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Hasse principle for simply connected groups over function fields of surfaces

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Abstract. Let K be the function field of a p-adic curve, G a semi-simple simply connected group over K and X a G-torsor over K. A conjecture of Colliot-Thélène, Parimala and Suresh predicts that if for every discrete valuation v of K, X has a point over the completion K_v, then X has a K-rational point. The main result of this paper is the proof of this conjecture for groups of some classical types. In particular, we prove the conjecture when G is of one of the following types: (1) ${}^2A_n^*$, i.e. $G = \mathbf{SU}(h)$ is the special unitary group of some hermitian form h over a pair (D, τ), where D is a central division algebra of square-free index over a quadratic extension L of K and τ is an involution of the second kind on D such that $L^\tau = K$; (2) B_n, i.e., $G = \mathbf{Spin}(q)$ is the spinor group of quadratic form of odd dimension over K; (3) D_n^*, i.e., $G = \mathbf{Spin}(h)$ is the spinor group of a hermitian form h over a quaternion K-algebra D with an orthogonal involution. Our method actually yields a parallel local-global result over the fraction field of a 2-dimensional, henselian, excellent local domain with finite residue field, under suitable assumption on the residue characteristic.

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1. Introduction

Let K be a field and G a smooth connected linear algebraic group over K. The cohomology set $H^1(K, G)$ classifies up to isomorphism G-torsors over K, and a class $\xi \in H^1(K, G)$ is trivial if and only if the corresponding G-torsor has a K-rational point. Let Ω_K denote the set of (normalized) discrete valuations (of rank 1) of the field K. For each $v \in \Omega_K$, let K_v denote the completion of K at v. The restriction maps $H^1(K, G) \to H^1(K_v, G)$, $v \in \Omega_K$ induce a natural map of pointed sets

$$H^1(K, G) \longrightarrow \prod_{v \in \Omega_K} H^1(K_v, G).$$

If the kernel of this map is trivial, we say that the *Hasse principle* with respect to Ω_K holds for G-torsors over K.

In the case of a p-adic function field, by which we mean the function field of an algebraic curve over a p-adic field (i.e., a finite extension of $\mathbb{Q}_p$), the following conjecture was made by Colliot-Thélène, Parimala and Suresh.

Conjecture 1.1 ([6]). Let K be the function field of an algebraic curve over a p-adic field and let G be a semisimple simply connected group over K.

Then the kernel of the natural map

$$H^1(K, G) \longrightarrow \prod_{v \in \Omega_K} H^1(K_v, G)$$

is trivial. In other words, if a G-torsor has points in all completions K_v, $v \in \Omega_K$, then it has a K-rational point.

1.2. Let K be a p-adic function field with field of constants F, i.e., K is the function field of a smooth projective geometrically integral curve over the p-adic field F. Let A be the ring of integers of F. It is in particular a henselian excellent local domain of dimension 1. By resolution of singularities, there exists a proper flat morphism $\mathcal{X} \to \operatorname{Spec} A$, where $\mathcal{X}$ is a connected regular 2-dimensional scheme with function field K. We will say that $\mathcal{X}$ is a p-adic *arithmetic surface* with function field K, or that $\mathcal{X} \to \operatorname{Spec} A$ is a *regular proper model* of the p-adic function field K.

An analog in the context of a 2-dimensional base is as follows. Let A be a henselian excellent 2-dimensional local domain with *finite* residue field k and let K be the field of fractions of A. Again by resolution of singularities, there exists a proper birational morphism $\mathcal{X} \to \operatorname{Spec} A$, where $\mathcal{X}$ is a connected regular 2-dimensional scheme with function field K. We will say that Spec A is a *local henselian surface* with function field K and that $\mathcal{X} \to \operatorname{Spec} A$ is a *regular proper model* of Spec A.

Experts have also been interested in the following analog of Conjecture 1.1:

Question 1.3. *Let K be the function field of a local henselian surface* Spec A *with finite residue field and let G be a semisimple simply connected group over K.*

Does the Hasse principle with respect to Ω_K hold for G-torsors over K?

Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field. For most quasi-split K-groups, the Hasse principle may be proved by combining an injectivity property of the Rost invariant map (cf. [6, Thm. 5.3]) and results from higher dimensional class field theory of Kato and Saito.

The goal of this paper is to prove the Hasse principle for groups of several types in the non-quasisplit case. To give precise statement of our main result, we will refine the usual classification of absolutely simple simply connected groups in some cases.

1.4. Let E be a field and let G be an absolutely simple simply connected group over E. We say that G is of type

(1) ${}^1A_n^*$, if $G = \mathbf{SL}_1(A)$ is the special linear group of some central simple E-algebra A *of square-free index*;
(2) ${}^2A_n^*$, if $G = \mathbf{SU}(h)$ is the special unitary group of some nonsingular hermitian form h over a pair (D, τ), where D is a central division algebra *of square-free index* over a separable quadratic field extension L of E and τ is an involution of the second kind on D such that $L^\tau = E$; when the index of division algebra D is odd (resp. even), we say the group $G = \mathbf{SU}(h)$ is of type ${}^2A_n^*$ of odd (resp. even) index;
(3) C_n^*, if $G = \mathbf{U}(h)$ is the unitary group (also called symplectic group) of a nonsingular hermitian form h over a pair (D, τ), where D is *quaternion algebra* over E and τ is a symplectic involution on D;
(4) D_n^* (in characteristic $\neq 2$), if $G = \mathbf{Spin}(h)$ is the spin group of a nonsingular hermitian form h over a pair (D, τ), where D is *quaternion algebra* over E and τ is an orthogonal involution on D;
(5) F_4^{red} (in characteristic different from 2, 3), if $G = \mathbf{Aut}_{alg}(J)$ is the group of algebra automorphisms of some *reduced* exceptional Jordan E-algebra J of dimension 27.
Recall also that G is of type
(6) B_n (in characteristic $\neq 2$), if $G = \mathbf{Spin}(q)$ is the spin group of a nonsingular quadratic form q of dimension $2n + 1$ over E;
(7) G_2 (in characteristic $\neq 2$), if $G = \mathbf{Aut}_{alg}(C)$ is the group of algebra automorphisms of a Cayley algebra C over E.

1.5. In the local henselian case, we shall exclude some possibilities for the residue characteristic. To this end, we define for any semisimple simply connected group G a set $S(G)$ of prime numbers as follows (cf. [30, §2.2] or [9, p. 44]):

$S(G) = \{2\}$, if G is of type G_2 or of classical type B_n, C_n or D_n (trialitarian D_4 excluded);

$S(G) = \{2, 3\}$, if G is of type E_6, E_7, F_4 or trialitarian D_4;

$S(G) = \{2, 3, 5\}$, if G is of type E_8;

$S(G)$ is the set of prime factors of the index $\mathrm{ind}(A)$ of A, if $G = \mathbf{SL}_1(A)$ for some central simple algebra A;

$S(G)$ is the set of prime factors of $2.\mathrm{ind}(D)$, if $G = \mathbf{SU}(h)$ for some nonsingular hermitian form h over a division algebra D with an involution of the second kind.

In the general case, define $S(G) = \cup S(G_i)$, where G_i runs over the almost simple factors of G.

When G is absolutely simple, let n_G be the order of the Rost invariant of G. Except for a few cases where $n_G = 1$, the set $S(G)$ coincides with the set of prime factors of n_G (cf. [21, Appendix B] or [17, §31.B]).

We summarize our main results in the following two theorems.

Theorem 1.6. **Let K be the function field of a p-adic arithmetic surface and G a semisimple simply connected group over K. Assume $p \neq 2$ if G contains an almost simple factor of type ${}^2A_n^*$ of even index.*

If every almost simple factor of G is of type

$$^1A_n^*,\ ^2A_n^*,\ B_n,\ C_n^*,\ D_n^*,\ F_4^{\mathrm{red}} \quad or \quad G_2,$$

then the natural map

$$H^1(K, G) \longrightarrow \prod_{v\in\Omega_K} H^1(K_v, G)$$

has a trivial kernel.

Theorem 1.7. *Let K be the function field of a local henselien surface with finite residue field of characteristic p. Let G be a semisimple simply connected group over K. Assume $p \notin S(G)$.*

If every almost simple factor of G is of type

$$^1A_n^*,\ ^2A_n^* \text{ of odd index},\ B_n,\ C_n^*,\ D_n^*,\ F_4^{\mathrm{red}} \quad or \quad G_2,$$

then the natural map

$$H^1(K, G) \longrightarrow \prod_{v\in\Omega_K} H^1(K_v, G)$$

has a trivial kernel.

If moreover the Hasse principle with respect to Ω_K holds for quadratic forms q of rank 6 over K (i.e., q has a nontrivial zero over K if and only if it has a nontrivial zero over every K_v, $v \in \Omega_K$), then the same result is also true for an absolutely simple group of type ${}^2A_n^*$ of even index.

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*R. Preeti [26] has proved results on the injectivity of the Rost invariant which overlap with the results in this paper. Our work was carried out independently.

In fact, it suffices to consider only divisorial discrete valuations in the above theorems.

Remark 1.8. Let K be as in Theorem 1.6 or 1.7. Assume the residue characteristic p is not 2.

(1) By [28, Thm. 3.4] (the arithmetic case) and [13, Thm. 3.4] (the local henselian case), a central division algebra of exponent 2 over the field K is either a quaternion algebra or a biquaternion algebra. So for a group of type C_n, say $G = \mathbf{U}(h)$ with h a hermitian form over a symplectic pair (D, τ), the only case not covered by our theorems is the case where D is a biquaternion algebra. Similarly, for a group of classical type D_n, say $D = \mathbf{Spin}(h)$ with h a hermitian form over an orthogonal pair (D, τ), the only remaining case is the one with D a biquaternion algebra.
(2) In Theorem 1.7, the hypothesis on the Hasse principle for quadratic forms of rank 6 is satisfied if $K = \mathrm{Frac}(\mathcal{O}[\![t]\!])$ is the fraction field of a formal power series ring over a complete discrete valuation ring $\mathcal{O}$ (whose residue field is finite), by [12, Thm. 1.2]. In the arithmetic case, this is established in [6, Thm. 3.1].

In the rest of the paper, after some preliminary reviews in Section 2, we will prove our main theorems case by case: the cases ${}^1A_n^*$, C_n, F_4^{red} and G_2 in Section 3; the cases B_n and D_n^* in Sections 4 and 5; and the case ${}^2A_n^*$ in Section 6.

Our proofs use ideas from Parimala and Preeti's paper [23]. In particular, two exact sequences of Witt groups, due to Parimala–Sridharan–Suresh and Suresh respectively, play a special role in some cases. Other important ingredients include Hasse principles for degree 3 cohomology of $\mathbb{Q}/\mathbb{Z}(2)$ coming from higher dimensional class field theory of Kato and Saito (cf. [15] and [27]), as well as the work of Merkurjev and Suslin on reduced norm criterion and norm principles ([31,20]). For spinor groups and groups of type ${}^2A_n^*$ of even index, we also make use of results on quadratic forms over the base field K obtained in [25,19] (see also [11]) in the p-adic case and in [13] in the local henselian case.

2. Some reviews and basic tools

In this section, we briefly review some basic notions which will be used frequently and we recall some known results that are essential in the proofs to come later.

Throughout this section, let L denote a field of characteristic different from 2.

2.1 Hermitian forms and Witt groups

We will assume the readers have basic familiarity with the theory of involutions and hermitian forms over central simple algebras (cf. [29,16,17]). For later use, we recall in this subsection some facts on Witt groups, the "key exact sequence" of Parimala, Sridharan and Suresh and the exact sequence of Suresh. The readers are referred to [3, §3 and Appendix 2], [4, §3] and [23, §8] for more information.

Unless otherwise stated, all hermitian forms and skew-hermitian forms (in particular all quadratic forms) in this paper are assumed to be nonsingular.

2.1. Let L be a field of characteristic different from 2, A a central simple algebra over L and σ an involution on A. Let $E = L^{\sigma}$. We say that σ is an *L/E-involution* on A. To each hermitian or skew-hermitian form (V, h) over (A, σ), one can associate an involution on $\mathrm{End}_A(V)$, called the *adjoint involution* on $\mathrm{End}_A(V)$ with respect to h. This is the unique involution σ_h on $\mathrm{End}_A(V)$ such that

$$h(x, f(y)) = h(\sigma_h(f)(x), y) \quad \forall\ x, y \in V, \qquad \forall\ f \in \mathrm{End}_A(V).$$

For a fixed finitely generated right A-module V, define an equivalence relation $\sim$ on the set of hermitian or skew-hermitian forms on V (with respect to the involution σ) by

$$h \sim h' \iff \text{there exists } \lambda \in E^* \text{ such that } h = \lambda.h'.$$

Let $\mathcal{H}^+(V)$ (resp. $\mathcal{H}^-(V)$) denote the set of equivalence classes of hermitian (resp. skew-hermitian) forms on V and let $\mathcal{H}^{\pm}(V) = \mathcal{H}^+(V) \cup \mathcal{H}^-(V)$. The assignment $h \mapsto \sigma_h$ defines a map from $\mathcal{H}^{\pm}(V)$ to the set of involutions on $\mathrm{End}_A(V)$. If σ is of the first kind, then the map $h \mapsto \sigma_h$ induces a bijection between $\mathcal{H}^{\pm}(V)$ and the set of involutions of the first kind on $\mathrm{End}_A(V)$, and the involutions σ_h and σ have the same type (orthogonal or symplectic) if h is hermitian and they have opposite types if h is skew-hermitian. If σ is of the second kind, then the map $h \mapsto \sigma_h$ induces a bijection between $\mathcal{H}^+(V)$ and the set of L/E-involutions on $\mathrm{End}_A(V)$. (cf. [17, p. 43, Thm. 4.2].)

If $A = L$ and $\sigma = \mathrm{id}$, a hermitian (resp. skew-hermitian) form h is simply a symmetric (resp. skew-symmetric) bilinear form b. In this case, $b \mapsto \sigma_b$ defines a bijection between equivalence classes of nonsingular symmetric or skew-symmetric bilinear forms on V modulo multiplication by a factor in L^* and involutions of the first kind on $\mathrm{End}_L(V)$. If q is the quadratic form associated to a symmetric bilinear form b, we also write σ_q for the adjoint involution σ_b.

2.2. Let (A, σ) be a pair consisting of a central simple algebra A over a field L of characteristic $\neq 2$ and an involution (of any kind) σ on A. The orthogonal sum of hermitian forms defines a semigroup structure on the set of isomorphism classes of hermitian forms over (A, σ). The quotient of the corresponding Grothendieck group by the subgroup generated by hyperbolic forms is called the *Witt group* of (A, σ) and denoted $W(A, \sigma) = W^1(A, \sigma)$. The same construction applies to skew-hermitian forms and the corresponding Witt group will be denoted $W^{-1}(A, \sigma)$.

If $A = L$ and $\sigma = \mathrm{id}$, then $W(A, \sigma)$ is the usual Witt group $W(L)$ of quadratic forms (cf. [18,29]). One has a ring structure on $W(L)$ induced by the tensor product of quadratic forms. The classes of even dimensional forms form an ideal $I(L)$ of the ring $W(L)$. For each $n \geq 1$, we write $I^n(L)$ for the n-th power of the ideal $I(L)$. As an abelian group, $I^n(L)$ is generated by the classes of n-fold Pfister forms.

2.3. Let D be a quaternion division algebra over a field L of characteristic $\neq 2$. Let τ_0 be the standard (symplectic) involution on D. The Witt group $W(D, \tau_0)$ has a nice description as follows (cf. [29, p. 352]).

If $h : V \times V \to D$ is a hermitian form over (D, τ_0), then the map

$$q_h : V \longrightarrow L, \quad q_h(x) := h(x, x)$$

defines a quadratic form on the L-vector space V, called the *trace form* of h. If h is isomorphic to the diagonal form $\langle \lambda_1, \dots, \lambda_r \rangle$, then q_h is isomorphic to the form $\langle \lambda_1, \dots, \lambda_r \rangle \otimes n_D$, where n_D denotes the norm form of the quaternion algebra D. By [29, p. 352, Thm. 10.1.7], the assignment $h \mapsto q_h$ induces an injective group homomorphism $W(D, \tau_0) \to W(L)$, whose image is the principal ideal of $W(L)$ generated by (the class of) the norm form n_D of D. In particular, two hermitian forms over (D, τ_0) are isomorphic if and only if their trace forms are isomorphic.

2.4. Let L/E be a quadratic extension of fields of characteristic different from 2. The nontrivial element ι of the Galois group $\mathrm{Gal}(L/E)$ may be viewed as a unitary involution on the L-algebra $A = L$. The Witt group $W(L, \iota)$ can be determined as follows (cf. [29, pp. 348–349]):

As in (2.3), to each hermitian form $h : V \times V \to L$ over (L, ι), one can associate a quadratic form q_h on the E-vector space V, called the *trace form* of h, by defining

$$q_h(x) := h(x, x) \in E, \quad \forall\ x \in V.$$

One can show that $h \mapsto q_h$ induces a group homomorphism $W(L, \iota) \to W(E)$ which identifies $W(L, \iota)$ with the kernel of the base change homomorphism $W(E) \to W(L)$. In particular, two hermitian forms over (L, ι) are isomorphic if and only if their trace forms are isomorphic. (cf. [29, Thm. 10.1.2].)

Let $\delta \in E$ be an element such that $L = E(\sqrt{\delta})$. Then for $a \in E^*$, the trace form of $h = \langle a \rangle$ is isomorphic to $\langle a, -a\delta \rangle = a.\langle 1, -\delta \rangle$. So the image of the map

$$W(L, \iota) \longrightarrow W(E)\,; \quad h \mapsto q_h$$

is the principal ideal generated by the form $\langle 1, -\delta \rangle$ (cf. [29, Remark 10.1.3]).

2.5. Let A be a central simple algebra over a field L of characteristic $\operatorname{char}(L) \neq 2$. Let σ be an involution on A and let $E = L^{\sigma}$. For any invertible element $u \in A^*$, let $\operatorname{Int}(u) : A \to A$ denote the inner automorphism $x \mapsto u.x.u^{-1}$. If $\sigma(u)u^{-1} = \pm 1$, then $\operatorname{Int}(u) \circ \sigma$ is an involution on A of the same kind as σ.

Conversely, let σ, τ be involutions of the same kind on A. If σ and τ are of the first kind, then there is a unit $u \in A^*$, uniquely determined up to a scalar factor in E^*, such that $\tau = \operatorname{Int}(u) \circ \sigma$ and $\sigma(u) = \pm u$. Moreover, the two involutions σ and $\tau = \operatorname{Int}(u) \circ \sigma$ are of the same type (orthogonal or symplectic) if and only if $\sigma(u) = u$. If σ and τ are of the second kind, then there exists a unit $u \in A^*$, uniquely determined up to a scalar factor in E^*, such that $\tau = \operatorname{Int}(u) \circ \sigma$ and $\sigma(u) = u$.

Let $\mathfrak{H}(A, \sigma) = \mathfrak{H}^1(A, \sigma)$ (resp. $\mathfrak{H}^{-1}(A, \sigma)$) denote the category of hermitian (resp. skew-hermitian) forms over (A, σ). Let $\varepsilon, \varepsilon' \in \{\pm 1\}$. Let $a \in A^*$ be an element such that $\sigma(a) = \varepsilon' a$. Then the functor

$$\Phi_a : \mathfrak{H}^{\varepsilon}(A, \operatorname{Int}(a^{-1}) \circ \sigma) \longrightarrow \mathfrak{H}^{\varepsilon\varepsilon'}(A, \sigma); \quad (V, h) \longmapsto (V, a.h)$$

is an equivalence of categories, called a *scaling*. There is also an induced isomorphism of Witt groups

$$\phi_a : W^{\varepsilon}(A, \operatorname{Int}(a^{-1}) \circ \sigma) \xrightarrow{\sim} W^{\varepsilon\varepsilon'}(A, \sigma).$$

In particular, if σ and τ are involutions of the same kind and type on A, then there is a scaling isomorphism of Witt groups $\phi_a : W(A, \tau) \xrightarrow{\sim} W(A, \sigma)$.

2.6. Let A be a central simple algebra over a field L of characteristic $\neq 2$ and σ an involution of any kind on A. Let (V, h) be a hermitian form over (A, σ). Let $B = \operatorname{End}_A(V)$ and let σ_h be the adjoint involution with respect to h. There is an equivalence of categories, called the *Morita equivalence*,

$$\Phi_h : \mathfrak{H}(B, \sigma_h) \longrightarrow \mathfrak{H}(A, \sigma)$$

defined as follows (cf. [3, §1.4], [16, §I.9]): For a hermitian form (M, f) over (B, σ_h), define a map

$$h * f : (M \otimes_B V) \times (M \otimes_B V) \longrightarrow A$$

by

$$(h * f)(m_1 \otimes v_1, m_2 \otimes v_2) := h(v_1, f(m_1, m_2)(v_2)).$$

One verifies that $\Phi_h(M, f) := (M \otimes_B V, h * f)$ yields a well-defined functor $\mathfrak{H}(B, \sigma_h) \to \mathfrak{H}(A, \sigma)$, which can be shown to be an equivalence (cf. [16, p. 56, Thm. I.9.3.5]). The Morita equivalence induces an isomorphism of Witt groups:

$$\phi_h : W(\mathrm{End}_A(V), \sigma_h) \xrightarrow{\sim} W(A, \sigma).$$

2.7. We briefly recall the construction of the key exact sequence of Parimala, Sridharan and Suresh. The readers are referred to [3, §3 and Appendix 2] for more details.

Let (A, σ) be a central simple algebra with involution over L. Let $E = L^\sigma$. Assume there is a subfield $M \subseteq A$ which is a quadratic extension of L such that $\sigma(M) = M$. Suppose $\sigma|_M = \mathrm{id}_M$ if σ is of the first kind. Let

$$\widetilde{A} := \{a \in A \mid a.m = m.a, \quad \forall\ m \in M\}$$

be the centralizer of M in A. This is a central simple algebra over M. By [3, Lemma 3.1.1], there exists $\mu \in A^*$ such that $\sigma(\mu) = -\mu$ and that the restriction of $\mathrm{Int}(\mu)$ to M is the nontrivial element of the Galois group $\mathrm{Gal}(M/L)$.

Set $\tau = \mathrm{Int}(\mu) \circ \sigma$ and let τ_1, τ_2 be the restrictions of τ and σ to $\widetilde{A}$ respectively. Then τ_1 is an involution of the second kind, τ_2 is of the same kind and type as σ, and τ is orthogonal (resp. symplectic) if and only if σ is symplectic (resp. orthogonal).

One has a decomposition $A = \widetilde{A} \oplus \mu.\widetilde{A}$ (as right M-modules). Let $\pi_1, \pi_2 : A \to \widetilde{A}$ be the M-linear projections

$$\pi_1(x + \mu y) = x, \quad \pi_2(x + \mu y) = y, \quad \forall\ x, y \in \widetilde{A}.$$

These induce well-defined group homomorphisms

$$\pi_1 : W(A, \tau) \longrightarrow W(\widetilde{A}, \tau_1) \quad \text{and} \quad \pi_2 : W^{-1}(A, \tau) \longrightarrow W(\widetilde{A}, \tau_2).$$

On the other hand, let $\lambda \in M$ be an element such that $\lambda^2 \in L$ and $M = L(\lambda)$. For a hermitian form $(\widetilde{V}, f)$ over $(\widetilde{A}, \tau_1)$, define $\rho(f)$ to be the unique skew-hermitian form on $V = \widetilde{V} \oplus \widetilde{V}\mu$ which extends $\lambda.f : \widetilde{V} \times \widetilde{V} \to \widetilde{A}$. This defines a group homomorphism

$$\rho : W(\widetilde{A}, \tau_1) \longrightarrow W^{-1}(A, \tau); \quad (\widetilde{V}, f) \mapsto (\widetilde{V} \oplus \widetilde{V}\mu, \rho(f)).$$

The sequence

$$W^\varepsilon(A, \tau) \xrightarrow{\pi_1} W^\varepsilon(\widetilde{A}, \tau_1) \xrightarrow{\rho} W^{-\varepsilon}(A, \tau) \xrightarrow{\pi_2} W^\varepsilon(\widetilde{A}, \tau_2) \tag{2.7.1}$$

turns out to be an exact sequence (cf. [3, Appendix 2]).

Since $\tau(\mu) = -\mu$, one has a scaling isomorphism (cf. (2.5))

$$\phi_\mu^{-1} : W^{-1}(A,\tau) \xrightarrow{\sim} W(A,\sigma).$$

We may thus replace $W^{-1}(A,\tau)$ in the exact sequence (2.7.1) by $W(A,\sigma)$ and rewrite it as

$$W(A,\tau) \xrightarrow{\pi_1} W(\widetilde{A},\tau_1) \xrightarrow{\widetilde{\rho}} W(A,\sigma) \xrightarrow{\widetilde{\pi}_2} W(\widetilde{A},\tau_2) \tag{2.7.2}$$

where $\widetilde{\rho} = \phi_\mu^{-1} \circ \rho$ and $\widetilde{\pi}_2 = \pi_2 \circ \phi_\mu$. This exact sequence is due to Parimala, Sridharan and Suresh and is referred to as the *key exact sequence* in [3].

We will only use the exact sequence (2.7.2) in the case where $A = D$ is a quaternion algebra and σ is an orthogonal involution. This special case was already discussed by Scharlau in [29, p. 359].

2.8. Now let D be a quaternion division algebra over a quadratic field extension L of E and let τ be a unitary L/E-involution on D (i.e. a unitary involution such that $L^\tau = E$). There is a unique quaternion E-algebra D_0 contained in D such that $D = D_0 \otimes_E L$ and $\tau = \tau_0 \otimes \iota$, where τ_0 is the canonical (symplectic) involution on D_0 and ι is the nontrivial element of the Galois group $\mathrm{Gal}(L/E)$. Write $L = E(\sqrt{d})$ with $d \in E^*$. Then $D = D_0 \oplus D_0\sqrt{d}$. For any hermitian form (V,h) over (D,τ), we may write

$$h(x,y) = h_1(x,y) + h_2(x,y)\sqrt{d} \quad \text{with} \quad h_i(x,y) \in D_0, \quad \text{for} \quad i = 1,2$$

for any $x, y \in V$.

The projection $h \mapsto h_2$ defines a group homomorphism

$$p_2 : W(D,\tau) \longrightarrow W^{-1}(D_0,\tau_0).$$

For a hermitian form (V_0, f) over (D_0,τ_0), set

$$V = V_0 \otimes_{D_0} D = V_0 \otimes_E L = V_0 \oplus V_0\sqrt{d}$$

and let $\widetilde{\rho}(f) : V \times V \to D$ be the map extending $f : V_0 \times V_0 \to D_0$ by τ-sesquilinearity. One checks that this defines a group homomorphism

$$\widetilde{\rho} : W(D_0,\tau_0) \longrightarrow W(D,\tau)\,; \quad (V_0, f) \longmapsto (V_0 \oplus V_0\sqrt{d}, \widetilde{\rho}(f)).$$

For any quadratic form q over $L = E(\sqrt{d})$, there are quadratic forms q_1, q_2 over k such that $q(x) = q_1(x) + q_2(x)\sqrt{d}$. We have thus group homomorphisms

$$\pi_i : W(L) \longrightarrow W(E)\,; q \longmapsto q_i, \quad i = 1,2.$$

We denote by $\widetilde{\pi}_1 : W(L) \to W(D_0,\tau_0)$ the composite map

$$W(L) \xrightarrow{\pi_1} W(E) \longrightarrow W(D_0,\tau_0)$$

where the map $W(E) \to W(D_0,\tau_0)$ is induced by base change.

Suresh (cf. [23, Prop. 8.1]) proved that the sequence

$$W(L) \xrightarrow{\widetilde{\pi}_1} W(D_0, \tau_0) \xrightarrow{\widetilde{\rho}} W(D, \tau) \xrightarrow{p_2} W^{-1}(D_0, \tau_0)$$

is exact. We will refer to this sequence as Suresh's exact sequence in the sequel.

2.2 Invariants of hermitian forms

In this subsection, we recall the definitions of some invariants of hermitian forms. For more details, see [3, §2], [4, §3] and [23, §5, §7].

2.9. Let (D, σ) be a central division algebra with involution over L. Let $E = L^\sigma$. Let (V, h) be a hermitian form over (D, σ). The *rank* of (V, h), denoted $\operatorname{rank}(V, h)$ or simply $\operatorname{rank}(h)$, is by definition the rank of the D-module V:

$$\operatorname{rank}(h) := \operatorname{rank}_D(V).$$

2.10. With notation as in (2.9), let $e_1, \dots, e_n$ be a basis of the D-module V (so that $\operatorname{rank}(h) = \operatorname{rank}_D(V) = n$). Let $M(h) := (h(e_i, e_j))$ be the matrix of the hermitian form h with respect to this basis. The matrix algebra $A = \mathrm{M}_n(D)$ has dimension

$$\dim_L A = n^2 \dim_L D = (\operatorname{rank}(h) \cdot \deg_L D)^2.$$

Put

$$m = \sqrt{\dim_L A} = \operatorname{rank}(h) \cdot \deg_L D = \frac{\dim_L V}{\deg_L D}.$$

We define the *discriminant* $\operatorname{disc}(h) = \operatorname{disc}(V, h)$ of the hermitian form (V, h) by

$$\operatorname{disc}(h) = (-1)^{\frac{m(m-1)}{2}} \operatorname{Nrd}_A(M(h))$$

$$\in \begin{cases} E^*/E^{*2} & \text{if } \sigma \text{ is of the first kind} \\ E^*/N_{L/E}(E^*) & \text{if } \sigma \text{ is of the second kind} \end{cases}$$

If h is a hermitian form over (D, σ), the image of the canonical map

$$H^1(E, \mathbf{SU}(h)) \longrightarrow H^1(E, \mathbf{U}(h))$$

consists of classes $[h'] \in H^1(E, \mathbf{U}(h))$ of hermitian forms h' which have the same rank and discriminant as h.

2.11. Let D be a cenral division algebra over L and let σ be an orthogonal involution on D. Note that the Brauer class of D in the Brauer group $\mathrm{Br}(L)$ lies in the subgroup

$$ {}_2\mathrm{Br}(L) := \{\alpha \in \mathrm{Br}(L) \mid 2.\alpha = 0\}. $$

Let h be a hermitian form over (D, σ). Let

$$ \delta : H^1(L, \mathbf{SU}(h)) \longrightarrow H^2(L, \mu_2) = {}_2\mathrm{Br}(L) $$

be the connecting map associated to the exact sequence of algebraic groups

$$ 1 \longrightarrow \mu_2 \longrightarrow \mathbf{Spin}(h) \longrightarrow \mathbf{SU}(h) \longrightarrow 1. $$

Let h' be a hermitian form over (D, σ) such that $\mathrm{rank}(h') = \mathrm{rank}(h)$ and $\mathrm{disc}(h') = \mathrm{disc}(h)$. Then there is an element $c(h') \in H^1(L, \mathbf{SU}(h))$ which lifts $[h'] \in H^1(L, \mathbf{U}(h))$. The class of $\delta(c(h'))$ in the quotient ${}_2\mathrm{Br}(L)/\langle [D] \rangle$ is independent of the choice of $c(h')$ (cf. [3, §2.1]). Following [2], we define the *relative Clifford invariant* $\mathscr{C}\ell_h(h')$ by

$$ \mathscr{C}\ell_h(h') := [\delta(c(h'))] \in \frac{{}_2\mathrm{Br}(L)}{\langle [D] \rangle}. $$

When h has even rank $2n$ and trivial discriminant, the *Clifford invariant* $\mathscr{C}\ell(h)$ of h is defined as

$$ \mathscr{C}\ell(h) := \mathscr{C}\ell_{H_{2n}}(h) \in \frac{{}_2\mathrm{Br}(L)}{\langle [D] \rangle}, $$

where H_{2n} denotes a hyperbolic hermitian form of rank $2n = \mathrm{rank}(h)$ over (D, σ). If $D = L$ and $h = q$ is a nonsingular quadratic form over L, then $\mathscr{C}\ell(h)$ coincides with the usual Clifford invariant of the quadratic form q.

2.12. Let (D, σ) be a central division algebra with an orthogonal involution over L. We denote by $\mathbf{U}_{2n}(D, \sigma)$, $\mathbf{SU}_{2n}(D, \sigma)$ and $\mathbf{Spin}_{2n}(A, \sigma)$ respectively the unitary group, the special unitary group and the spin group of the hyperbolic form over (D, σ) defined by the matrix $H_{2n} = \left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$.

Let h be a hermitian form of even rank $2n$, trivial discriminant and trivial Clifford invariant. There is an element $\xi \in H^1(L, \mathbf{Spin}_{2n}(D, \sigma))$ which is mapped to the class $[h] \in H^1(L, \mathbf{U}_{2n}(D, \sigma))$ under the composite map

$$ H^1(L, \mathbf{Spin}_{2n}(D, \sigma)) \longrightarrow H^1(L, \mathbf{SU}_{2n}(D, \sigma)) \longrightarrow H^1(L, \mathbf{U}_{2n}(D, \sigma)). $$

Let

$$ R_{\mathbf{Spin}_{2n}(D,\sigma)} : H^1(L, \mathbf{Spin}_{2n}(D, \sigma)) \longrightarrow H^3(L, \mathbb{Q}/\mathbb{Z}(2)) $$

be the usual Rost invariant map of the simply connected group $\mathbf{Spin}_{2n}(D,\sigma)$ (cf. [17, §31.B]). It is shown in [4, p. 664] that the class of $R_{\mathbf{Spin}_{2n}(D,\sigma)}(\xi)$ in the quotient

$$\frac{H^3(L,\mathbb{Q}/\mathbb{Z}(2))}{H^1(L,\mu_2)\cup(D)}$$

is well-defined. The *Rost invariant* $\mathscr{R}(h)$ of the form h is defined as

$$\mathscr{R}(h):=[R_{\mathbf{Spin}_{2n}(D,\sigma)}(\xi)]\in\frac{H^3(L,\mathbb{Q}/\mathbb{Z}(2))}{H^1(L,\mu_2)\cup(D)}.$$

2.13. Let (D,σ) be a quaternion algebra with an orthogonal involution over L. We will need some further analysis on the map $\widetilde{\rho}:W(\widetilde{D},\tau_1)\to W(D,\sigma)$ in the exact sequence (2.7.2). Note that in this case $\widetilde{D}=M$ is a quadratic field extension of L and τ_1 is the nontrivial element ι of the Galois group $\mathrm{Gal}(M/L)$. Let $\mathbf{U}_{2n}(M,\iota)$ and $\mathbf{SU}_{2n}(M,\iota)$ denote the unitary group and the special unitary group of the hyperbolic form over (M,ι) defined by the matrix $H_{2n}=\left(\begin{smallmatrix}0 & I_n\\ I_n & 0\end{smallmatrix}\right)$. We have

$$\mathbf{U}_{2n}(M,\iota)(L)=\{A\in \mathrm{M}_{2n}(M)\mid A.H_{2n}\iota(A)^t=H_{2n}\}.$$

Note that for $A\in\mathrm{M}_{2n}(M)$, $\iota(A)=\mathrm{Int}(\mu)\circ\sigma(A)=\mu A\mu^{-1}$ (cf. (2.7)) and

$$\begin{aligned}A.H_{2n}.\iota(A)^t=H_{2n}&\iff(A.H_{2n}.\iota(A)^t)^t\\ &=(H_{2n})^t\iff\iota(A).H_{2n}.A^t=H_{2n}.\end{aligned}$$

Therefore, for $A\in\mathbf{U}_{2n}(M,\iota)(L)$, we have

$$\begin{aligned}A.\mu^{-1}\lambda H_{2n}.\sigma(A)^t&=A.\mu^{-1}\lambda H_{2n}.A^t\\ &=\mu^{-1}(\mu A\mu^{-1})\lambda H_{2n}.A^t\\ &=\mu^{-1}\lambda\iota(A).H_{2n}A^t=\mu^{-1}\lambda H_{2n}\end{aligned}$$

inside $\mathrm{M}_{2n}(D)$. So we have a natural inclusion

$$\begin{aligned}\mathbf{U}_{2n}(M,\iota)(L)&\subseteq\mathbf{U}(\mu^{-1}\lambda H_{2n})(L)\\ &=\{B\in\mathrm{M}_{2n}(D)\mid B.\mu^{-1}\lambda H_{2n}.\sigma(B)^t=\mu^{-1}\lambda H_{2n}\}.\end{aligned}$$

In fact, this defines an inclusion of algebraic groups over L:

$$\rho':\mathbf{U}_{2n}(M,\iota)\longrightarrow\mathbf{U}(\mu^{-1}\lambda H_{2n});\quad A\longmapsto A.$$

By [17, p. 402, Example 29.19], any element ξ of $H^1(L,\mathbf{U}_{2n}(M,\iota))$ is represented by a matrix $S\in GL_{2n}(M)$ which is symmetric with respect to the adjoint involution $\iota_{H_{2n}}$ on $\mathrm{M}_{2n}(M)$, and ξ is the isomorphism class of the hermitian form $H_{2n}S^{-1}$. The natural map

$$H^1(L, \mathbf{U}_{2n}(M, \iota)) \longrightarrow H^1(L, \mathbf{U}(\mu^{-1}\lambda H_{2n}))$$

induced by the homomorphism ρ' maps ξ to the class of the hermitian form $\mu^{-1}\lambda H_{2n}S^{-1}$. On the other hand, by the construction of the homomorphism $\widetilde{\rho} : W(M, \iota) \to W(D, \sigma)$, the form $H_{2n}S'$ over (M, ι) is mapped to the form $\mu^{-1}\lambda H_{2n}S^{-1}$ over (D, σ). Hence the natural map

$$H^1(L, \mathbf{U}_{2n}(M, \iota)) \longrightarrow H^1(L, \mathbf{U}(\mu^{-1}\lambda H_{2n}))$$

is compatible with the restriction of ρ to forms of rank $2n$.

Clearly, the inclusion $\rho' : \mathbf{U}_{2n}(M, \iota) \to \mathbf{U}(\mu^{-1}\lambda H_{2n})$ induces an inclusion $\mathbf{SU}_{2n}(M, \iota) \to \mathbf{SU}(\mu^{-1}\lambda H_{2n})$ (cf. [4, p. 671]). A choice of isomorphism of hermitian forms $\mu^{-1}\lambda H_{2n} \cong H_{2n}$ over (D, σ) yields an injection

$$\mathbf{SU}_{2n}(M, \iota) \longrightarrow \mathbf{SU}(H_{2n}) = \mathbf{SU}_{2n}(D, \sigma).$$

This lifts to a homomorphism

$$\rho_0 : \mathbf{SU}_{2n}(M, \iota) \longrightarrow \mathbf{Spin}_{2n}(D, \sigma).$$

The composition

$$\mathbf{SU}_{2n}(M, \iota) \xrightarrow{\rho_0} \mathbf{Spin}_{2n}(D, \sigma) \longrightarrow \mathbf{U}_{2n}(D, \sigma)$$

induces a commutative diagram

$$\begin{array}{ccc} H^1(L, \mathbf{SU}_{2n}(M, \iota)) & \xrightarrow{\rho_0} & H^1(L, \mathbf{Spin}_{2n}(D, \sigma)) \\ & \searrow^{\rho'} \quad \swarrow & \\ & H^1(L, \mathbf{U}_{2n}(D, \sigma)) & \end{array}$$

such that the map $\widetilde{\rho} : W(M, \iota) \to W(D, \sigma)$ restricted to forms of rank $2n$ and of trivial discriminant is compatible with the map ρ' at the level of cohomology sets. Moreover, for any $\xi \in H^1(L, \mathbf{SU}_{2n}(M, \iota))$, one has by [4, Prop. 3.20]

$$R_{\mathbf{Spin}_{2n}(D,\sigma)}(\rho_0(\xi)) = R_{\mathbf{SU}_{2n}(M,\iota)}(\xi) \in H^3(L, \mathbb{Q}/\mathbb{Z}(2)),$$

i.e., $\rho_0(\xi) \in H^1(L, \mathbf{Spin}_{2n}(D, \sigma))$ has the same Rost invariant as ξ. If h is a hermitian form over (D, σ) representing the class $\rho'(\xi) \in H^1(L, \mathbf{U}_{2n}(D, \sigma))$, then the Rost invariant of the form h is

$$\mathscr{R}(h) = [R_{\mathbf{Spin}_{2n}(D,\sigma)}(\rho_0(\xi))] = [R_{\mathbf{SU}_{2n}(M,\iota)}(\xi)] \in \frac{H^3(L, \mathbb{Q}/\mathbb{Z}(2))}{H^1(L, \mu_2) \cup (D)}$$

by definition (cf. (2.12)).

2.14. We shall also use the notion of Rost invariant of hermitian forms over an algebra with unitary involution. The definition is as follows. Let E be a field of characteristic $\neq 2$, L/E a quadratic field extension and (D, τ) a central division algebra over L with a unitary L/E-involution. Let $\mathbf{U}_{2n}(D, \tau)$ and $\mathbf{SU}_{2n}(D, \tau)$ denote respectively the unitary group and the special unitary group of the hyperbolic form $\left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$ over (D, τ). For a hermitian form h of rank $2n$ and trivial discriminant over (D, τ), we may define its *Rost invariant* $\mathscr{R}(h)$ by

$$\mathscr{R}(h) := [R_{\mathbf{SU}_{2n}(D,\tau)}(\xi)] \in \frac{H^3(E, \mathbb{Q}/\mathbb{Z}(2))}{\mathrm{Cores}_{L/E}((L^{*1}) \cup (D))},$$

where $\xi \in H^1(E, \mathbf{SU}_{2n}(D, \tau))$ is any lifting of the class $[h] \in H^1(E, \mathbf{U}_{2n}(D, \tau))$ and

$$L^{*1} = (R^1_{L/E}\mathbb{G}_m)(E) = \{a \in L^* \mid N_{L/E}(a) = 1\}.$$

Indeed, by [23, Appendix, Remark B], the class $[R_{\mathbf{SU}_{2n}(D,\tau)}(\xi)]$ is independent of the choice of the lifting ξ, so that this Rost invariant $\mathscr{R}(h)$ is well defined. Note that if $D = D_0 \otimes_E L$ for some central division algebra D_0 over E, then

$$\mathrm{Cores}_{L/E}((L^{*1}) \cup (D)) = 0$$

and hence the Rost invariant of h is simply the usual Rost invariant of any lifting $\xi \in H^1(E, \mathbf{SU}_{2n}(D, \tau))$ of the isomorphism class of h.

2.3 Spinor norms

2.15. Let E be a field of characteristic different from 2, A a central simple algebra over E and σ an orthogonal involution on A. Let h be a nonsingular hermitian form over (A, σ). The exact sequence of algebraic groups

$$1 \longrightarrow \mu_2 \longrightarrow \mathbf{Spin}(h) \longrightarrow \mathbf{SU}(h) \longrightarrow 1,$$

induces a connecting map

$$\delta : \mathbf{SU}(h)(E) \longrightarrow H^1(E, \mu_2) = E^*/E^{*2}$$

which we call the *spinor norm* map. We will write

$$\mathrm{Sn}(h_E) := \mathrm{Im}(\delta : \mathbf{SU}(h)(E) \longrightarrow E^*/E^{*2})$$

for the image of the above spinor norm map. If $A = E$, $\sigma = \mathrm{id}$ and $h = q$ is a quadratic form, the spinor norm map $\delta : \mathbf{SO}(q)(E) \to E^*/E^{*2}$ has an explicit description as follows (cf. [18, p. 108]): Any element $\theta \in \mathbf{SO}(q)(E)$ can be written as the product of an even number of hyperplan reflections associated with anisotropic vectors $v_1, \ldots, v_{2r}$. The spinor norm $\delta(\theta)$ is equal to the class of the product $q(v_1)\ldots q(v_{2r})$ in E^*/E^{*2}.

A deep theorem of Merkurjev is the following norm principle for spinor norms.

Theorem 2.16 (Merkurjev, [20, 6.2]). *With notation as in* (2.15), *assume that* $\deg(A).\mathrm{rank}(h)$ *is even and at least* 4.

Then the image $\mathrm{Sn}(h_E)$ *of the spinor norm map is equal to the subgroup of* E^*/E^{*2} *generated by the canonical images of the norm groups* $N_{L/E}(L^*)$ *over all finite field extensions* L/E *such that* A_L *is split and* h_L *is isotropic.*

The following corollary is immediate from the above theorem.

Corollary 2.17. *With notation and hypotheses as in Theorem* 2.16, *for any finite field extension* E'/E, *one has*

$$N_{E'/E}(\mathrm{Sn}(h_{E'})) \subseteq \mathrm{Sn}(h_E).$$

2.18. With notation and hypotheses as in Theorem 2.16, the well-known norm principle for reduced norms states that the subgroup $\mathrm{Nrd}(A^*) \subseteq E^*$ of reduced norms is generated by the norm groups $N_{L/E}(L^*)$, where L/E runs over all finite field extensions such that A_L is split. So Theorem 2.16 implies that $\mathrm{Sn}(h_E)$ is contained in the canonical image of $\mathrm{Nrd}(A^*)$ in E^*/E^{*2}.

2.19. Let (A, σ) be a central simple algebra with an orthogonal involution over a field E of characteristic $\neq 2$. Let L/E be a field extension which splits A and let $\phi : (A, \sigma) \otimes_E L \cong (\mathrm{M}_n(L), \sigma_{q_0})$ be an isomorphism of L-algebras with involution, where σ_{q_0} is the adjoint involution of a quadratic form q_0 of rank $n = \deg(A)$ over L. Let h be a hermitian form over $(A, \sigma) \otimes_E L$. Then by Morita theory (cf. (2.6)), h corresponds via the above isomorphism ϕ to a quadratic form q of rank $n.\mathrm{rank}(h) = \deg(A).\mathrm{rank}(h)$ over L. The similarity class $[q] \in W(L)$ of q is uniquely determined by h and is independent of the choice of ϕ and q_0. The hermitian form h_L is isotropic if and only if the quadratic form q_L is isotropic. So, if $\deg(A).\mathrm{rank}(h)$ is even and at least 4, one has $\mathrm{Sn}(q_L) = \mathrm{Sn}(h_L)$ by Theorem 2.16.

3. Some easy cases

We shall now start the proofs of our main theorems. In a few cases, as may be already well-known to specialists, the results basically follow by combining a general injectivity result for the Rost invariant and a Hasse principle coming from higher dimensional class field theory.

3.1. Recall that our base field K is the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field (cf. (1.2)). Namely, K is either (the case of p-adic arithmetic surface) the function field $F(C)$ of a smooth projective geometrically integral curve C over F, where F is a p-adic field with ring of integers A and residue field k;

or

(the case of local henselian surface) the field of fractions $\mathrm{Frac}(A)$ of a 2-dimensional, henselian, excellent local domain A with finite residue field k of characteristic p.

In either case, by abuse of language we say k is the residue field of K and $p = \mathrm{char}(k)$ is the *residue characteristic* of K.

In our proofs of the main theorems, we only use local conditions at *divisorial valuations*, i.e., valuations corresponding to codimension 1 points of regular proper models (cf. (1.2)). More precisely, the set Ω_A of divisorial valuations of the field K is the subset of Ω_K defined as follows:

In p-adic arithmetic case, define

$$\Omega_A = \bigcup_{\mathcal{X} \to \mathrm{Spec}\, A} \mathcal{X}^{(1)},$$

where $\mathcal{X} \to \mathrm{Spec}\, A$ runs over proper flat morphisms from a regular integral scheme $\mathcal{X}$ with function field K and $\mathcal{X}^{(1)}$ denotes the set of codimension 1 points of $\mathcal{X}$ identified with a subset of Ω_K.

In the local henselian case, define

$$\Omega_A = \bigcup_{\mathcal{X} \to \mathrm{Spec}\, A} \mathcal{X}^{(1)},$$

where $\mathcal{X} \to \mathrm{Spec}\, A$ runs over proper birational morphisms from a regular integral scheme $\mathcal{X}$ with function field K and $\mathcal{X}^{(1)}$ denotes the set of codimension 1 points of $\mathcal{X}$ identified with a subset of Ω_K.

3.2. Let L/K be a finite field extension. Then L is a field of the same type as K if K is the function field of a p-adic arithemetic surface or a local henselian surface with finite residue field. In the p-adic arithmetic case, let F' be the field of constants of L and let A' be the integral closure of A in F'. In the local henselian case, let A' be the integral closure of A in L. Then the set $\Omega_{A'}$ of divisorial discrete valuations of L is precisely the set of discrete valuations $w \in \Omega_L$ lying over valuations in $\Omega_A \subseteq \Omega_K$.

3.3. By the general theory of semisimple groups (see e.g. [17, p. 365, Thm. 26.8]), any semisimple simply connected group G over K is a finite product of groups of the form $R_{L/K}(G')$, where L/K is a finite separable field extension, G' is an absolutely simple simply connected group over L and $R_{L/K}$ denotes the Weil restriction functor. For each $v \in \Omega_A$, one has

$L \otimes_K K_v \cong \prod_{w|v} L_w$ and by Shapiro's lemma,

$$H^1(K, R_{L/K}G') \cong H^1(L, G') \quad \text{and} \quad H^1(K_v, R_{L/K}G') \cong \prod_{w|v} H^1(L_w, G').$$

Therefore, to prove the Hasse principle for semisimpe simply connected groups we may reduce to the case where G is an absolutely simple simply connected group.

3.1 The quasi-split case

We recall the proof of the Hasse principle for quasi-split groups without E_8 factors (cf. [6, Thm. 5.4]).

The following theorem is of particular importance to us.

Theorem 3.4. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Let Ω_A be the set of divisorial discrete valuations of K (as defined in (3.1)).*

(i) *(Kato, [15]) In the p-adic arithmetic case, the natural map*

$$H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \prod_{v \in \Omega_A} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$$

is injective.

(ii) *(Saito, [27], cf. [13, Prop. 4.1]) In the local henselian case, let $n > 0$ be an integer prime to p. Then the natural map*

$$H^3(K, \mu_n^{\otimes 2}) \longrightarrow \prod_{v \in \Omega_A} H^3(K_v, \mu_n^{\otimes 2})$$

is injective.

The next result is an injectivity statement for the Rost invariant of quasi-split groups.

Theorem 3.5 (cf. [6, Thm. 5.3]). †*Let E be a field of cohomological 2-dimension ≤ 3 and let G be an absolutely simple simply connected quasi-split group over E. Assume that G is not of type E_8. Assume further the characteristic of E is not 2 if G is of classical type B_n or D_n.*

Then the kernel of the Rost invariant map $R_G : H^1(E, G) \to H^3(E, \mathbb{Q}/\mathbb{Z}(2))$ is trivial.

† See [26] for a recent improvement of this theorem.

Proof. For a quasi-split group of type ${}^1A_n$ or C_n, it is well-known that $H^1(E, G) = 1$ over an arbitrary field E. For exceptional groups (not of type E_8), the kernel of the Rost invariant is trivial over an arbitrary field by the work of Chernousov, Garibaldi and Gille (cf. [9, Thm. 5.2], [5,7] and [8]). If G is of type ${}^2A_n$, B_n or classical type D_n, the proof can be done as in [6, Thm. 5.3], by passing to a quadratic form argument. □

The p-adic case of the following result is [6, Thm. 5.4].

Theorem 3.6. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Let G be an absolutely simple simply connected quasi-split group not of type E_8 over K. Assume $p \notin S(G)$ in the local henselian case (see (1.5) for the definition of $S(G)$).*

Then the natural map

$$H^1(K, G) \longrightarrow \prod_{v \in \Omega_A} H^1(K_v, G)$$

has a trivial kernel.

Proof. The result follows from the following commutative diagram

$$\begin{array}{ccc} H^1(K, G) & \longrightarrow & \prod_{v \in \Omega_A} H^1(K_v, G) \\ \downarrow & & \downarrow \\ H^3(K, \mu_n^{\otimes 2}) & \longrightarrow & \prod_{v \in \Omega_A} H^3(K_v, \mu_n^{\otimes 2}) \end{array}$$

where the vertical maps have trivial kernel by Theorem 3.5 and the bottom horizontal map is injective by Theorem 3.4. □

3.2 Groups of type ${}^1A_n^*$

For groups of inner type A_n^*, the proof is essentially the same as the quasi-split case.

Theorem 3.7. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Let A a central simple K-algebra of square-free index n and $G = \mathbf{SL}_1(A)$. Assume $p \nmid n$ in the local henselian case.*

Then the natural map

$$H^1(K, G) \longrightarrow \prod_{v \in \Omega_A} H^1(K_v, G)$$

is injective.

Proof. A well-known theorem of Suslin ([31, Thm. 24.4]) implies that under the assumptions of the theorem, the Rost invariant map

$$H^1(E, \mathbf{SL}_1(A)) = E^*/\mathrm{Nrd}(A^*) \longrightarrow H^3(E, \mu_n^{\otimes 2}); \quad \lambda \longmapsto (\lambda) \cup (A)$$

is injective for $E = K$ or K_v. An argument similar to the proof of Theorem 3.6 yields the result. □

3.3 Groups of type C_n^*

Lemma 3.8. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Assume $p \neq 2$ in the local henselian case.*

Then the natural map

$$I^3(K) \longrightarrow \prod_{v \in \Omega_A} I^3(K_v)$$

is injective.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} I^3(K) & \longrightarrow & \prod_{v \in \Omega_A} I^3(K_v) \\ {\scriptstyle e_3}\downarrow & & \downarrow \\ H^3(K, \mathbb{Z}/2) & \longrightarrow & \prod_{v \in \Omega_A} H^3(K_v, \mathbb{Z}/2) \end{array}$$

where the vertical maps are induced by the Arason invariants. Since $\mathrm{cd}_2(K) \leq 3$, we have $I^4(K) = 0$. So the map

$$e_3 : I^3(K) \longrightarrow H^3(K, \mathbb{Z}/2)$$

is injective by [1, Prop. 3.1]. The map

$$H^3(K, \mathbb{Z}/2) \longrightarrow \prod_{v \in \Omega_A} H^3(K_v, \mathbb{Z}/2)$$

is injective by Theorem 3.4. The lemma then follows from the above commutative diagram. □

Theorem 3.9. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Let D be a quaternion division algebra over K with standard involution τ_0 and h a nonsingular hermitian form over (D, τ_0). Assume $p \neq 2$ in the local henselian case. Let $G = \mathbf{U}(h)$ be the unitary group of the hermitian form h.*

Then the natural map

$$H^1(K,G) \longrightarrow \prod_{v\in\Omega_A} H^1(K_v,G)$$

is injective.

Proof. The pointed set $H^1(K,G) = H^1(K,\mathrm{U}(h))$ classifies up to isomorphism hermitian forms over (D,τ_0) of the same rank as h. Let h_1 and h_2 be hermitian forms over (D,τ_0) of the same rank as h. Put $h' = h_1 \perp (-h_2)$. Note that h' has even rank, so the class of $q_{h'}$ in the Witt group $W(K)$ lies in the subgroup $I^3(K) = I(K)\cdot I^2(K)$ (cf. (2.3)). Thus

$$[q_{h_1}] - [q_{h_2}] = [q_{h'}] \in I^3(K).$$

If $(h_1)_v \cong (h_2)_v$ for all $v \in \Omega_A$, then by Lemma 3.8, $[q_{h'}] = 0 \in I^3(K)$. This implies that $q_{h_1} \cong q_{h_2}$ over K. Two hermitian forms over (D,τ_0) are isomorphic if and only if their trace forms are isomorphic as quadratic forms (cf. (2.3)). So we get from the above that $h_1 \cong h_2$, proving the theorem. □

3.4 Groups of type G_2 or F_4^{red}

Theorem 3.10. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Let G be an absolutely simple simply connected group of type G_2 over K. Assume $p \neq 2$ in the local henselian case.*

Then the natural map

$$H^1(K,G) \longrightarrow \prod_{v\in\Omega_A} H^1(K_v,G)$$

has a trivial kernel.

Proof. The group G is isomorphic to $\mathbf{Aut}_{alg}(C)$ for some Cayley algebra C over K. Let $\xi \in H^1(K,G)$ be a locally trivial class and let C' be a Cayley algebra which represents ξ. We have $C_{K_v} \cong C'_{K_v}$ for every $v \in \Omega_A$ by hypothesis and we want to show $C \cong C'$ over K. Since two Cayley algebras are isomorphic if and only if their norm forms are isomorphic and since the norm form of a Cayley algebra is a 3-fold Pfister form (cf. [17, p. 460]), the result follows easily from Lemma 3.8. □

Theorem 3.11. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Assume*

$p \nmid 6$ in the local henselian case. Let $G = \mathbf{Aut}_{alg}(J)$ be the automorphism group of a reduced *27-dimensional exceptional Jordan algebra over K.*

Then the natural map

$$H^1(K,G) \longrightarrow \prod_{v\in\Omega_A} H^1(K_v,G)$$

has a trivial kernel.

Proof. Recall that (cf. [30, §9]) to each exceptional Jordan algebra J' of dimension 27 over a field F of characteristic not 2 or 3, one can associate three invariants

$$f_3(J') \in H^3(F,\mathbb{Z}/2),\, f_5(J') \in H^5(F,\mathbb{Z}/2) \quad \text{and} \quad g_3(J') \in H^3(F,\mathbb{Z}/3).$$

One has $g_3(J') = 0$ if and only if J' is reduced. Two reduced exceptional Jordan algebras are isomorphic if and only if their f_3 and f_5 invariants are the same.

Now our base field K has cohomological 2-dimension $\mathrm{cd}_2(K) = 3$. So the invariant $f_5(J')$ is always zero. Let $\xi \in H^1(K,G)$ correspond to the isomorphism class of an exceptional Jordan algebra J' over K. Assume that ξ is locally trivial in $H^1(K_v,G)$ for every $v \in \Omega_A$. By Theorem 3.4, we have $f_3(J) = f_3(J')$ and $g_3(J) = g_3(J')$. Since J is reduced by assumption, we have $g_3(J') = 0$ and hence J' is reduced. Thus it follows that $J \cong J'$ over K, showing that ξ is trivial in $H^1(K,G)$ as desired. □

4. Spin groups of quadratic forms

4.1. Let E be a field of characteristic different from 2 and q a nonsingular quadratic form of rank ≥ 3 over E. Recall that $\mathrm{Sn}(q_E)$ denotes the image of the spinor norm map

$$\mathbf{SO}(q)(E) \longrightarrow E^*/E^{*2},$$

i.e., the connecting map associated to the cohomology of the exact sequence

$$1 \longrightarrow \mu_2 \longrightarrow \mathbf{Spin}(q) \longrightarrow \mathbf{SO}(q) \longrightarrow 1.$$

Proposition 4.2. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Assume $p \neq 2$ in the local henselian case. Let q be a nonsingular quadratic form of rank 3 or 4 over K.*

Then the natural map

$$\frac{K^*/K^{*2}}{\mathrm{Sn}(q_K)} \longrightarrow \prod_{v\in\Omega_A} \frac{K_v^*/K_v^{*2}}{\mathrm{Sn}(q_{K_v})}$$

is injective.

Proof. If $\mathrm{rank}(q) = 3$, we may assume $q = \langle 1, a, b\rangle$ after scaling. Let D be the quaternion algebra $(-a, -b)_K$ over K. Then $\mathrm{Sn}(q) = \mathrm{Nrd}(D^*)$ modulo squares. The result then follows from Theorem 3.7.

Assume next $\mathrm{rank}(q) = 4$. If $\mathrm{disc}(q) = 1$, we may assume after scaling $q = \langle 1, a, b, ab\rangle$. Put $D = (-a, -b)_K$. Then $\mathrm{Sn}(q) = \mathrm{Nrd}(D^*)$ and the result follows again from Theorem 3.7. If $d = \mathrm{disc}(q)$ is nontrivial in K^*/K^{*2}, we may assume $q = \langle 1, a, b, abd\rangle$. Then

$$\mathrm{Sn}(q_K) = \mathrm{Nrd}(D^*_{K(\sqrt{d})}) \cap K^* \quad \text{modulo squares}$$

by [17, p. 214, Coro. 15.11]. The field $K(\sqrt{d})$ is a field of the same type as K (cf. (3.2)). Let $\Omega_{A'}$ denote the set of divisorial valuations of $K' = K(\sqrt{d})$. If $\alpha \in K^*$ lies in $\mathrm{Sn}(q_{K_v})$ for all $v \in \Omega_A$, then α is a reduced norm from $D_{K'_w}$ for all $w \in \Omega_{A'}$. By Theorem 3.7, α is a reduced norm from $D_{K'} = D_{K(\sqrt{d})}$. This finishes the proof. □

Recall that the u-invariant $u(E)$ of a field E of characteristic $\neq 2$ is the supremum of dimensions of anisotropic quadratic forms over E (so $u(E) = \infty$, if such dimensions can be arbitrarily large).

Proposition 4.3. *Let E be a field of characteristic $\neq 2$ and q a nonsingular quadratic form of rank r over E. Assume $u(E) < 2r$.*

Then $\mathrm{Sn}(q_E) = E^*/E^{*2}$, i.e., the spinor norm map

$$\mathbf{SO}(q)(E) \longrightarrow E^*/E^{*2}$$

is surjective.

Proof. The image $\mathrm{Sn}(q_E)$ of the spinor norm map consists of elements of the form $\prod_i^{2m} q(v_i)$, where v_i are anisotropic vectors for q (cf. (2.15)). If q is isotropic over E, then for every $\alpha \in E^*$, there is a vector v_α such that $q(v_\alpha) = \alpha$. Let v_1 be a vector such that $q(v_1) = 1$. Then we have $\alpha = q(v_\alpha) \cdot q(v_1) \in \mathrm{Sn}(q_E)$.

Assume next q is anisotropic. For any $\alpha \in E^*$, the form $q \perp (-\alpha \cdot q)$ is isotropic over E by the assumption on the u-invariant. Hence there are vectors x, y such that $q(x) - \alpha \cdot q(y) = 0$. Since q is anisotropic, we have $\lambda := q(y) \in E^*$ and $q(x) \in E^*$. It follows that

$$\alpha = q(x) \cdot q(y)^{-1} = \lambda^{-2} q(x) \cdot q(y) = q(x) \cdot q(\lambda^{-1} y) \in \mathrm{Sn}(q_E)$$

whence the desired result. □

Corollary 4.4. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Assume*

$p \neq 2$ in the local henselian case. Let q be a nonsingular quadratic form of rank ≥ 5 over K.

Then $\mathrm{Sn}(q_K) = K^*/K^{*2}$, i.e., the spinor norm map

$$\mathbf{SO}(q)(K) \longrightarrow K^*/K^{*2}$$

is surjective.

Proof. In the p-adic arithmetic case, we have $u(K) = 8$ by [25] (if $p \neq 2$) or [19] (see also [11]). In the local henselian case, it is proved in [13, Thm. 1.2] that $u(K) = 8$. The result then follows immediately from Proposition 4.3. □

Theorem 4.5. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Assume $p \neq 2$ in the local henselian case. Let q be a nonsingular quadratic form of rank ≥ 3 over K and $G = \mathbf{Spin}(q)$.*

(i) *The natural map*

$$H^1(K, G) \longrightarrow \prod_{v \in \Omega_A} H^1(K_v, G)$$

has a trivial kernel.

(ii) *The Rost invariant*

$$R_G : H^1(K, G) \longrightarrow H^3(K, \mathbb{Q}/\mathbb{Z}(2))$$

has a trivial kernel if $\mathrm{rank}(q) \geq 5$.

Proof. Consider the exact sequence of algebraic groups

$$1 \longrightarrow \mu_2 \longrightarrow \mathbf{Spin}(q) = G \longrightarrow \mathbf{SO}(q) \longrightarrow 1$$

which gives rise to an exact sequence of pointed sets

$$\mathbf{SO}(q)(K) \xrightarrow{\delta} K^*/K^{*2} \xrightarrow{\psi} H^1(K, \mathbf{Spin}(q)) \xrightarrow{\eta} H^1(K, \mathbf{SO}(q)). \tag{4.5.1}$$

The image of the map η is in bijection with isomorphism classes of nonsingular quadratic forms q' with the same rank, discriminant and Clifford invariant as q.

Let $\xi \in H^1(K, G) = H^1(K, \mathbf{Spin}(q))$ with $\eta(\xi) \in H^1(K, \mathbf{SO}(q))$ corresponding to a quadratic form q'. Then in the Witt group $W(K)$ the class of $q \perp (-q')$ lies in $I^3(K)$ by Merkurjev's theorem (cf. [29, p. 89, Thm. 2.14.3]) and its Arason invariant $e_3([q \perp (-q')]) \in H^3(K, \mathbb{Z}/2)$ coincides with Rost invariant $R_G(\xi)$ of ξ when $\mathrm{rank}(q) \geq 5$ ([17, p. 437]).

For (i), assume the canonical image ξ_v of ξ in $H^1(K_v, G)$ is trivial for every $v \in \Omega_A$. We have

$$[q\perp(-q')]_v = 0 \in I^3(K_v), \quad \forall\ v \in \Omega_A.$$

By Lemma 3.8, we have $q \cong q'$ over K. This means that $\xi \in H^1(K, G)$ lies in the kernel of

$$\eta : H^1(K, G) \longrightarrow H^1(K, \mathbf{SO}(q)).$$

By the exactness of the sequence (4.5.1), $\xi = \psi(\alpha)$ for some $\alpha \in \mathrm{Coker}(\delta) = \frac{K^*/K^{*2}}{\mathrm{Sn}(q_K)}$. Consider now the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 1 \longrightarrow & \frac{K^*/K^{*2}}{\mathrm{Sn}(q_K)} & \xrightarrow{\psi} & H^1(K, G) & \xrightarrow{\eta} & H^1(K, \mathbf{SO}(q)) \\ & \downarrow & & \downarrow & & \downarrow \\ 1 \longrightarrow & \prod_{v \in \Omega_A} \frac{K_v^*/K_v^{*2}}{\mathrm{Sn}(q_{K_v})} & \xrightarrow{\psi} & \prod_{v \in \Omega_A} H^1(K_v, G) & \xrightarrow{\eta} & \prod_{v \in \Omega_A} H^1(K_v, \mathbf{SO}(q)) \end{array}$$

The canonical image α_v of α in $\frac{K_v^*/K_v^{*2}}{\mathrm{Sn}(q_{K_v})}$ is trivial for all $v \in \Omega_A$. From Proposition 4.2 and Corollary 4.4, it follows that $\alpha = 1$ and hence $\xi = \psi(\alpha)$ is trivial.

For (ii), assume the Rost invariant $R_G(\xi)$ of ξ is trivial. Then the Arason invariant $e_3([q\perp(-q')])$ is zero. Since $\mathrm{cd}_2(K) \le 3$, the map $e_3 : I^3(K) \to H^3(K, \mathbb{Z}/2)$ is injective. So we get $q \cong q'$ over K and therefore $\xi = \psi(\alpha)$ for some $\alpha \in \frac{K^*/K^{*2}}{\mathrm{Sn}(q_K)}$. When the rank of q is ≥ 5, we have $K^*/K^{*2} = \mathrm{Sn}(q_K)$ by Corollary 4.4. So $\alpha = 1$ and ξ is trivial. □

Remark 4.6. Assertion (ii) of Theorem 4.5 may be compared with the following result, which was already known to experts (cf. [6, Prop. 5.2]): Let E be a field of characteristic $\neq 2$ and of cohomological 2-dimension $\mathrm{cd}_2(E) \le 3$. Let q be an *isotropic* quadratic form of rank ≥ 5 over E. Then the Rost invariant

$$H^1(E, \mathbf{Spin}(q)) \longrightarrow H^3(E, \mathbb{Q}/\mathbb{Z}(2))$$

for the spinor group $\mathbf{Spin}(q)$ has a trivial kernel.

5. Groups of type D_n^*

Proposition 5.1. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Assume $p \neq 2$ in the local henselian case. Let (D, σ) be a quaternion division algebra with an orthogonal involution over K and let h be a hermitian form of rank ≥ 2 over (D, σ).*

Then the natural map

$$\frac{K^*/K^{*2}}{\mathrm{Sn}(h_K)} \longrightarrow \prod_{v\in\Omega_A} \frac{K_v^*/K_v^{*2}}{\mathrm{Sn}(h_{K_v})}$$

is injective.

Proof. First assume $\mathrm{rank}(h) = 2$. Put $d = \mathrm{disc}(h) \in K^*/K^{*2}$. If $d = 1 \in K^*/K^{*2}$, then h is isotropic and $\mathrm{Sn}(h) = \mathrm{Nrd}(D^*)$ modulo squares by Merkurjev's norm principle (Theorem 2.16). The result then follows from Theorem 3.7. Let us assume $d = \mathrm{disc}(h) \in K^*/K^{*2}$ is nontrivial. Let $(A, \tilde{\sigma}) = (\mathrm{M}_2(D), \sigma_h)$, where σ_h denotes the adjoint involution of h on $A = \mathrm{M}_2(D)$. The even Clifford algebra $C = C_0(A, \tilde{\sigma})$ of the pair $(A, \tilde{\sigma})$ (cf. [17, §8]) is a quaternion algebra over the field $K(\sqrt{d})$ and one has

$$\mathrm{Sn}(h_K) = \mathrm{Nrd}(C^*) \cap K^* \pmod{K^{*2}}.$$

(cf. [17, p. 94, Thm. 8.10 and p. 214, Coro. 15.11].) As in the proof of Proposition 4.2, it follows from Theorem 3.7 that an element $\lambda \in K^*/K^{*2}$ is a spinor norm for h_K if and only if it is a spinor norm for h_{K_v} for all $v \in \Omega_A$.

Assume next $\mathrm{rank}(h) \geq 3$. Let $\lambda \in K^*$ and assume λ is a local spinor norm for h_{K_v} for every $v \in \Omega_A$. Merkurjev's norm principle (Theorem 2.16) implies that $\lambda \in \mathrm{Nrd}(D_{K_v}^*)$ for every $v \in \Omega_A$. Hence $\lambda \in \mathrm{Nrd}(D^*)$ by Theorem 3.7. (Note that $K^{*2} \subseteq \mathrm{Nrd}(D^*)$ since D is a quaternion algebra.) Let K'/K be a field extension such that $D_{K'}$ is split and $\lambda = N_{K'/K}(\mu)$ for some $\mu \in (K')^*$. By Corollary 2.17, $N_{K'/K}(\mathrm{Sn}(h_{K'})) \subseteq \mathrm{Sn}(h_K)$. Since $\lambda \in K^*/K^{*2}$ lies in the image of $N_{K'/K} : (K')^*/(K')^{*2} \to K^*/K^{*2}$, to show λ is a spinor norm for h_K it suffices to show that the map

$$\delta' : \mathbf{SU}(h)(K') \longrightarrow (K')^*/(K')^{*2}$$

is surjective. Note that D splits over K' by the choice of K'. So we see from (2.19) that $\mathrm{Im}(\delta') = \mathrm{Sn}(h_{K'}) = \mathrm{Sn}(q_{K'})$, where $q_{K'}$ is a quadratic form of rank $2 \cdot \mathrm{rank}(h) \geq 6$ over K'. Now the result follows immediately from Corollary 4.4. □

Proposition 5.2. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Assume $p \neq 2$ in the local henselian case. Let (D, σ) be a quaternion division algebra with an orthogonal involution over K. Let h be a nonsingular hermitian form of even rank ≥ 2 over (D, σ). Assume that h has trivial discriminant, trivial Clifford invariant and trivial Rost invariant (cf. §2.2).*

If the form h_{K_v} over $(D_{K_v}, \sigma) = (D \otimes_K K_v, \sigma)$ is hyperbolic for every $v \in \Omega_A$, then the form h over (D, σ) is hyperbolic.

Proof. Let $L \subseteq D$ be a subfield which is a quadratic extension over K such that $\sigma(L) = L$ and $\sigma|_L = \mathrm{id}_L$. Such an L exists since σ is an orthogonal involution. Let $\mu \in D^*$ be an element such that $\sigma(\mu) = -\mu$, $\mathrm{Int}(\mu)(L) = L$ and $\mathrm{Int}(\mu)|_L = \iota$, where ι denotes the nontrivial element of the Galois group $\mathrm{Gal}(L/K)$. The involution $\tau := \mathrm{Int}(\mu) \circ \sigma$ is a symplectic involution on D (and hence coincides with the canonical involution on the quaternion algebra D). The "key exact sequence" of Parimala-Sridharan-Suresh (cf. (2.7.2)) yields the following commutative diagram with exact rows

$$\begin{array}{ccccccc} W(D,\tau) & \xrightarrow{\pi_1} & W(L,\iota) & \xrightarrow{\widetilde{\rho}} & W(D,\sigma) & \xrightarrow{\widetilde{\pi}_2} & W(L) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \prod_{v\in\Omega_A} W(D_v,\tau) & \xrightarrow{\pi_1} & \prod_{v\in\Omega_A} W(L_v,\iota) & \xrightarrow{\widetilde{\rho}} & \prod_{v\in\Omega_A} W(D_v,\sigma) & \xrightarrow{\widetilde{\pi}_2} & \prod_{v\in\Omega_A} W(L_v) \end{array}$$

where for any K-algebra B we denote $B_v = B \otimes_K K_v$ for each $v \in \Omega_A$. (Here L_v need not be a field. It can be a Galois K_v-algebra of the form $L_{w_1} \times L_{w_2}$, where w_1, w_2 are discrete valuations of L lying over v. But this does not affect the construction of the key exact sequence for D_v. Indeed, the same choice of $\mu \in D^* \subseteq D_v^*$ satisfies the condition that $\mathrm{Int}(\mu)|_{L_v}$ is the nontrivial automorphism of the K_v-algebra L_v. It is not difficult to check that the key exact sequence for D_v is still well defined.)

The form $\widetilde{\pi}_2(h) \in W(L)$ has even rank, trivial discriminant and trivial Clifford invariant by [3, Prop. 3.2.2]. Hence $\widetilde{\pi}_2(h) \in I^3(L) \subseteq W(L)$. Let $\Omega_{A'}$ denote the set of divisorial valuations of L. Then for every $w \in \Omega_{A'}$ one has $\widetilde{\pi}_2(h) = 0$ in $W(L_w)$. By Lemma 3.8, $\widetilde{\pi}_2(h) = 0$ in $W(L)$. So by the exactness of the first row in the above diagram, there exists a hermitian form of even rank h_0 over (L,ι) such that $\widetilde{\rho}(h_0) = h \in W(D,\sigma)$.

Let $\alpha = \mathrm{disc}(h_0) \in K^*/N_{L/K}(L^*)$ be the discriminant of h_0. One has

$$\mathscr{C}\ell(\widetilde{\rho}(h_0)) = (L,\alpha) \in {}_2\mathrm{Br}(K)/(D)$$

by [3, Prop. 3.2.3]. Since $\mathscr{C}\ell(\widetilde{\rho}(h_0)) = \mathscr{C}\ell(h) = 0$ by assumption, one has either $(L,\alpha) = 0$ or $(L,\alpha) = (D)$ in $\mathrm{Br}(K)$. If $(L,\alpha) = 0 \in \mathrm{Br}(K)$ then α is a norm for the extension L/K so that $\mathrm{disc}(h_0) = 1 \in K^*/N_{L/K}(L^*)$. If $(L,\alpha) = D$, writing $L = K(\sqrt{a})$ such that $D = (a,\alpha)_K$, one has $\mathrm{disc}(\langle 1, -\alpha\rangle) = \alpha \in K^*/N_{L/K}(L^*)$. By the construction of the map π_1, one has $\pi_1(\langle 1\rangle) = \langle 1, -\alpha\rangle \in W(L,\iota)$ (since $D = L \oplus \mu L$ with $\mu^2 = \alpha$). Replacing h_0 by $h_0 - \pi_1(\langle 1\rangle)$, we may assume that $\mathrm{disc}(h_0) = 1 \in K^*/N_{L/K}(L^*)$. Let $2n = \mathrm{rank}(h_0)$ and let $\mathbf{SU}_{2n}(L,\iota)$ denote the special unitary group of the hyperbolic form $\left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$ over (L,ι). The form h_0, having trivial discriminant, now determines a class in $H^1(K, \mathbf{SU}_{2n}(L,\iota))$.

Let H_{2n} be the hyperbolic form $\left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$ over (D, σ) and let $\mathbf{U}_{2n}(D, \sigma)$, $\mathbf{SU}_{2n}(D, \sigma)$ and $\mathbf{Spin}_{2n}(D, \sigma)$ denote respectively the unitary group, the special unitary group and the spin group of the form H_{2n}. By (2.13), there is a homomorphism

$$\rho_0 : \mathbf{SU}_{2n}(L, \iota) \longrightarrow \mathbf{Spin}_{2n}(D, \sigma).$$

which induces a commutative diagram

$$\begin{array}{ccc} H^1(K, \mathbf{SU}_{2n}(L, \iota)) & \xrightarrow{\quad\rho_0\quad} & H^1(K, \mathbf{Spin}_{2n}(D, \sigma)) \\ & {\scriptstyle\rho'}\searrow \quad \swarrow & \\ & H^1(K, \mathbf{U}_{2n}(D, \sigma)) & \end{array}$$

such that the map $\widetilde{\rho} : W(L, \iota) \to W(D, \sigma)$ in the "key exact sequence" (2.7.2) restricted to forms of rank $2n$ and of trivial discriminant is compatible with the map ρ' at the level of cohomology sets.

By [4, Prop. 3.20], one has

$$R_{\mathbf{Spin}_{2n}(D,\sigma)}(\rho_0([h_0])) = R_{\mathbf{SU}_{2n}(L,\iota)}([h_0]) \in H^3(K, \mathbb{Q}/\mathbb{Z}(2)).$$

Thus by the definition of the Rost invariant $\mathscr{R}$ (cf. (2.12)),

$$\begin{aligned} 0 = \mathscr{R}(h) &= [R_{\mathbf{Spin}_{2n}(D,\sigma)}(\rho_0([h_0]))] \\ &= [R_{\mathbf{SU}_{2n}(L,\iota)}([h_0])] \in \frac{H^3(K, \mathbb{Q}/\mathbb{Z}(2))}{H^1(K, \mu_2) \cup (D)}. \end{aligned}$$

Therefore, there is an element $\beta \in K^*/K^{*2} = H^1(K, \mu_2)$ such that

$$R_{\mathbf{SU}_{2n}(L,\iota)}([h_0]) = (\beta) \cup (D) \in H^3(K, \mathbb{Q}/\mathbb{Z}(2)).$$

A direct computation shows that the element $\widetilde{h}_0 := \pi_1(\langle 1, -\beta \rangle) \in W(L, \iota)$ has associated trace form $q_{\widetilde{h}_0} = \langle 1, -\beta \rangle \otimes n_D$, where n_D denotes the norm form of the quaternion algebra D. By [17, p. 438, Example 31.44], the class of $\widetilde{h}_0$ has Rost invariant

$$R_{\mathbf{SU}_4(L,\iota)}([\widetilde{h}_0]) = e_3(q_{\widetilde{h}_0}) = (\beta) \cup (D) \in H^3(K, \mathbb{Q}/\mathbb{Z}(2)).$$

Modifying h_0 by $\widetilde{h}_0 = \pi_1(\langle 1, -\beta \rangle)$, we may further assume that the class $[h_0] \in H^1(K, \mathbf{SU}_{2n}(L, \iota))$ has trivial Rost invariant, i.e., $e_3(q_{h_0}) = 0$. Since $\mathrm{cd}_2(K) \le 3$, the Arason invariant $e_3 : I^3(K) \to H^3(K, \mathbb{Z}/2)$ is injective. Hence $[q_{h_0}] = 0 \in W(K)$ and $[h_0] = 0 \in W(L, \iota)$ by (2.4) (cf. [29, p. 348, Thm. 10.1.1]). It then follows immediately that $[h] = \widetilde{\rho}([h_0]) = 0 \in W(D, \sigma)$. □

Corollary 5.3. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Assume $p \neq 2$ in the local henselian case. Let (D, σ) be a quaternion division algebra with an orthogonal involution over K. Let h_1, h_2 be hermitian forms over (D, σ) with the same rank and discriminant such that*

$$\mathscr{C}\ell(h_1 \perp (-h_2)) = 0 \in {}_2\mathrm{Br}(K)/(D)$$

and

$$\mathscr{R}(h_1 \perp (-h_2)) = 0 \in H^3(K, \mathbb{Q}/\mathbb{Z}(2))/H^1(K, \mu_2) \cup (D).$$

Then $h_1 \cong h_2$ if and only if $(h_1)_{K_v} \cong (h_2)_{K_v}$ for every $v \in \Omega_A$.

Proof. Apply Proposition 5.2 to the form $h = h_1 \perp (-h_2)$ and use Witt's cancellation theorem. □

Theorem 5.4. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Assume $p \neq 2$ in the local henselian case. Let (D, σ) be a quaternion division algebra with an orthogonal involution over K, h a nonsingular hermitian form of rank ≥ 2 over (D, σ) and $G = \mathbf{Spin}(h)$.*

Then the natural map

$$H^1(K, G) \longrightarrow \prod_{v \in \Omega_A} H^1(K_v, G)$$

has a trivial kernel.

Proof. Let $\xi \in H^1(K, \mathbf{Spin}(h))$ be a class which is trivial in $H^1(K_v, \mathbf{Spin}(h))$ for all $v \in \Omega_A$. The image of ξ under the composite map

$$H^1(K, G) = H^1(K, \mathbf{Spin}(h)) \longrightarrow H^1(K, \mathbf{SU}(h)) \longrightarrow H^1(K, \mathbf{U}(h))$$

is the class of a hermitian form h' which has the same rank and discriminant as h such that

$$\mathscr{C}\ell(h \perp (-h')) = 0 \in {}_2\mathrm{Br}(K)/(D).$$

Let $n = \mathrm{rank}(h)$. Let $\mathbf{Spin}_{2n}(D, \sigma)$ and $\mathbf{U}_{2n}(D, \sigma)$ denote respectively the spin group and the unitary group of the hyperbolic form $\left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$ over (D, σ). Then the class $[h \perp (-h')] \in H^1(K, \mathbf{U}_{2n}(D, \sigma))$ lifts to an element $\xi' \in H^1(K, \mathbf{Spin}_{2n}(D, \sigma))$. By [23, Lemma 5.1], we have

$$[R_G(\xi)] = \mathscr{R}(h \perp (-h')) = [R_{\mathbf{Spin}_{2n}(D,\sigma)}(\xi')] \in \frac{H^3(K, \mathbb{Q}/\mathbb{Z}(2))}{H^1(K, \mu_2) \cup (D)}. \tag{5.4.1}$$

Since ξ is locally trivial, the commutative diagram

$$\begin{array}{ccc} H^1(K,G) & \xrightarrow{R_G} & H^3(K,\mathbb{Q}/\mathbb{Z}(2)) \\ \downarrow & & \downarrow \\ \prod_{v\in\Omega_A} H^1(K_v,G) & \xrightarrow{R_G} & \prod_{v\in\Omega_A} H^3(K_v,\mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

shows that the Rost invariant $R_G(\xi)$ is locally trivial. By Theorem 3.4, noticing that the Rost invariant R_G takes values in the subgroup $H^3(K,\mu_4^{\otimes 2})$, we get $R_G(\xi)=0\in H^3(K,\mathbb{Q}/\mathbb{Z}(2))$. Thus, by (5.4.1),

$$\mathscr{R}(h\perp(h'))=0\in\frac{H^3(K,\mathbb{Q}/\mathbb{Z}(2))}{H^1(K,\mu_2)\cup(D)}.$$

Now Corollary 5.3 implies that $h\cong h'$ and hence the image of $\xi\in H^1(K,G)$ in $H^1(K,\mathbf{U}(h))$ is trivial. By [4, Lemma 7.11], the canonical image of ξ in $H^1(K,\mathbf{SU}(h))$ is also trivial.

Now consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 1 & \longrightarrow & \frac{K^*/K^{*2}}{\mathrm{Sn}(h_K)} & \xrightarrow{\varphi} & H^1(K,G) & \longrightarrow & H^1(K,\mathbf{SU}(h)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \prod_{v\in\Omega_A}\frac{K_v^*/K_v^{*2}}{\mathrm{Sn}(h_{K_v})} & \longrightarrow & \prod_{v\in\Omega_A} H^1(K_v,G) & \longrightarrow & \prod_{v\in\Omega_A} H^1(K_v,\mathbf{SU}(h)) \end{array}$$

which is induced by the natural exact sequence of algebraic groups

$$1\longrightarrow\mu_2\longrightarrow G=\mathbf{Spin}(h)\longrightarrow\mathbf{SU}(h)\longrightarrow 1.$$

The exactness of the first row yields $\xi=\varphi(\theta)$ for some $\theta\in\frac{K^*/K^{*2}}{\mathrm{Sn}(h_K)}$. The commutative diagram then shows that θ is locally trivial since ξ is locally trivial. From Proposition 5.1 it follows that $\theta=1\in\frac{K^*/K^{*2}}{\mathrm{Sn}(h_K)}$ and hence $\xi=\varphi(\theta)$ is trivial in $H^1(K,G)$. This completes the proof. □

6. Groups of type ${}^2A_n^*$

6.1 Case of odd index

Proposition 6.1. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Assume $p\neq 2$ in the local henselian case. Let L/K be a quadratic field extension, (D,τ) a central division algebra of odd degree over L with an*

L/K-involution τ (i.e., a unitary involution τ such that $L^\tau = K$). Let h_1, h_2 be nonsingular hermitian forms over (D, τ) which have the same rank and discriminant.

If the forms $(h_1)_{K_v} \cong (h_2)_{K_v}$ over $(D_{K_v}, \tau) = (D \otimes_L L \otimes_K K_v, \tau)$ are isomorphic for all $v \in \Omega_A$, then the forms h_1, h_2 over (D, τ) are isomorphic.

Proof. Let M/K be a field extension of odd degree such that $D_M = D \otimes_L (L \otimes_K M)$ is split over the field $LM = L \otimes_K M$. (Such an extension M/K exists by [3, Lemma 3.3.1].) The base extension τ_M of τ is a unitary involution on the central simple (LM)-algebra D_M such that $(LM)^{\tau_M} = M$. Let ι denote the nontrivial element of the Galois group $\mathrm{Gal}(L/K)$ and regard $\iota_M \in \mathrm{Gal}(LM/M)$ as a unitary involution on LM. There is a nonsingular hermitian form (V, f) over (LM, ι_M) such that $(D_M, \tau_M) \cong (\mathrm{End}_{LM}(V), \iota_f)$, where ι_f denotes the adjoint involution on $\mathrm{End}_{LM}(V)$ with respect to f (cf. [17, p. 43, Thm. 4.2 (2)]). We have a Morita equivalence between the category of hermitian forms over (D_M, τ_M) and the category of hermitian forms over (LM, ι_M) (cf. (2.6)), which induces an isomorphism of Witt groups

$$\phi_f : W(D_M, \tau_M) \xrightarrow{\sim} W(LM, \iota_M).$$

Let $h = h_1 \perp (-h_2)$ and let h_M be its base extension over (D_M, τ_M). Via the Morita equivalence mentioned above, h_M corresponds to a hermitian form $\widetilde{h}_M$ over (LM, ι_M). Let $q_M := q_{\widetilde{h}_M}$ be the trace form of $\widetilde{h}_M$ (which is a quadratic form over the field M). Since h has even rank and trivial discriminant, the class $[q_M] \in W(M)$ of the quadratic form q_M lies in $I^3(M)$. The hypothesis on the local triviality (with respect to Ω_A) of $[h] = [h_1 \perp (-h_2)]$ implies that $[q_M] \in I^3(M)$ is locally trivial (with respect to the set of discrete valuations of M defined in the same way as Ω_A). By Lemma 3.8, we have $[q_M] = 0$ and hence $[\widetilde{h}_M] = 0$ in $W(LM, \iota_M)$. Since $W(D_M, \tau_M) \cong W(LM, M)$, $[h_M] = 0$ in $W(D_M, \tau_M)$. Since M/K is an odd degree extension, the natural map $W(D, \tau) \to W(D_M, \tau_M)$ is injective by a theorem of Bayer-Fluckiger and Lenstra (cf. [17, p. 80, Coro. 6.18]). So we get $[h] = 0$ in $W(D, \tau)$, thus proving the proposition. □

Lemma 6.2. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Let L/K be a separable quadratic field extension and (D, τ) a central division L-algebra of square-free index $\mathrm{ind}(D)$ with a unitary involution τ such that $L^\tau = K$. Assume $p \nmid \mathrm{ind}(D)$ in the local henselian case.*

Then for any nonsingular hermitian form h over (D, τ), the natural map

$$\frac{(R^1_{L/K}\mathbb{G}_m)(K)}{\mathrm{Nrd}(\mathbf{U}(h)(K))} \longrightarrow \prod_{v \in \Omega_A} \frac{(R^1_{L/K}\mathbb{G}_m)(K_v)}{\mathrm{Nrd}(\mathbf{U}(h)(K_v))}$$

is injective.

Proof. First assume $\mathrm{ind}(D) = 2$ so that D is a quaternion division algebra over L. By [17, p. 202, Exercise III.12 (a)], we have

$$\mathrm{Nrd}(\mathbf{U}(h)(K)) = \{z\tau(z)^{-1} \mid z \in \mathrm{Nrd}(D^*)\} = \mathrm{Nrd}(\mathbf{U}_2(D,\tau)(K)),$$

where $\mathbf{U}_2(D,\tau)$ denotes the unitary group of the rank 2 hyperbolic form $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ over (D,τ). So we may assume that $h = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. The exact sequence of algebraic groups

$$1 \longrightarrow \mathbf{SU}_2(D,\tau) \longrightarrow \mathbf{U}_2(D,\tau) \xrightarrow{\mathrm{Nrd}} (R^1_{L/K}\mathbb{G}_m) \longrightarrow 1$$

gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccc}
1 & \longrightarrow & \dfrac{(R^1_{L/K}\mathbb{G}_m)(K)}{\mathrm{Nrd}(\mathbf{U}(h)(K))} & \xrightarrow{\varphi} & H^1(K,\mathbf{SU}_2(D,\tau)) \\
 & & \downarrow & & \downarrow \\
1 & \longrightarrow & \prod\limits_{v\in\Omega_A} \dfrac{(R^1_{L/K}\mathbb{G}_m)(K_v)}{\mathrm{Nrd}(\mathbf{U}(h)(K_v))} & \longrightarrow & \prod\limits_{v\in\Omega_A} H^1(K_v,\mathbf{SU}_2(D,\tau))
\end{array}$$

We need only to show that the vertical map on the right in the above diagram is injective.

By [17, p. 26, Prop. 2.22], there is a unique quaternion K-algebra D_0 contained in D such that $D = D_0 \otimes_K L$ and $\tau = \tau_0 \otimes \iota$, where τ_0 is the canonical involution on D_0 and ι is the nontrivial element in the Galois group $\mathrm{Gal}(L/M)$. Write $L = K(\sqrt{d})$ and let n_{D_0} be the norm form of the quaternion K-algebra D_0. Then by [17, p. 229], we have $\mathbf{SU}_2(D,\tau) = \mathbf{Spin}(q)$, where $q = \langle 1, -d\rangle \otimes n_{D_0}$. Now the result follows from Theorem 4.5.

Assume next $\mathrm{ind}(D)$ is odd (and square-free). By [17, p. 202, Exercise III.12(b)],

$$\mathrm{Nrd}(\mathbf{U}(h)(K)) = \mathrm{Nrd}(D^*) \cap (R^1_{L/K}\mathbb{G}_m)(K).$$

Let $\lambda \in (R^1_{L/K}\mathbb{G}_m)(K) = \{z \in L^* \mid N_{L/K}(z) = 1\}$ be such that for every $v \in \Omega_A$, $\lambda \in \mathrm{Nrd}(\mathbf{U}(h)(K_v)) = \mathrm{Nrd}((D\otimes_K K_v)^*) \cap (R^1_{L/K}\mathbb{G}_m)(K_v)$. Since $\mathrm{ind}(D)$ is square-free, it follows from Theorem 3.7 that $\lambda \in \mathrm{Nrd}(D^*)$. Hence

$$\lambda \in \mathrm{Nrd}(\mathbf{U}(h)(K)) = \mathrm{Nrd}(D^*) \cap (R^1_{L/K}\mathbb{G}_m)(K).$$

Now assume $\mathrm{ind}(D)$ is even such that $\mathrm{ind}(D)/2$ is odd and square-free. In this case we have $D = H \otimes_L D'$ for some quaternion division algebra H over L and some central division algebra D' of odd index over L. By [3, Lemma 3.3.1], there is an odd degree separable extension K'/K such that $D' \otimes_K K' = D' \otimes_L LK'$ is split. By Morita theory, there is a unitary LK'/K'-involution σ on $H \otimes_L LK'$ and a hermitian form f over

$(H \otimes_L LK', \sigma)$ such that the involution τ on $D \otimes_L LK'$ is adjoint to f, and moreover, the form $h_{K'}$ over $(D \otimes_L LK', \tau)$ corresponds to a hermitian form h' over $(H \otimes_L LK', \sigma)$. Consider the commutative diagram

$$\begin{array}{ccc} \dfrac{R^1_{L/K}\mathbb{G}_m(K)}{\mathrm{Nrd}(\mathbf{U}(h)(K))} & \xrightarrow{\ \eta\ } & \prod_{v\in\Omega_A} \dfrac{R^1_{L/K}\mathbb{G}_m(K_v)}{\mathrm{Nrd}(\mathbf{U}(h)(K_v))} \\ \downarrow & & \downarrow \\ \dfrac{R^1_{LK'/K'}\mathbb{G}_m(K')}{\mathrm{Nrd}(\mathbf{U}(h')(K'))} & \xrightarrow{\ \eta'\ } & \prod_{v\in\Omega_A} \dfrac{R^1_{LK'/K'}\mathbb{G}_m(K'_v)}{\mathrm{Nrd}(\mathbf{U}(h')(K'_v))} \end{array}$$

The map η' is already shown to be injective. Let $\lambda \in R^1_{L/K}\mathbb{G}_m(K) \subseteq L^*$ be an element which is a reduced norm for $\mathbf{U}(h)(K_v)$ for every v. Then, considered as an element of $R^1_{LK'/K'}(K') \subseteq (LK')^*$, λ lies in $\mathrm{Nrd}(\mathbf{U}(h')(K'))$. By [23, Prop. 10.2], we have

$$N_{LK'/K'}(\mathrm{Nrd}(\mathbf{U}(h')(K'))) \subseteq \mathrm{Nrd}(\mathbf{U}(h)(K)).$$

Hence, $\lambda^{2r+1} \in \mathrm{Nrd}(\mathbf{U}(h)(K))$, where $2r+1 = [K' : K]$. It is sufficient to show that $\lambda^2 \in \mathrm{Nrd}(\mathbf{U}(h)(K))$. For this, we choose a quadratic extension M/K such that $H \otimes_K M = H \otimes_L LM$ is split. A similar argument as above, using the result in the case of odd index this time, shows that $\lambda \in \mathrm{Nrd}(\mathbf{U}(h_M)(M))$. Thus,

$$\lambda^2 = N_{LM/M}(\lambda) \in N_{LM/M}(\mathrm{Nrd}(\mathbf{U}(h_M)(M))) \subseteq \mathrm{Nrd}(\mathbf{U}(h)(K)).$$

This completes the proof of the lemma. □

Theorem 6.3. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Let L/K be a separable quadratic field extension and (D, τ) a central division L-algebra with a unitary L/K-involution whose index* $\mathrm{ind}(D)$ *is odd and square-free. Assume further that $p \nmid 2 \cdot \mathrm{ind}(D)$ in the local henselian case.*

Then for any nonsingular hermitian form h over (D, τ), the natural map

$$H^1(K, \mathbf{SU}(h)) \longrightarrow \prod_{v\in\Omega_A} H^1(K_v, \mathbf{SU}(h))$$

has a trivial kernel.

Proof. Let $\xi \in H^1(K, \mathbf{SU}(h))$ be a class that is locally trivial in $H^1(K_v, \mathbf{SU}(h))$ for every $v \in \Omega_A$. Let h' be a hermitian form whose class $[h'] \in H^1(K, \mathbf{U}(h))$ is the image of ξ under the natural map $H^1(K, \mathbf{SU}(h)) \to H^1(K, \mathbf{U}(h))$. The two forms h' and h have the same rank and discriminant, and they are locally isomorphic since ξ is locally trivial. So by Proposition 6.1,

$h' \cong h$ as hermitian forms over (D, τ). This means that $\xi \in H^1(K, \mathbf{SU}(h))$ maps to the trivial element in $H^1(K, \mathbf{U}(h))$.

Consider now the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
1 \longrightarrow & \frac{(R^1_{L/K}\mathbb{G}_m)(K)}{\mathrm{Nrd}(\mathbf{U}(h)(K))} & \xrightarrow{\varphi} & H^1(K, \mathbf{SU}(h)) & \longrightarrow & H^1(K, \mathbf{U}(h)) \\
& \eta\downarrow & & \downarrow & & \downarrow \\
1 \longrightarrow & \prod_{v\in\Omega_A} \frac{(R^1_{L/K}\mathbb{G}_m)(K_v)}{\mathrm{Nrd}(\mathbf{U}(h)(K_v))} & \longrightarrow & \prod_{v\in\Omega_A} H^1(K_v, \mathbf{SU}(h)) & \longrightarrow & \prod_{v\in\Omega_A} H^1(K_v, \mathbf{U}(h))
\end{array}$$

There is an element $\theta \in (R^1_{L/K}\mathbb{G}_m)(K)/\mathrm{Nrd}(\mathbf{U}(h)(K))$ such that $\varphi(\theta) = \xi$. The map η is injective by Lemma 6.2. So we have $\theta = 1$ and $\xi = \varphi(\theta)$ is trivial. The theorem is thus proved. □

6.2 Some observations on Suresh's exact sequence

6.4. Let E be a field of characteristic $\neq 2$. Let D be a quaternion division algebra over a quadratic field extension L of E. Let τ be a unitary L/E-involution on D. There is a unique quaternion E-algebra D_0 contained in D such that $D = D_0 \otimes_E L$ and $\tau = \tau_0 \otimes \iota$, where τ_0 is the canonical (symplectic) involution on D_0 and ι is the nontrivial element of the Galois group $\mathrm{Gal}(L/E)$. Then we have Suresh's exact sequence (cf. (2.8))

$$W(L) \xrightarrow{\widetilde{\pi}_1} W(D_0, \tau_0) \xrightarrow{\widetilde{\rho}} W(D, \tau) \xrightarrow{p_2} W^{-1}(D_0, \tau_0).$$

The goal of this subsection is to analyze the image of the map $\widetilde{\pi}_1$ in this sequence.

6.5. With notation as in (6.4), let h_0 be a hermitian form of rank m over (D_0, τ_0). Let $M(h_0) \in A := \mathrm{M}_m(D_0)$ be a representation matrix of h_0. One can define the pfaffian norm $\mathrm{Pf}(h_0)$ as the pfaffian norm of $M(h_0) \in A$ with respect to the adjoint involution of h_0 on A (cf. [17, p. 19]). This is a well defined element of the group $E^*/\mathrm{Nrd}(D_0^*)$. If $h_0 = \langle \alpha_1, \dots, \alpha_m \rangle$ with $\alpha_i \in E^*$, then $\mathrm{Pf}(h_0)$ is represented by the discriminant of the quadratic form $\langle \alpha_1, \dots, \alpha_m \rangle$ over E.

Lemma 6.6. *With notation as in* (6.4), *write $L = E(\sqrt{d})$ with $d \in E^*$. Let h_0 be a hermitian form of even rank over (D_0, τ_0).*

(i) *If the class $[h_0] \in W(D_0, \tau_0)$ lies in the image of $\widetilde{\pi}_1$, then its pfaffian norm $\mathrm{Pf}(h_0) \in E^*/\mathrm{Nrd}(D_0^*)$ lies in the subgroup generated by $N_{L/E}(L^*)$.*

(ii) *The converse of* (i) *is true if h_0 is of rank* 2.

Proof. (i) For $a + b\sqrt{d} \in L^*$ with $a, b \in E$, the form $\widetilde{\pi}_1(\langle a + b\sqrt{d}\rangle)$ is represented by the matrix

$$\begin{pmatrix} a & bd \\ bd & ad \end{pmatrix}.$$

One can then verify that

$$\widetilde{\pi}_1(\langle a + b\sqrt{d}\,\rangle) = \begin{cases} \langle a, ad(a^2 - b^2 d)\,\rangle & \text{if } a \neq 0 \\ \langle 2bd, -2bd\rangle & \text{if } a = 0 \neq b \end{cases}.$$

So it follows easily that $\mathrm{Pf}(\widetilde{\pi}_1(\langle a + b\sqrt{d}\,\rangle))$ is represented by an element of $N_{L/E}(L^*)$.

(ii) Conversely, let h_0 be a hermitian form of rank 2 whose pfaffian norm $\mathrm{Pf}(h_0)$ is represented by an element of $N_{L/E}(L^*)$. We want to show $[h_0] \in \mathrm{Im}(\widetilde{\pi}_1)$. By Suresh's exact sequence, it suffices to show that the form $\widetilde{\rho}(h_0)$ is hyperbolic over (D, τ).

We may assume $h_0 = \langle \alpha, -\gamma\alpha\rangle$ with $\alpha, \gamma \in E^*$. The assumption on the pfaffian norm implies that

$$\tau_0(u)u\gamma = \mathrm{Nrd}_{D_0}(u)\gamma = a^2 - b^2 d$$

for some $u \in D_0^*$ and some $a, b \in E$. Since

$$\langle \alpha, -\gamma\alpha\rangle \cong \langle \alpha, -\gamma\alpha\tau_0(u)u\rangle \quad \text{over } (D_0, \tau_0),$$

replacing γ by $\gamma\tau_0(u)u = \gamma \cdot \mathrm{Nrd}_{D_0}(u)$ if necessary, we may assume $\gamma = a^2 - b^2 d$ for some $a, b \in E$. From the definition of the map $\widetilde{\rho}$, it follows easily that the form $\widetilde{\rho}(h_0)$ over (D, τ) is also represented by the diagonal matrix $\langle \alpha, -\gamma\alpha\rangle$. But then for $v = (a + b\sqrt{d}, 1) \in D^2$, one has

$$\begin{aligned} \widetilde{\rho}(h_0)(v, v) &= (\tau(a + b\sqrt{d}), \tau(1)) \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma\alpha \end{pmatrix} \begin{pmatrix} a + b\sqrt{d} \\ 1 \end{pmatrix} \\ &= \alpha(a^2 - b^2 d - \gamma) = 0. \end{aligned}$$

This show that the rank 2 form $\widetilde{\rho}(h_0)$ is isotropic and hence hyperbolic. □

Lemma 6.7. *With notation as above, assume that the field E has finite u-invariant $u(E) = r$. Then for any hermitian form h_0 of rank $m > r/3$ over (D_0, τ_0), the form $\widetilde{\rho}(h_0)$ over (D, τ) is isotropic.*

Proof. We may assume D_0^m is the underlying space of the form h_0 and $h_0 = \langle \alpha_1, \dots, \alpha_m\rangle$ with $\alpha_i \in E^*$. Then the underlying space of $\widetilde{\rho}(h_0)$ is $D^m = D_0^m \oplus D_0^m\sqrt{d}$. We fix a quaternion basis $\{1, i, j, ij\}$ for the quaternion algebra D_0. The subspace $\mathrm{Sym}(D, \tau) \subseteq D$ consisting of τ-invariant elements is a 4-dimensional E-vector space with basis

$$1, i\sqrt{d}, j\sqrt{d}, ij\sqrt{d}.$$

Let $V \subseteq \mathrm{Sym}(D, \tau)$ be the subspace generated by $i\sqrt{d}$, $j\sqrt{d}$ and $ij\sqrt{d}$. For $w = x_1 \cdot i\sqrt{d} + x_2 \cdot j\sqrt{d} + x_3 \cdot ij\sqrt{d}$ with $x_i \in E$, a straightforward calculation yields

$$w^2 = di^2 \cdot x_1^2 + dj^2 \cdot x_2^2 + d(ij)^2 \cdot x_3^2 \in E.$$

So the map

$$\phi : V^m \longrightarrow E; \quad v = (v_1, \ldots, v_m) \longmapsto \widetilde{\rho}(h_0)(v, v) = \sum \alpha_i v_i^2$$

defines a quadratic form of rank $3m$ over E. By the assumption on the u-invariant of E, the quadratic form ϕ is isotropic and hence the hermitian form $\widetilde{\rho}(h_0)$ is isotropic. □

Lemma 6.8. *Assume that $u(E) < 12$. Then for any hermitian form h_0 of even rank $2n$ over (D_0, τ_0), one has*

$$[h_0] \in \mathrm{Im}(\widetilde{\pi}_1) \iff \mathrm{Pf}(h_0) \in N_{L/E}(L^*).\mathrm{Nrd}(D_0^*).$$

Proof. In view of Lemma 6.6, we need only to prove that if $\mathrm{Pf}(h_0) \in N_{L/E}(L^*)\mathrm{Nrd}(D_0^*)$, then $[h_0] \in \mathrm{Im}(\widetilde{\pi}_1)$.

To prove this, we use induction on $n = \mathrm{rank}(h_0)/2$, the case $n = 1$ being treated in Lemma 6.6. Now we assume $\mathrm{rank}(h_0) = 2n \geq 4$ and h_0 is anisotropic. Let V_0 be the underlying space of h_0. Then the underlying space of the form $\widetilde{\rho}(h_0)$ is $V = V_0 \oplus V_0\sqrt{d}$. By Lemma 6.7, the form $\widetilde{\rho}(h_0)$ is isotropic, that is, there is a nonzero vector $x_1 + y_1\sqrt{d} \in V = V_0 \oplus V_0\sqrt{d}$ such that

$$\begin{aligned} 0 &= \widetilde{\rho}(h_0)(x_1 + y_1\sqrt{d}, x_1 + y_1\sqrt{d}) \\ &= (h_0(x_1, x_1) - h_0(y_1, y_1)d) + (h_0(x_1, y_1) - h_0(y_1, x_1))\sqrt{d}. \end{aligned}$$

Thus

$$h_0(x_1, x_1) = d.h_0(y_1, y_1) \quad \text{and} \quad h_0(x_1, y_1) = h_0(y_1, x_1). \tag{6.8.1}$$

Since h_0 is anisotropic, $h_0(x_1, x_1)$ and $h_0(y_1, y_1)$ are both nonzero and hence lie in

$$E^* = \{x \in D_0^* \mid \tau_0(x) = x\}.$$

In particular, $x_1 \neq 0$, $y_1 \neq 0$ and

$$h_0(x_1, y_1) = h_0(y_1, x_1) \in E = \{x \in D_0 \mid \tau_0(x) = x\}.$$

If $x_1 = y_1\lambda$ for some $\lambda \in D_0^*$, then (6.8.1) yields

$$\tau_0(\lambda)\lambda = d \quad \text{and} \quad \tau_0(\lambda) = \lambda$$

whence $d = \lambda^2 \in E^{*2}$. Since d is not a square in E, the two vectors x_1, $y_1 \in V_0$ generate a D_0-submodule $W_0 := x_1 D_0 + y_1 D_0 \subseteq V_0$ of rank 2. Put $a = h_0(y_1, y_1) \in E^*$ and $bd = h_0(x_1, y_1) = h_0(y_1, x_1) \in E$. Then the restriction f_0 of h_0 to W_0 is represented by the matrix

$$\begin{pmatrix} ad & bd \\ bd & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & bd \\ bd & ad \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A direct computation then gives

$$\widetilde{\pi}_1(\langle a + b\sqrt{d}\rangle) = [f_0] \in W(D_0, \tau_0).$$

This means that h_0 contains a subform f_0 of rank 2, which lies in the image of $\widetilde{\pi}_1$. Writing $h_0 = f_0 \perp g_0$, we get $\mathrm{Pf}(g_0) \in N_{L/E}(L^*)\mathrm{Nrd}(D_0^*)$ since $\mathrm{Pf}(f_0)$ and $\mathrm{Pf}(h_0)$ lie in $N_{L/E}(L^*)\mathrm{Nrd}(D_0^*)$. Now the induction hypothesis yields $[g_0] \in \mathrm{Im}(\widetilde{\pi}_1)$, whence $[h_0] = [f_0] + [g_0] \in \mathrm{Im}(\widetilde{\pi}_1)$. □

6.3 A Hasse principle for H^4 of function fields of conics

Lemma 6.9. *Let F be a field of characteristic $\neq 2$, $\overline{F}$ a separable closure of F and $C \subseteq \mathbb{P}^2_F$ a smooth projective conic over F. Put $\overline{C} = C \times_F \overline{F}$ and let $F(C)$, $\overline{F}(C)$ denote the function fields of C and $\overline{C}$ respectively.*

Then the natural exact sequence

$$0 \longrightarrow \overline{F}(C)^* \otimes \mathbb{Q}_2/\mathbb{Z}_2(2) \longrightarrow \mathrm{Div}(\overline{C}) \otimes \mathbb{Q}_2/\mathbb{Z}_2(2) \longrightarrow \mathrm{Pic}(\overline{C}) \otimes \mathbb{Q}_2/\mathbb{Z}_2(2) \longrightarrow 0$$

induces an injection

$$H^3(F, \overline{F}(C) \otimes \mathbb{Q}_2/\mathbb{Z}_2(2)) \longrightarrow H^3(F, \mathrm{Div}(\overline{C}) \otimes \mathbb{Q}_2/\mathbb{Z}_2(2)).$$

Proof. Let $C^{(1)}$ be the set of closed points of C. For each $P \in C^{(1)}$, let G_P be the absolute Galois group of the residue field $F(P)$ of P. This is an open subgroup of $G = \mathrm{Gal}(\overline{F}/F)$. Write $M_P = \mathrm{Hom}_{G_P}(\mathbb{Z}[G], \mathbb{Z})$. We have an isomorphism of abelian groups $M_P \cong \bigoplus_{Q \mapsto P} \mathbb{Z}$, where the notation $Q \mapsto P$ means that Q runs over the closed points of $\overline{C}$ lying over P. On the other hand, we have an isomorphism of G-modules:

$$\mathrm{Div}(\overline{C}) \cong \bigoplus_{P \in C^{(1)}} M_P.$$

Since C is a smooth projective conic, $\mathrm{Pic}(\overline{C}) \cong \mathbb{Z}$ as G-modules. The natural map $\mathrm{Div}(\overline{C}) \to \mathrm{Pic}(\overline{C})$ can be identified with the summation map

$$\sigma : \bigoplus_{P \in C^{(1)}} \bigoplus_{Q \mapsto P} \mathbb{Z} \longrightarrow \mathbb{Z}$$

So the exact sequence in the lemma may be identified with the following

$$0 \longrightarrow \overline{F}(C)^* \otimes \mathbb{Q}_2/\mathbb{Z}_2(2) \longrightarrow \bigoplus_{P \in C^{(1)}} M_P \otimes \mathbb{Q}_2/\mathbb{Z}_2(2) \stackrel{\sigma}{\longrightarrow} \mathbb{Q}_2/\mathbb{Z}_2(2) \longrightarrow 0.$$

For any $i \geq 0$,

$$H^i\,(F, M_P \otimes \mathbb{Q}_2/\mathbb{Z}_2(2)) = H^i(F(P), \mathbb{Q}_2/\mathbb{Z}_2(2)).$$

It is thus sufficient to prove that the map

$$\bigoplus_{P \in C^{(1)}} H^2(F(P), \mathbb{Q}_2/\mathbb{Z}_2(2)) \longrightarrow H^2(F, \mathbb{Q}_2/\mathbb{Z}_2(2))$$

is surjective. In fact, we can choose a closed point $P \in C^{(1)}$ of degree 2 and consider the corresponding map

$$\psi : H^2(F(P), \mathbb{Q}_2/\mathbb{Z}_2(2)) \longrightarrow H^2(F, \mathbb{Q}_2/\mathbb{Z}_2(2)),$$

which coincides with the corestriction map. We claim that this map is already surjective. To see this, consider for each $n \in \mathbb{N}$ the corestriction map

$$\psi_n : H^2(F(P), \mathbb{Z}/2^n(2)) \longrightarrow H^2(F, \mathbb{Z}/2^n(2)).$$

By the Merkurjev–Suslin theorem, the map ψ_n may be identified with the norm map

$$N_{F(P)/F} : K_2(F(P))/2^n \longrightarrow K_2(F)/2^n$$

in Milnor's K-theory. The cokernel of this norm map is killed by $2 = [F(P) : F]$. So taking limits yields the surjectivity of the map ψ. This proves the lemma. □

Theorem 6.10. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Assume $p \neq 2$ in the local henselian case. Let C be a smooth projective conic in $\mathbb{P}^2_K$.*

Then the natural map

$$H^4(K(C), \mathbb{Z}/2) \longrightarrow \prod_{v \in \Omega_A} H^4(K_v(C), \mathbb{Z}/2)$$

is injective, where v runs over all divisorial valuations of K.

Proof. By the Merkurjev–Suslin theorem, we may replace $\mathbb{Z}/2$ by $\mathbb{Q}_2/\mathbb{Z}_2(3)$. Also, we may replace the completion K_v by the henselisation $K_{(v)}$ for each v (cf. [14, Thm. 2.9 and its proof]). Let $\overline{K}$ be a separable closure of K. Then we have a diagram of field extensions

$$\begin{array}{ccc} \overline{K}(C) & \longleftarrow & \overline{K} \\ \uparrow & & \uparrow \\ K_{(v)}(C) & \longleftarrow & K_{(v)} \\ \uparrow & & \uparrow \\ K(C) & \longleftarrow & K \end{array}$$

which identifies the Galois groups

$$\mathrm{Gal}(\overline{K}/K) = \mathrm{Gal}(\overline{K}(C)/K(C)) \quad \text{and} \quad \mathrm{Gal}(\overline{K}/K_{(v)}) = \mathrm{Gal}(\overline{K}(C)/K_{(v)}(C)).$$

This induces Hochschild-Serre spectral sequences

$$E_2^{pq}(K) = H^p(K, H^q(\overline{K}(C), \mathbb{Q}_2/\mathbb{Z}_2(3))) \Longrightarrow H^{p+q}(K(C), \mathbb{Q}_2/\mathbb{Z}_2(3))$$

and

$$E_2^{pq}(K_{(v)}) = H^p(K_{(v)}, H^q(\overline{K}(C), \mathbb{Q}_2/\mathbb{Z}_2(3))) \\ \Longrightarrow H^{p+q}(K_{(v)}(C), \mathbb{Q}_2/\mathbb{Z}_2(3)).$$

Using

$$\mathrm{cd}_2(\overline{K}(C)) \leq 1 \quad \text{and} \quad \mathrm{cd}_2(K_{(v)}) \leq \mathrm{cd}_2(K) \leq 3,$$

one finds easily that the above spectral sequences induce canonical isomorphisms

$$H^4(K(C), \mathbb{Q}_2/\mathbb{Z}_2(3)) \cong H^3(K, H^1(\overline{K}(C), \mathbb{Q}_2/\mathbb{Z}_2(3)))$$

and

$$H^4(K_{(v)}(C), \mathbb{Q}_2/\mathbb{Z}_2(3)) \cong H^3(K_{(v)}, H^1(\overline{K}(C), \mathbb{Q}_2/\mathbb{Z}_2(3))).$$

Since $H^1(\overline{K}(C), \mathbb{Q}_2/\mathbb{Z}_2(3)) \cong \overline{K}(C)^* \otimes \mathbb{Q}_2/\mathbb{Z}_2(2)$, we need only prove the injectivity of the natural map

$$H^3(K, \overline{K}(C)^* \otimes \mathbb{Q}_2/\mathbb{Z}_2(2)) \longrightarrow \prod_{v \in \Omega_A} H^3(K_{(v)}, \overline{K}(C)^* \otimes \mathbb{Q}_2/\mathbb{Z}_2(2))$$

is injective.

By Lemma 6.9, we have an injection

$$H^3(K,\overline{K}(C)^*\otimes\mathbb{Q}_2/\mathbb{Z}_2(2))\hookrightarrow\bigoplus_{p\in C^{(1)}}H^3(K(P),\mathbb{Q}_2/\mathbb{Z}_2(2)).$$

For each v, let $C_{(v)}=C\times_K K_{(v)}$ be the base extension of C and let $K_{(v)}(C)$ denote the function field of $C_{(v)}$. By functoriality, we may reduce to proving the injectivity of the map

$$\varphi:\bigoplus_{P\in C^{(1)}}H^3(K(P),\mathbb{Q}_2/\mathbb{Z}_2(2))\longrightarrow\prod_{v\in\Omega_A}\bigoplus_{Q\in C^{(1)}_{(v)}}H^3(K_{(v)}(Q),\mathbb{Q}_2/\mathbb{Z}_2(2)).$$

For fixed v and $P\in C^{(1)}$, the corresponding component $\varphi_{v,P}$ of the map φ is given by

$$\varphi_{v,P}:H^3(K(P),\mathbb{Q}_2/\mathbb{Z}_2(2))\longrightarrow\bigoplus_{Q|P}H^3(K_{(v)}(Q),\mathbb{Q}_2/\mathbb{Z}_2(2)),$$

where Q runs over the points of the fiber $C_{(v)}\times_C P=\operatorname{Spec}(K_{(v)}\otimes_K K(P))$. An element $\alpha=(\alpha_P)\in\oplus_P H^3(K(P),\mathbb{Q}_2/\mathbb{Z}_2(2))$ lies in $\ker(\varphi)$ if and only if for each $P\in C^{(1)}$, α_P lies in the kernel of

$$\begin{aligned}\varphi_P=\prod_v\varphi_{v,P}&:H^3(K(P),\mathbb{Q}_2/\mathbb{Z}_2(2))\\&\longrightarrow\prod_v H^3_{\text{ét}}(K_{(v)}\otimes_K K(P),\mathbb{Q}_2/\mathbb{Z}_2(2)).\end{aligned}$$

It suffices to prove that for every P, the map φ_P is injective.

Replacing $K(P)$ by the separable closure of K in $K(P)$ if necessary, we may assume that $K(P)/K$ is a finite separable extension. Then we have

$$K_{(v)}\otimes_K K(P)\cong\prod_{w|v}K(P)_{(w)},$$

(cf. [10, IV.18.6.8]). So the map φ_P gets identified with the natural map

$$H^3(K(P),\mathbb{Q}_2/\mathbb{Z}_2(2))\longrightarrow\prod_w H^3(K(P)_{(w)},\mathbb{Q}_2/\mathbb{Z}_2(2)),$$

where w runs over divisorial valuations of $K(P)$. This map is injective by Theorem 3.4. The theorem is thus proved. □

Corollary 6.11. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of charachteristic p. Assume $p\neq 2$ in the local henselian case. Let C a smooth projective conic in $\mathbb{P}^2_K$.*

Then the natural map

$$I^4(K(C)) \longrightarrow \prod_{v\in\Omega_A} I^4(K_v(C))$$

is injective, where v runs over the set Ω_A of divisorial valuations of K.

Proof. For $F = K(C)$ or $K_v(C)$, we have $\mathrm{cd}_2(F) \le 4$. By the degree 4 case of the Milnor conjecture (cf. [32] and [22]), we have an isomorphism $I^4(F) \cong H^4(F, \mathbb{Z}/2)$. (In the p-adic arithmetic case, we can also deduce this isomorphism from [1, p. 655, Prop. 2] together with [19, Thm. 3.4].) The result then follows immediately from Theorem 6.10. □

6.4 Case of even index

Proposition 6.12. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Let L/K be a quadratic field extension, (D, τ) a central division algebra over L with a unitary L/K-involution whose index is not divisible by 4. Let h be a nonsingular hermitian form over (D, τ) which has even rank, trivial discriminant and trivial Rost invariant (cf. (2.14)). Assume $p \neq 2$ if $\mathrm{ind}(D)$ is even. In the local henselian case, assume further that the Hasse principle with respect to divisorial valuations holds for quadratic forms of rank 6 over K.*

Then we have $[h] = 0 \in W(D, \tau)$ if and only if $[h \otimes_K K_v] = 0 \in W(D \otimes_K K_v, \tau)$ for every $v \in \Omega_A$.

Proof. If the index $\mathrm{ind}(D)$ is odd, the result is already proved in Proposition 6.1. We assume next that $\mathrm{ind}(D)$ is even and not divisible by 4.

We first consider the case where D is a quaternion algebra. As in (6.4), we write $D = D_0 \otimes_K L$ with D_0 a quaternion division algebra over K and $L = K(\sqrt{d})$ with $d \in K^*$, and we have Suresh's exact sequence

$$W(L) \xrightarrow{\widetilde{\pi}_1} W(D_0, \tau_0) \xrightarrow{\widetilde{p}} W(D, \tau) \xrightarrow{p_2} W^{-1}(D_0, \tau_0) \qquad (6.12.1)$$

Let $C \subseteq \mathbb{P}^2_K$ be the smooth projective conic associated to the quaternion algebra D_0. Then the algebra $D \otimes_K K(C) = D_0 \otimes_K L(C)$ is a split central simple algebra over $L(C)$ with a unitary $L(C)/K(C)$-involution τ. By Morita theory, the hermitian form $h \otimes_K K(C)$ over $(D \otimes_K K(C), \tau)$ corresponds to a hermitian form h'_C over $(L(C), \iota)$, where ι denotes the nontrivial element of the Galois group $\mathrm{Gal}(L(C)/K(C))$. The trace form $q_{h,C}$ of h'_C gives a quadratic form over $K(C)$. By [17, Example 31.44], the quadratic form $q_{h,C}$ has even rank, trivial discriminant, trivial Clifford invariant and trivial Rost invariant, since h'_C has even rank, trivial discriminant and trivial Rost invariant (these invariants being invariant under Morita equivalence). Hence in the

Witt group $W(K(C))$ we have $[q_{h,C}] \in I^4(K(C))$. Since h is locally hyperbolic, it follows from Corollary 6.11 that $[q_{h,C}] = 0 \in W(K(C))$, whence $[h \otimes_K K(C)] = 0 \in W(D \otimes_K K(C), \tau)$. In the commutative diagram

$$\begin{array}{ccc} W(D,\tau) & \xrightarrow{p_2} & W^{-1}(D_0,\tau_0) \\ \downarrow & & \downarrow \\ W(D\otimes_K K(C),\tau) & \xrightarrow{p_2} & W^{-1}(D_0\otimes_K K(C),\tau_0) \end{array}$$

the right vertical map is injective by [24]. So we have $p_2(h) = 0 \in W^{-1}(D_0, \tau_0)$. The exactness of the sequence (6.12.1) implies that $[h] = \tilde{\rho}([h_0])$ for some hermitian form h_0 over (D_0, τ_0) of even rank.

Let $\lambda = \mathrm{Pf}(h_0) \in K^*/\mathrm{Nrd}(D_0^*)$ be the pfaffian norm of h_0. Since h is locally hyperbolic, by considering Suresh's exact sequence locally, we see that $(h_0)_v$ lies in the image of $(\widetilde{\pi}_1)_v$ for every v. By Lemma 6.6, this implies that $\lambda \in \mathrm{Nrd}((D_0)_v^*) \cdot N_{L_v/K_v}(L_v^*)$ for every v. In other words, the quadratic form

$$\phi := \lambda \cdot n_{D_0} - \langle 1, -d \rangle,$$

where n_{D_0} denotes the norm form of the quaternion algebra D_0, is isotropic over every K_v. By the assumption on the Hasse principle for quadratic forms of rank 6 (and [6, Thm. 3.1] in the p-adic arithmetic case), ϕ is isotropic over K, which shows $\lambda \in \mathrm{Nrd}(D_0^*) \cdot N_{L/K}(L^*)$. As was mentioned in the proof of Corollary 4.4, the field K has u-invariant 8. So by Lemma 6.8, we have $[h_0] \in \mathrm{Im}(\widetilde{\pi}_1)$. Hence $[h] = \tilde{\rho}([h_0]) = 0 \in W(D, \tau)$ as desired.

Consider next the general case where $\mathrm{ind}(D)$ is even and not divisible by 4. In this case we have $D = Q \otimes_L D'$ for some quaternion division algebra Q over L and some central division algebra D' of odd index over L. By [3, Lemma 3.3.1], there is an odd degree separable extension K'/K such that $D' \otimes_K K' = D' \otimes_L LK'$ is split. By Morita theory, there is a unitary LK'/K'-involution σ on $H \otimes_L LK'$ and a hermitian form f over $(H \otimes_L LK', \sigma)$ such that the involution τ on $D \otimes_L LK'$ is adjoint to f, and moreover, the form $h_{K'}$ over $(D \otimes_L LK', \tau)$ corresponds to a hermitian form h' over $(H \otimes_L LK', \sigma)$, which has even rank, trivial discriminant and trivial Rost invariant. The hypothesis that h is locally hyperbolic over every K_v implies that h' is locally hyperbolic over every K'_w, where w runs over the set of divisorial valuations of K'. By the previous case, $[h'] = 0 \in W(H \otimes_L LK', \sigma)$ and hence $[h] = 0 \in W(D \otimes_L LK', \tau)$. Since the degree $[LK' : L] = [K' : K]$ is odd, the natural map $W(D, \tau) \to W(D \otimes_L LK', \tau)$ is injective by a theorem of Bayer-Fluckiger and Lenstra (cf. [17, p. 80, Coro. 6.18]). So we get $[h] = 0 \in W(D, \tau)$. This completes the proof. □

Theorem 6.13. *Let K be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic p. Let L/K*

be a quadratic field extension, (D, τ) a central division algebra over L with a unitary L/K-involution whose index $\mathrm{ind}(D)$ *is square-free. Let h be a non-singular hermitian form over (D, τ).*

Assume $p \neq 2$ if $\mathrm{ind}(D)$ *is even. In the local henselian case, assume further that $p \nmid \mathrm{ind}(D)$ and that the Hasse principle with respect to divisorial valuations holds for quadratic forms of rank 6 over K.*

Then the natural map

$$H^1(K, \mathbf{SU}(h)) \longrightarrow \prod_{v \in \Omega_A} H^1(K_v, \mathbf{SU}(h))$$

has trivial kernel.

Proof. Let $\xi \in H^1(K, \mathbf{SU}(h))$ be a class that is locally trivial. Let the image of ξ in $H^1(K, \mathbf{U}(h))$ correspond to a hermitian form h'. The form $h' \perp (-h)$ has even rank, trivial discriminant and is locally hyperbolic. We claim that the Rost invariant $\mathscr{R}(h' \perp (-h))$ is trivial. Indeed, as ξ is locally trivial, $R_{\mathbf{SU}(h)}(\xi)$ is locally trivial in $H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$ for every v. By Theorem 3.4, $R_{\mathbf{SU}(h)}(\xi) = 0$. There is a group homomorphism

$$\mathbf{SU}(h) \longrightarrow \mathbf{SU}(h \perp (-h)),\, f \longmapsto (f, \mathrm{id})$$

which induces a map

$$\alpha : H^1(K, \mathbf{SU}(h)) \longrightarrow H^1(K, \mathbf{SU}(h \perp (-h))).$$

The image $\alpha(\xi)$ of ξ lifts the class $[h' \perp (-h)] \in H^1(K, \mathbf{U}(h \perp (-h)))$. By general property of the (usual) Rost invariant, there is an integer n_α such that

$$R_{\mathbf{SU}(h \perp (-h))}(\alpha(\xi)) = n_\alpha R_{\mathbf{SU}(h)}(\xi).$$

We have thus $\mathscr{R}(h' \perp (-h)) = R_{\mathbf{SU}(h \perp (-h))}(\alpha(\xi)) = 0$ since ξ has trivial Rost invariant. Now Proposition 6.12 implies that the two forms h', h over (D, τ) are isomorphic.

Consider the cohomology exact sequence

$$1 \longrightarrow \frac{R^1_{L/K}\mathbb{G}_m(K)}{\mathrm{Nrd}(\mathbf{U}(h)(K))} \xrightarrow{\varphi} H^1(K, \mathbf{SU}(h)) \longrightarrow H^1(K, \mathbf{U}(h)) \qquad (6.13.1)$$

arising from the exact sequence of algebraic groups

$$1 \longrightarrow \mathbf{SU}(h) \longrightarrow \mathbf{U}(h) \xrightarrow{\mathrm{Nrd}} R^1_{L/K}\mathbb{G}_m \longrightarrow 1.$$

The fact that $h' \cong h$ implies that ξ lies in the image of the map φ in the above cohomology exact sequence (6.13.1). Considering the sequence (6.13.1) locally and using Lemma 6.2, we conclude that ξ is trivial in $H^1(K, \mathbf{SU}(h))$, thus proving the theorem. □

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