}I\\ &+ (-1)^{k_1+(n+1)}\{r^{k_1+\{k_1+(n+1)\}-1} - \ldots\}X\\ &+ (-1)^{k_1+(n+1)}\{r^{k_1+\{k_1+(n+1)\}-2}\Delta - \ldots\}Y\\ &+ (-1)^{k_1+(n+1)}\{r^{k_1+\{k_1+(n+1)\}-1} - \ldots\}XY.\end{aligned}$$

Hence the result is true for $(XY)^{q_1}(XY^{-1})^{q_2}$ such that $q_1 < q_2$. □

Theorem 2. *If $w = (x\ y)^{q_1}(xy^{-1})^{q_2}$, where $q_1, q_2 \in \mathbb{Z}^+$, then the corresponding matrix can be expressed linearly as $W = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY$, such that*

$$\begin{aligned}\max(\mu_1) &= (-1)^{q_2} r^{\tau_1 - 1},\\ \max(\mu_2) &= (-1)^{q_2} r^{\tau_1 - 2}\Delta,\\ \max(\mu_3) &= (-1)^{q_2} r^{\tau_1 - 1}.\end{aligned}$$

The Proof is an immediate consequence of Propositions 1 and **??**.

Total number of triangles in the circuit $(xy^{-1})^{s_1}(xy)^{s_2}$ are $s_1 + s_2$, let $s_1 + s_2 = \tau_2$ and $\alpha = \begin{cases} 0 & \text{if } s_2 = 1 \\ 1 & \text{otherwise} \end{cases}$. Since $(xy^{-1})^{s_1}(xy)^{s_2}$ can be expressed linearly as

$$(XY^{-1})^{s_1}(XY)^{s_2} = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$$

where λ_i, for $i = 0, 1, 2, 3$ is polynomial in r and Δ, we use $\max(\lambda_i)$ for the term containing the highest power of r, in λ_i.

Again by using mathematical induction, we have the following Theorem.

Theorem 3. *If $w = (xy^{-1})^{s_1}(xy)^{s_2}$ where $s_1, s_2, \in \mathbb{Z}^+$, then the corresponding matrix can be expressed linearly as $W = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$, such that*

$$\max(\lambda_1) = (-1)^{s_1+1} a r^{\tau_2-3}\Delta,$$

$$\max(\lambda_2) = (-1)^{s_1+1} r^{\tau_2-2}\Delta,$$

$$\max(\lambda_3) = (-1)^{s_1} r^{\tau_2-1}.$$

Theorem 4. *Let $\gamma \in \Omega$ then degree of the polynomial $f(\theta)$ obtained from γ is $s_1 + s_2 + q_1 + q_2 - 2$. Moreover $f(\theta)$ is monic.*

Proof. Since $\gamma \in \Omega$, therefore its one vertex v, is a fixed point of the circuits $(xy^{-1})^{s_1}(xy)^{s_2}$ and $(xy)^{q_1}(xy^{-1})^{q_2}$, where $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$. The matrices corresponding to $(xy^{-1})^{s_1}(xy)^{s_2}$ and $(xy)^{q_1}(xy^{-1})^{q_2}$ are $(XY^{-1})^{s_1}(XY)^{s_2}$ and $(XY)^{q_1}(XY^{-1})^{q_2}$ respectively, and these can be written as a linear combination of I, X, Y and XY, that is

$$(XY^{-1})^{s_1}(XY)^{s_2} = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$$

and

$$(XY)^{q_1}(XY^{-1})^{q_2} = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY$$

where λ_i and μ_i for $i = 0, 1, 2, 3$ are polynomials in r and Δ. By Theorems 2 and 3, we have

$$\max(\mu_1) = (-1)^{q_2} r^{\tau_1-1},$$

$$\max(\mu_2) = (-1)^{q_2} r^{\tau_1-2}\Delta,$$

$$\max(\mu_3) = (-1)^{q_2} r^{\tau_1-1}.$$

$$\max(\lambda_1) = (-1)^{s_1+1} a r^{\tau_2-3}\Delta,$$

$$\max(\lambda_2) = (-1)^{s_1+1} r^{\tau_2-2}\Delta,$$

$$\max(\lambda_3) = (-1)^{s_1} r^{\tau_2-1}.$$

Now

$$\max(\lambda_2\mu_3) = (-1)^{s_1+q_2+1} r^{\tau_1+\tau_2-3}\Delta$$

and

$$\max(\lambda_3\mu_2) = (-1)^{s_1+q_2} r^{\tau_1+\tau_2-3}\Delta.$$

Then

$$\max(\lambda_2\mu_3 - \lambda_3\mu_2) = (-1)^{s_1+q_2+1} 2 r^{\tau_1+\tau_2-3}\Delta \tag{4.1}$$

shows that

$$\max(\lambda_2\mu_3 - \lambda_3\mu_2)^2 = 4r^{2(\tau_1+\tau_2-3)}\Delta^2. \tag{4.2}$$

Now

$$\max(\lambda_3\mu_1) = (-1)^{q_2+s_1} r^{\tau_1+\tau_2-2} \tag{4.3}$$

and

$$\max(\lambda_1\mu_3) = (-1)^{s_1+q_2+1} r^{\tau_1+\tau_2-4}\Delta \tag{4.4}$$

together imply that

$$\max(\lambda_3\mu_1 - \lambda_1\mu_3) = (-1)^{s_1+q_2} r^{\tau_1+\tau_2-2} \tag{4.5}$$

or

$$\max(\Delta(\lambda_3\mu_1 - \lambda_1\mu_3)^2) = r^{2(\tau_1+\tau_2-2)}\Delta. \tag{4.6}$$

Now

$$\max(\lambda_1\mu_2) = (-1)^{s_1+q_2+1} a r^{\tau_1+\tau_2-5}\Delta^2 \tag{4.7}$$

and

$$\max(\lambda_2\mu_1) = (-1)^{s_1+q_2+1} r^{\tau_1+\tau_2-3}\Delta. \tag{4.8}$$

So

$$\max(\lambda_1\mu_2 - \lambda_2\mu_1) = (-1)^{s_1+q_2} r^{\tau_1+\tau_2-3}\Delta. \tag{4.9}$$

and

$$\max(\lambda_1\mu_2 - \lambda_2\mu_1)^2 = r^{2(\tau_1+\tau_2-3)}\Delta^2. \tag{4.10}$$

By using Equations 4.1 and 4.5, we obtain

$$\max(r(\lambda_2\mu_3 - \lambda_3\mu_2)(\lambda_3\mu_1 - \lambda_1\mu_3)) = -2r^{2(\tau_1+\tau_2-2)}\Delta. \tag{4.11}$$

Also by using Equations 4.1 and 4.9, we get

$$\max((\lambda_2\mu_3 - \lambda_3\mu_2)(\lambda_1\mu_2 - \lambda_2\mu_1)) = -2r^{2(\tau_1+\tau_2-3)}\Delta^2. \tag{4.12}$$

The term containing the highest power of θ, in the polynomial equation 3.13 yields degree and leading coefficient of the polynomial obtained from γ. By using Equations 4.2 to 4.12, we have

$$\max\begin{pmatrix} -(\lambda_2\mu_3 - \mu_2\lambda_3)^2 - \Delta(\lambda_3\mu_1 - \mu_3\lambda_1)^2 - (\lambda_1\mu_2 - \mu_1\lambda_2)^2 \\ -r(\lambda_2\mu_3 - \mu_2\lambda_3)(\lambda_3\mu_1 - \mu_3\lambda_1) - (\lambda_2\mu_3 - \mu_2\lambda_3)(\lambda_1\mu_2 - \mu_1\lambda_2) \end{pmatrix} = r^{2(\tau_1+\tau_2-2)}\Delta.$$

Since $r^2 = \Delta\theta$, therefore

$$\max\begin{pmatrix} -(\lambda_2\mu_3 - \mu_2\lambda_3)^2 - \Delta(\lambda_3\mu_1 - \mu_3\lambda_1)^2 - (\lambda_1\mu_2 - \mu_1\lambda_2)^2 \\ -r(\lambda_2\mu_3 - \mu_2\lambda_3)(\lambda_3\mu_1 - \mu_3\lambda_1) - (\lambda_2\mu_3 - \mu_2\lambda_3)(\lambda_1\mu_2 - \mu_1\lambda_2) \end{pmatrix} = \theta^{\tau_1+\tau_2-2}\Delta^{\tau_1+\tau_2-1}.$$

Since polynomial equation 3.13 is homogeneous in Δ and r^2 [3], so after replacing r^2 by $\Delta\theta$, we get the same degree, that is $\tau_1 + \tau_2 - 1$, of Δ in all the terms of polynomial equation. Also $\Delta = \det(X) \neq 0$, therefore, we can omit $\Delta^{\tau_1+\tau_2-1} = \Delta^{s_1+s_2+q_1+q_2-1}$. Hence the degree of the polynomial obtained from γ is $s_1 + s_2 + q_1 + q_2 - 2$. Also this polynomial is monic. □

Theorem 5. *Let $\gamma \in \Omega$ and $T(\gamma)$ and Deg (f) denote the number of triangles in γ and the degree of the polynomial obtained from γ respectively. Then Deg $(f) = T(\gamma)$.*

The Proof is an immediate consequence of the Theorems 1 and 4.

Theorem 6. *For a fixed degree n, there are finitely many polynomials in Ω.*

Proof. By Theorem 4, degree of polynomials in the family Ω of fragments, containing a vertex fixed by $(xy^{-1})^{s_1}(xy)^{s_2}, (xy)^{q_1}(xy^{-1})^{q_2}$, where $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ is $q_1 + q_2 + s_1 + s_2 - 2$. Since there are finite number of possibilities for $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1 + q_2 + s_1 + s_2 - 2 = n$, therefore there are finitely many polynomials of a fixed degree n, evolved from the fragments in Ω. □

5. Method to find the number of polynomials of a fixed degree

By Theorem 4, the degree of all the polynomials obtained from the family Ω of fragments containing one vertex fixed by $(xy^{-1})^{s_1}(xy)^{s_2}, (xy)^{q_1}(xy^{-1})^{q_2}$, where $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ is $q_1 + q_2 + s_1 + s_2 - 2$. So in order to find the number of polynomials of a fixed degree n, we first have to find the number of possibilities for $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1 + q_2 + s_1 + s_2 = n + 2$. For this, here we reproduce the statement of a well known combinatorics result.

Theorem 7. $\binom{n}{r}$ *is the number of possibilities for $x_1, x_2, x_3, \ldots, x_{r+1} \in \mathbb{Z}^+$ such that $\sum_{i=1}^{r+1} x_i = n + 1$.*

Remark 1. By using above Theorem we have $\binom{n+1}{3}$ possibilities for $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1+q_2+s_1+s_2 = n + 2$.

Each possibility ρ_i for $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1 + q_2 + s_1 + s_2 = n + 2$, gives a pair of words $(xy)^{q_1}(xy^{-1})^{q_2}, (xy^{-1})^{s_1}(xy)^{s_2}$ that fixes a vertex in a fragment $\gamma \in \Omega$ and hence a polynomial is evolved.

Definition 1. *Two distinct possibilities ρ_i and ρ_j for $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1 + q_2 + s_1 + s_2 = n + 2$, are equivalent if and only if they have the same polynomial.*

Proposition 3. *For the possibility $q_1 = c_1, q_2 = c_2, s_1 = e_1, s_2 = e_2$; there is no equivalent possibility but $q_1 = e_1, q_2 = e_2, s_1 = c_1, s_2 = c_2$.*

Proof. The pair of words obtained from the possibility $q_1 = c_1, q_2 = c_2, s_1 = e_1, s_2 = e_2$ is $(xy)^{c_1}(xy^{-1})^{c_2}$, $(xy^{-1})^{e_1}(xy)^{e_2}$ and the pair of words obtained from the possibility $q_1 = e_1, q_2 = e_2, s_1 = c_1, s_2 = c_2$ is $(xy)^{e_1}(xy^{-1})^{e_2}, (xy^{-1})^{c_1}(xy)^{c_2}$. Let $(xy)^{c_1}(xy^{-1})^{c_2}, (xy^{-1})^{e_1}(xy)^{e_2}$ fix a vertex in γ and $(xy)^{e_1}(xy^{-1})^{e_2}$, $(xy^{-1})^{c_1}(xy)^{c_2}$ fix a vertex in δ. Then by Remark 1 in [6], γ and δ are the mirror images of each other, and hence have the same polynomial. This shows that $q_1 = c_1, q_2 = c_2, s_1 = e_1, s_2 = e_2$ and $q_1 = e_1, q_2 = e_2, s_1 = c_1, s_2 = c_2$ are equivalent possibilities. Next we show that there is no other equivalent possibility for the possibility $q_1 = c_1, q_2 = c_2, s_1 = e_1, s_2 = e_2$.

Let v and u be the vertices in γ and δ respectively, such that u is fixed by $(xy)^{c_1}(xy^{-1})^{c_2}, (xy^{-1})^{e_1}(xy)^{e_2}$ and v is fixed by $(xy)^{c_1'}(xy^{-1})^{c_2'}, (xy^{-1})^{e_1'}(xy)^{e_2'}$. Now if at-least one of the following is true

$$(i)\quad c_1 \neq e_1' \quad (ii)\quad c_2 \neq e_2' \quad (iii)\quad e_1 \neq c_1' \quad (iv)\quad e_2 \neq c_2'.$$

Then it is obvious from the figure 9 that, γ and δ are neither the same nor the mirror image of each other. That is, they are distinct fragments and therefore have different conditions for the existence in $D(\theta, q)$, implying that they have different polynomials. Hence for the possibility $q_1 = c_1, q_2 = c_2, s_1 = e_1, s_2 = e_2$, there is no equivalent possibility but $q_1 = e_1, q_2 = e_2, s_1 = c_1, s_2 = c_2$. □

Definition 2. *Let $q_1 = c_1, q_2 = c_2, s_1 = c_1, s_2 = c_2$ be one of the possible values of $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1 + q_2 + s_1 + s_2 = n + 2$. Then this possibility for $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1 + q_2 + s_1 + s_2 = n + 2$ is called symmetric.*

Remark 2.

(i) Symmetric possibility exists only if $q_1 + q_2 + s_1 + s_2 \in 2\mathbb{Z}^+$.
(ii) Symmetric possibility does not have any equivalent possibility.
(iii) Corresponding to each non-symmetric possibility ρ_i for $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1 + q_2 + s_1 + s_2 = n + 2$, there is a unique equivalent possibility ρ_j for $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1 + q_2 + s_1 + s_2 = n + 2$.

Proposition 4. *The number of symmetric possibilities for $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1 + q_2 + s_1 + s_2 = n + 2$ are $\begin{cases} \frac{n}{2} & \text{if } n + 2 \text{ is even positive integer} \\ 0 & \text{if } n + 2 \text{ is odd positive integer} \end{cases}$.*

Proof. Let $n + 2$ be an even positive integer, and ρ_i be a symmetric possibilities for s_1, s_2, q_1, q_2 such that $q_1 + q_2 + s_1 + s_2 = n + 2$. Then $q_1 = s_1, q_2 = s_2$ that is $q_1 + q_2 = s_1 + s_2 = \frac{n+2}{2}$. By Theorem 7 the number of possibilities for $s_1, s_2 \in \mathbb{Z}^+$ such that $s_1 + s_2 = \frac{n+2}{2}$ are $\binom{\frac{n}{2}}{1} = \frac{n}{2}$. Hence if $n + 2$ is even, then there are $\frac{n}{2}$ symmetric possibilities for $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1 + q_2 + s_1 + s_2 = n + 2$.

If $n + 2$ is odd then it is quite trivial that there is no symmetric possibility. □

Theorem 8. *For a fixed degree n, there are*

$$\begin{cases} \frac{\binom{n+1}{3}}{2} & \text{if } n \text{ is odd} \\ \frac{\binom{n+1}{3} + \frac{n}{2}}{2} & \text{if } n \text{ is even} \end{cases}$$

polynomials, obtained from the fragments in Ω.

Proof. By Remark 1, the total number of possibilities for $s_1, s_2, q_1, q_2 \in \mathbb{Z}^+$ such that $q_1 + q_2 + s_1 + s_2 = n + 2$ are $\binom{n+1}{3}$.

If n is odd, then by Proposition 4, all of these possibilities are non-symmetric. Since for each non-symmetric possibility there is a unique equivalent possibility. Thus there are $\frac{\binom{n+1}{3}}{2}$ non-equivalent possibilities, which implies that there are $\frac{\binom{n+1}{3}}{2}$ polynomials.

If n is even, then by Proposition 4, $\frac{n}{2}$ possibilities are symmetric and the remaining $\binom{n+1}{3} - \frac{n}{2}$ possibilities are non-symmetric. Since for each non-symmetric possibility there is a unique equivalent possibility and for the symmetric possibility there is no equivalent possibility. Thus there are

$$\frac{\binom{n+1}{3} - \frac{n}{2}}{2} + \frac{n}{2} = \frac{\binom{n+1}{3} + \frac{n}{2}}{2}$$

non-equivalent possibilities, so there are

$$\frac{\binom{n+1}{3} - \frac{n}{2}}{2} + \frac{n}{2} = \frac{\binom{n+1}{3} + \frac{n}{2}}{2}$$

distinct polynomials. □

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