J. Ramanujan Math. Soc. **29**, No.4 (2014) 413–458

Subtle invariants of F-crystals

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Communicated by: Dipendra Prasad

Received: November 15th, 2013

Abstract. Vasiu proved that the level torsion $\ell_{\mathcal{M}}$ of an F-crystal $\mathcal{M}$ over an algebraically closed field of characteristic $p > 0$ is a non-negative integer that is an effectively computable upper bound of the isomorphism number $n_{\mathcal{M}}$ of $\mathcal{M}$ and expected that in fact one always has $n_{\mathcal{M}} = \ell_{\mathcal{M}}$. In this paper, we prove that this equality holds.

1. Introduction

1.1 Notations

Let p be a prime number and k an algebraically closed field of characteristic p. For every k-algebra R, let $W(R)$ be the ring of p-typical Witt vectors with coefficients in R. For every integer $s \geq 1$, let $W_s(R)$ be the ring of truncated p-typical Witt vectors of length s with coefficients in R. Let σ_R be the Frobenius of $W(R)$ and $W_s(R)$. Let θ_R be the Verschiebung of $W(R)$ and $W_s(R)$. Recall that $\sigma_R\theta_R = \theta_R\sigma_R = p$. When there is no confusion of the base ring, we also denote σ_R by σ and θ_R by θ. Set $B(R) = W(R)[1/p]$. When $R = k$, $B(k)$ is the field of fractions of $W(k)$. An F-crystal $\mathcal{M}$ over k is a pair (M, φ) where M is a free $W(k)$-module of finite rank and $\varphi : M \to M$ is a σ-linear monomorphism. Unless mentioned otherwise, all F-crystals in this paper are over k. We denote by $\mathcal{M}_R$ the pair $(M \otimes_{W(k)} W(R), \varphi \otimes \sigma_R)$. For every $W(k)$-linear automorphism g of M, we denote by $\mathcal{M}(g)$ the F-crystal $(M, g\varphi)$ over k.

1.2 Aim and scope

The isomorphism number $n_{\mathcal{M}}$ of an F-crystal $\mathcal{M} = (M, \varphi)$ is the smallest non-negative integer such that for every $W(k)$-linear automorphism g of M

with the property that $g \equiv 1_M$ modulo $p^{n_{\mathcal{M}}}$, the F-crystal $\mathcal{M}(g)$ is isomorphic to $\mathcal{M}$. This is the generalization of the isomorphism number n_D of a p-divisible group D over k, which is defined to be the smallest non-negative integer such that for every p-divisible group D' over k with the same dimension and codimension as D, $D'[p^{n_D}]$ and $D[p^{n_D}]$ are isomorphic if and only if D' is isomorphic to D. The isomorphism numbers of p-divisible groups are known to exist as early as in [8], as a consequence of Theorems 3.4 and 3.5 of the loc. cit. Recently, the isomorphism numbers of F-crystals are known to exist by [13, Main Theorem A].

Traverso proved that $n_D \leq cd + 1$ in [11, Theorem 3], where c and d are the codimension and the dimension (respectively) of the p-divisible group D. He later conjectured that $n_D \leq \min\{c, d\}$ in [12, Section 40, Conjecture 4]. In search of optimal upper bounds of n_D, the following theorem plays an important role:

Theorem 1.1 ([7, Theorem 1.6]). *If D is a non-ordinary p-divisible group over an algebraically closed field k, then its isomorphism number n_D is equal to its level torsion ℓ_D.*

For the definition of ℓ_D, see [16, Subsection 1.4] and [7, Definition 8.3]. We point out that the two definitions are slightly different. In the case when D is a direct sum of two or more isoclinic ordinary p-divisible groups of different Newton slopes, we get $\ell_D = 1$ by the definition in [16, Subsection 1.4]; on the other hand, we get $\ell_D = 0$ by [7, Definition 8.3]. If we assume that D is non-ordinary, then the two definitions coincide.

Vasiu proved that $n_D \leq \ell_D$ in [16, Main Theorem A], and that $n_D = \ell_D$ provided D is a direct sum of isoclinic p-divisible groups, that is, of p-divisible groups whose Newton polygons are straight lines. Later Lau, Nicole and Vasiu proved the equality $n_D = \ell_D$ in [7] for all p-divisible groups D over k. Theorem 1.1 builds a bridge between the isomorphism number n_D and other invariants of D, such as the level torsion ℓ_D, the endomorphism number e_D, and the coarse endomorphism number f_D, which turn out to be all equal by [7, Theorem 8.11]; see [7, Definitions 2.2 and 7.2] for their definitions. Using Theorem 1.1, Lau, Nicole and Vasiu were able to find the optimal upper bound of $n_D \leq \lfloor 2cd/(c+d) \rfloor$ (see [7, Theorem 1.4]), which provides a corrected version of Traverso's conjecture.

The level torsion $\ell_{\mathcal{M}}$ of an F-crystal $\mathcal{M}$ is well-defined; see [16, Section 1.2] or Subsection 4.4 for its definition. Therefore it is natural to ask if the similar equality $n_{\mathcal{M}} = \ell_{\mathcal{M}}$ holds or not in general. As mentioned before, Vasiu has already proved that $n_{\mathcal{M}} \leq \ell_{\mathcal{M}}$ and the equality holds when $\mathcal{M}$ is a direct sum of isoclinic F-crystals. He expressed the expectation that the equality is true in general; see the paragraph after [16, 1.3 Main Theorem A]. In this paper, we confirm this expectation.

Theorem 1.2 (Main Theorem). *If $\mathcal{M}$ is a non-ordinary F-crystal over an algebraically closed field k, then its isomorphism number $n_{\mathcal{M}}$ is equal to its level torsion $\ell_{\mathcal{M}}$.*

See Theorem 5.5 for its proof. The definition of the level torsion $\ell_{\mathcal{M}}$ in our paper is slightly different from the definition in [16, Subsection 1.2]; see Remark 4.9. When $\mathcal{M}$ is a non-ordinary F-crystal, the two definitions are exactly the same just as in the case of p-divisible groups.

1.3 On the proof of the Main Theorem

The proof of the Main Theorem uses many ideas from [7], [14], and [4]. It involves two major steps:

Step 1. Generalize the level torsion $\ell_{\mathcal{M}}$, the homomorphism number $e_{\mathcal{M}}$, and the coarse homomorphism number $f_{\mathcal{M}}$ to F-crystals $\mathcal{M}$ over k. Then prove that they are all equal via a sequence of inequalities $f_{\mathcal{M}} \leq e_{\mathcal{M}} \leq \ell_{\mathcal{M}} \leq f_{\mathcal{M}}$ that are the generalization of the inequalities $f_D \leq e_D \leq \ell_D \leq f_D$ obtained in [7].

The main difficulty in Step 1 is to have the right generalizations of $\ell_{\mathcal{M}}$, $e_{\mathcal{M}}$ and $f_{\mathcal{M}}$ so that they remain unchanged under extensions of algebraically closed fields. This requires the constructions of suitable groups schemes $\mathbf{End}_s(\mathcal{M})$ (resp. $\mathbf{Aut}_s(\mathcal{M})$) whose k-valued points are the endomorphisms (resp. automorphisms) of F-truncations modulo p^s of $\mathcal{M}$ for all $s \geq 1$. The F-truncations modulo p^s of F-crystals are the generalization of truncated Barsotti–Tate groups of level s associated to p-divisible groups. They are first introduced by Vasiu in [13] and will be recalled in Section 2; see Definition 2.1. We will show that $\ell_{\mathcal{M}}$, $e_{\mathcal{M}}$ and $f_{\mathcal{M}}$ are invariant under extensions of algebraically closed fields. This allows us to generalize the proof in [7, Section 8] to our case.

Step 2. Prove that $f_{\mathcal{M}} = n_{\mathcal{M}}$ by showing that both $f_{\mathcal{M}}$ and $n_{\mathcal{M}}$ are equal to the smallest number m defined by the property that the image of the natural reduction homomorphism $\pi_{s,1} : \mathbf{End}_s(\mathcal{M}) \to \mathbf{End}_1(\mathcal{M})$ has zero dimension if and only if $s - 1 \geq m$.

In Step 2, the main result (see Theorem 3.15) is to show that $n_{\mathcal{M}}$ is the place where the non-decreasing sequence $(\dim(\mathbf{Aut}_s(\mathcal{M})))_{s\geq 1}$ stabilizes, which generalizes a similar result for p-divisible groups in [4]. In order to show this, we construct a group action for each $s \geq 1$ whose orbits parametrize isomorphism classes of F-truncations modulo p^s; see Subsection 3.2. It turns out that the dimension of the stabilizer of the identity element of this action is equal to the dimension of $\mathbf{Aut}_s(\mathcal{M})$ (Lemma 3.11). This allows us to use the machinery of group actions to work with the sequence $(\dim(\mathbf{Aut}_s(\mathcal{M})))_{s\geq 1}$ in a way similar to [4] and [14].

We note that the proof of our Main Theorem does not rely on the known fact that $n_{\mathcal{M}} \leq \ell_{\mathcal{M}}$ proved in [16].

Notes. After this manuscript was finished, we learned that Sian Nie had a proof of the fact $\ell_{\mathcal{M}} \leq n_{\mathcal{M}}$ where $\mathcal{M}$ is defined over the ring $k[[\epsilon]]$ of formal power series instead of over $W(k)$; see [9]. He expressed the hope that the same strategy might be used to prove Theorem 1.2.

2. F-truncations of F-crystals

In this section, we recall F-truncations modulo p^s of an F-crystal $\mathcal{M}$ over k and provide several equivalent descriptions of homomorphisms and isomorphisms between them.

2.1 Filtrations of F-crystals

Let r be the rank of $\mathcal{M}$. Throughout this paper, the integers $e_1 \leq \cdots \leq e_r$ will always be the Hodge slopes of $\mathcal{M}$ and the integers $f_1 < \cdots < f_t$ will always be all the distinct Hodge slopes of $\mathcal{M}$; thus $\{f_1, \ldots, f_t\} = \{e_1, \ldots, e_r\}$ as sets. Clearly $f_1 = e_1$ and $f_t = e_r$. For each integer $s \geq 0$, let h_s be the Hodge number of $\mathcal{M}$, that is, $h_s = \#\{e_i \mid e_i = s, 1 \leq i \leq r\}$. Clearly, $h_{f_i} \geq 1$ for all $1 \leq i \leq t$. We say that a $W(k)$-basis $\{v_1, v_2, \ldots, v_r\}$ of M is an *F-basis* of $\mathcal{M}$ if $\{p^{-e_1}\varphi(v_1), p^{-e_2}\varphi(v_2), \ldots, p^{-e_r}\varphi(v_r)\}$ is as well a $W(k)$-basis of M. Every F-basis of $\mathcal{M}$ is also an F-basis of $\mathcal{M}(g)$ for all $g \in \mathrm{GL}_M(W(k))$. For each isomorphism of F-crystals $h : \mathcal{M}_1 \to \mathcal{M}_2$ and an F-basis $\mathcal{B}$ of $\mathcal{M}_1$, it is easy to see that $h(\mathcal{B})$ is an F-basis of $\mathcal{M}_2$.

For each positive integer $1 \leq j \leq t$, we define $I_j = \{i \mid e_i = f_j, 1 \leq i \leq r\}$. For an F-basis $\mathcal{B}$ of $\mathcal{M}$, let $\widetilde{F}^j_{\mathcal{B}}(M)$ be the free $W(k)$-submodule of M generated by all v_i with $i \in I_j$. We obtain two direct sum decompositions of M that depend on $\mathcal{B}$ (and thus on $\mathcal{M}$):

$$M = \bigoplus_{j=1}^{t} \widetilde{F}^j_{\mathcal{B}}(M) = \bigoplus_{j=1}^{t} \frac{1}{p^{f_j}}\varphi(\widetilde{F}^j_{\mathcal{B}}(M)).$$

For each $1 \leq i \leq t$, by letting $F^i_{\mathcal{B}}(M) := \bigoplus_{j=i}^{t} \widetilde{F}^j_{\mathcal{B}}(M)$, we get a decreasing and exhaustive filtration of M

$$F^{\bullet}_{\mathcal{B}}(M) : \widetilde{F}^t_{\mathcal{B}}(M) = F^t_{\mathcal{B}}(M) \subset F^{t-1}_{\mathcal{B}}(M) \subset \cdots \subset F^1_{\mathcal{B}}(M) = M.$$

For each $F^i_{\mathcal{B}}(M)$, let $\varphi_{F^i_{\mathcal{B}}(M)} : F^i_{\mathcal{B}}(M) \to M$ be the restriction of $p^{-f_i}\varphi$ to $F^i_{\mathcal{B}}(M)$. For every integer $s > 0$, let $F^{\bullet}_{\mathcal{B}}(M)_s$ be the reduction modulo p^s of the filtration $F^{\bullet}_{\mathcal{B}}(M)$, namely

$$F^t_{\mathcal{B}}(M)/p^s F^t_{\mathcal{B}}(M) \subset F^{t-1}_{\mathcal{B}}(M)/p^s F^{t-1}_{\mathcal{B}}(M) \subset \cdots \subset F^1_{\mathcal{B}}(M)/p^s F^1_{\mathcal{B}}(M).$$

For each $1 \leq i \leq t$, we denote by $\varphi_{F^i_{\mathcal{B}}(M)}[s]$ the σ-linear monomorphism $\varphi_{F^i_{\mathcal{B}}(M)}$ modulo p^s, and by $\varphi_{F^{\bullet}_{\mathcal{B}}(M)}[s]$ the sequence of the σ-linear monomorphisms $\varphi_{F^i_{\mathcal{B}}(M)}[s]$ with $1 \leq i \leq t$. By a *filtered F-crystal modulo p^s of an* F-crystal $\mathcal{M}$, we mean a triple of the form

$$(M/p^s M, F^{\bullet}_{\mathcal{B}}(M)_s, \varphi_{F^{\bullet}_{\mathcal{B}}(M)}[s]).$$

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two F-crystals with the same Hodge polygons as $\mathcal{M}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ two F-bases of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. By an *isomorphism of filtered F-crystals modulo p^s* from a filtered F-crystal modulo p^s of $\mathcal{M}_1$ to a filtered F-crystal modulo p^s of $\mathcal{M}_2$, we mean a $W_s(k)$-linear isomorphism $f : M_1/p^s M_1 \to M_2/p^s M_2$ such that for all $1 \leq i \leq t$ we have $f(F^i_{\mathcal{B}_1}(M_1)/p^s F^i_{\mathcal{B}_1}(M_1)) = F^i_{\mathcal{B}_2}(M_2)/p^s F^i_{\mathcal{B}_2}(M_2)$ and $\varphi_{F^i_{\mathcal{B}_2}(M_2)}[s] f = f \varphi_{F^i_{\mathcal{B}_1}(M_1)}[s]$.

2.2 F-truncations

In this subsection, we recall the F-truncation modulo p^s of an F-crystal defined in [13, Sect. 3.2.9]. It is the generalization of the D-truncation $(M/p^s M, \varphi[s], \theta[s])$ of a Dieudonné module (M, φ, θ); see [13, Sect. 3.2.1] for the definition of D-truncations.

Definition 2.1. *For every integer $s > 0$, the F-*truncation modulo p^s *of an F-crystal $\mathcal{M}$ is the set $F_s(\mathcal{M})$ of isomorphism classes of filtered F-crystals modulo p^s of $\mathcal{M}$ as $\mathcal{B}$ varies among all possible F-bases of $\mathcal{M}$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two F-crystals with the same Hodge polygon. A $W_s(k)$-linear isomorphism $f : M_1/p^s M_1 \to M_2/p^s M_2$ is an* isomorphism of F-truncations modulo p^s *from $F_s(\mathcal{M}_1)$ to $F_s(\mathcal{M}_2)$ if for every F-basis $\mathcal{B}_1$ of $\mathcal{M}_1$, there exists an F-basis $\mathcal{B}_2$ of $\mathcal{M}_2$ such that*

$$f : (M_1/p^s M_1, F^{\bullet}_{\mathcal{B}_1}(M_1)_s, \varphi_{F^{\bullet}_{\mathcal{B}_1}(M_1)}[s]) \to (M_2/p^s M_2, F^{\bullet}_{\mathcal{B}_2}(M_2)_s, \varphi_{F^{\bullet}_{\mathcal{B}_2}(M_2)}[s])$$

is an isomorphism of filtered F-crystals modulo p^s.

Suppose f is an isomorphism of F-truncations modulo p^s from $F_s(\mathcal{M})$ to $F_s(\mathcal{M}(g))$. Define a set function $\Gamma_{f,s} : F_s(\mathcal{M}) \to F_s(\mathcal{M}(g))$ as follows: the image of the isomorphism class represented by $(M/p^sM, F^{\bullet}_{\mathcal{B}_1}(M)_s, \varphi_{F^{\bullet}_{\mathcal{B}_1}(M)}[s])$ under $\Gamma_{f,s}$ is the isomorphism class represented by $(M/p^sM, F^{\bullet}_{\mathcal{B}_2}(M)_s, (g\varphi)_{F^{\bullet}_{\mathcal{B}_2}(M)}[s])$ if

$$f : (M/p^sM, F^{\bullet}_{\mathcal{B}_1}(M)_s, \varphi_{F^{\bullet}_{\mathcal{B}_1}(M)}[s]) \to (M/p^sM, F^{\bullet}_{\mathcal{B}_2}(M)_s, (g\varphi)_{F^{\bullet}_{\mathcal{B}_2}(M)}[s])$$

is an isomorphism of filtered F-crystals modulo p^s. It is easy to see that this function is well-defined and we shall prove that $\Gamma_{f,s}$ is a bijection of sets in Corollary 2.4.

The following lemma is a generalization of [13, Lemma 3.2.2] to F-crystals for $G = \mathbf{GL}_M$.

Lemma 2.2. *For each F-crystal $\mathcal{M}$ and every $g \in \mathrm{GL}_M(W(k))$, the following two statements are equivalent:*

(1) There exist $h \in \mathrm{GL}_M(W(k))$, F-bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{M}$ and $\mathcal{M}(g)$ respectively, such that the reduction $h[s]$ of h modulo p^s induces an isomorphism

$$\begin{aligned} h[s] : (M/p^sM, F^{\bullet}_{\mathcal{B}_1}(M)_s, \varphi_{F^{\bullet}_{\mathcal{B}_1}(M)}[s]) \\ \to (M/p^sM, F^{\bullet}_{\mathcal{B}_2}(M)_s, (g\varphi)_{F^{\bullet}_{\mathcal{B}_2}(M)}[s]) \end{aligned} \tag{2.1}$$

of filtered F-crystals modulo p^s.

(2) There exists an element $g_s \in \mathrm{GL}_M(W(k))$ with the property that it is congruent to 1_M modulo p^s such that $\mathcal{M}(g_s)$ is isomorphic to $\mathcal{M}(g)$.

Proof. To prove that (2) implies (1), suppose $h \in \mathrm{GL}_M(W(k))$ is an isomorphism from $\mathcal{M}(g_s)$ to $\mathcal{M}(g)$. For every F-basis $\mathcal{B}$ of $\mathcal{M}(g_s)$, there is an F-basis $h(\mathcal{B})$ of $\mathcal{M}(g)$, and the reduction of h modulo p^s is an isomorphism of filtered F-crystals modulo p^s:

$$\begin{aligned} h[s] : (M/p^sM, F^{\bullet}_{\mathcal{B}}(M)_s, (g_s\varphi)_{F^{\bullet}_{\mathcal{B}}(M)}[s]) \\ \to (M/p^sM, F^{\bullet}_{h(\mathcal{B})}(M)_s, (g\varphi)_{F^{\bullet}_{h(\mathcal{B})}(M)}[s]). \end{aligned}$$

As $g_s \equiv 1_M$ modulo p^s and $\mathcal{B}$ is also an ordered F-basis of $\mathcal{M}$, we have a canonical identification of filtered F-crystals modulo p^s:

$$\mathrm{id}[s] : (M/p^sM, F^{\bullet}_{\mathcal{B}}(M)_s, \varphi_{F^{\bullet}_{\mathcal{B}}(M)}[s]) \cong (M/p^sM, F^{\bullet}_{\mathcal{B}}(M)_s, (g_s\varphi)_{F^{\bullet}_{\mathcal{B}}(M)}[s]).$$

Composing the two isomorphisms $h[s] \circ \mathrm{id}[s] = h[s]$, we get the desired isomorphism (2.1) by taking $\mathcal{B}_1 = \mathcal{B}$ and $\mathcal{B}_2 = h(\mathcal{B})$.

To prove that (1) implies (2), let $g_s = h^{-1}g\varphi h\varphi^{-1}$. We claim that g_s belongs to $\mathrm{GL}_M(W(k))$, which is equivalent to $h(\varphi^{-1}(M)) \subset \varphi^{-1}(M)$. As $M = \bigoplus_{j=1}^{t} p^{-f_j}\varphi(\widetilde{F}^j_{\mathcal{B}_1}(M))$, it is enough to show that

$$h(\widetilde{F}^j_{\mathcal{B}_1}(M)) \subset \bigoplus_{i=1}^{t} p^{\max(0,f_j-f_i)}\widetilde{F}^i_{\mathcal{B}_1}(M) = \varphi^{-1}(p^{f_j}M) \cap M.$$

Indeed, for each $v \in \widetilde{F}^j_{\mathcal{B}_1}(M) \subset F^j_{\mathcal{B}_1}(M)$, we have $h\varphi_{F^j_{\mathcal{B}_1}(M)}(v) - g\varphi_{F^j_{\mathcal{B}_2}(M)}h(v) \in p^sM$, therefore $h\varphi(v) - g\varphi h(v) \in p^{s+f_j}M$. As $v \in \widetilde{F}^j_{\mathcal{B}_1}(M)$, we know that $\varphi(v) \in p^{f_j}M$ and thus $h\varphi(v) \in p^{f_j}M$. By the last two sentences, we know that $g\varphi h(v) \in p^{f_j}M$, whence $\varphi h(v) \in p^{f_j}M$. This implies that $h(v) \in \varphi^{-1}(p^{f_j}M) \cap M$. As

$$h^{-1} : (M, g\varphi) \cong (M, h^{-1}g\varphi h) = (M, g_s\varphi),$$

it remains to prove that g_s is congruent to 1_M modulo p^s. As $\mathcal{B}_2$ is an F-basis of $\mathcal{M}(g)$, $h^{-1}(\mathcal{B}_2)$ is an F-basis of $\mathcal{M}(g_s)$. We have an isomorphism of filtered F-crystals modulo p^s as follows:

$$\begin{aligned} h^{-1}[s] : (M/p^sM, F^\bullet_{\mathcal{B}_2}(M)_s, (g\varphi)_{F^\bullet_{\mathcal{B}_2}(M)}[s]) \\ \longrightarrow (M/p^sM, F^\bullet_{h^{-1}(\mathcal{B}_2)}(M)_s, (g_s\varphi)_{F^\bullet_{h^{-1}(\mathcal{B}_2)}(M)}[s]). \end{aligned}$$

Composing the isomorphism (2.1) with the last isomorphism, we have an isomorphism

$$\begin{aligned} \mathrm{id}[s] : (M/p^sM, F^\bullet_{\mathcal{B}_1}(M)_s, \varphi_{F^\bullet_{\mathcal{B}_1}(M)}[s]) \\ \longrightarrow (M/p^sM, F^\bullet_{h^{-1}(\mathcal{B}_2)}(M)_s, (g_s\varphi)_{F^\bullet_{h^{-1}(\mathcal{B}_2)}(M)}[s]). \end{aligned}$$

For every $1 \le j \le t$, and for each $v \in \widetilde{F}^j_{\mathcal{B}_1}(M)$, we have

$$(g_s\varphi)_{F^j_{h^{-1}(\mathcal{B}_2)}(M)}(v) - \varphi_{F^j_{\mathcal{B}_1}(M)}(v) \in p^sM.$$

This means that $g_s(p^{-f_j}\varphi(v)) - p^{-f_j}\varphi(v) \in p^sM$, that is, g_s fixes every element of $p^{-j}\varphi(\widetilde{F}^j_{\mathcal{B}_1}(M))$ modulo p^s. Because $M = \bigoplus_{j=1}^{t} p^{-f_j}\varphi(\widetilde{F}^j_{\mathcal{B}_1}(M))$, we know that g_s fixes every element of M modulo p^s, whence $g_s \equiv 1_M$ modulo p^s. □

Proposition 2.3. *For all $g, h \in \mathrm{GL}_M(W(k))$, the reduction of h modulo p^s is an isomorphism from $F_s(\mathcal{M})$ to $F_s(\mathcal{M}(g))$ if and only if $h^{-1}g\varphi h\varphi^{-1} \equiv 1_M$ modulo p^s.*

Proof. For every $h \in \mathrm{GL}_M(W(k))$, if the reduction of h modulo p^s is an isomorphism from $F_s(\mathcal{M})$ to $F_s(\mathcal{M}(g))$, then $h : \mathcal{M}(g_s) \to \mathcal{M}(g)$ is an isomorphism of F-crystals where $g_s \equiv h^{-1}g\varphi h\varphi^{-1} \equiv 1_M$ modulo p^s by Lemma 2.2.

If $h^{-1}g\varphi h\varphi^{-1} \equiv 1_M$ modulo p^s, then there exists $g_s \equiv h^{-1}g\varphi h\varphi^{-1}$ congruent to 1_M modulo p^s such that h induces an isomorphism from $\mathcal{M}(g_s)$ to $\mathcal{M}(g)$. For every F-basis $\mathcal{B}$ of $\mathcal{M}$, which is also an F-basis of $\mathcal{M}(g_s)$, we get an isomorphism of filtered F-crystals modulo p^s:

$$\begin{aligned} h[s] : (M/p^sM, F^\bullet_{\mathcal{B}}(M)_s, \varphi_{F^\bullet_{\mathcal{B}}(M)}[s]) \\ \to (M/p^sM, F^\bullet_{h(\mathcal{B})}(M)_s, (g\varphi)_{F^\bullet_{h(\mathcal{B})}(M)}[s]). \end{aligned}$$ □

Corollary 2.4. *Let s be a positive integer. We recall that $\Gamma_{f,s} : F_s(\mathcal{M}) \to F_s(\mathcal{M}(g))$ is the function defined by an isomorphism f of F-truncations modulo p^s from $F_s(\mathcal{M})$ to $F_s(\mathcal{M}(g))$ (see the paragraph after Definition 2.1 for its definition). Then the function $\Gamma_{f,s}$ is a bijection.*

Proof. Let $h \in \mathrm{GL}_M(W(k))$ be a preimage of $f \in \mathrm{GL}_M(W_s(k))$ via the canonical surjection $\mathrm{GL}_M(W(k)) \to \mathrm{GL}_M(W_s(k))$. By Proposition 2.3, we have $h^{-1}g\varphi h\varphi^{-1} \equiv 1_M$ modulo p^s. Taking inverses on both hand sides, we have $\varphi h^{-1}\varphi^{-1}g^{-1}h \equiv 1_M$ modulo p^s. After multiplying h on the left and h^{-1} on the right on both hand sides, we get $h\varphi h^{-1}\varphi^{-1}g^{-1} \equiv 1$ modulo p^s, that is, $h\varphi h^{-1}(g\varphi)^{-1} \equiv 1_M$ modulo p^s. Hence h^{-1} defines an isomorphism of F-truncations modulo p^s from $F_s(\mathcal{M}(g))$ to $F_s(\mathcal{M})$. This implies that $\Gamma_{f,s}$ is a bijection. □

The next corollary justifies that the isomorphism number of F-crystals is the right generalization of the isomorphism number of p-divisible groups.

Corollary 2.5. *Let $t_{\mathcal{M}}$ be the smallest integer such that for all $g \in \mathrm{GL}_M(W(k))$, if $F_{t_{\mathcal{M}}}(\mathcal{M})$ is isomorphic to $F_{t_{\mathcal{M}}}(\mathcal{M}(g))$, then $\mathcal{M}$ is isomorphic to $\mathcal{M}(g)$. We have $t_{\mathcal{M}} = n_{\mathcal{M}}$.*

Proof. If $F_{n_{\mathcal{M}}}(\mathcal{M})$ is isomorphic to $F_{n_{\mathcal{M}}}(\mathcal{M}(g))$, then by Lemma 2.2, there exists $g_{n_{\mathcal{M}}} \in \mathrm{GL}_M(W(k))$ with the property that $g_{n_{\mathcal{M}}} \equiv 1_M$ modulo $p^{n_{\mathcal{M}}}$ such that $\mathcal{M}(g_{n_{\mathcal{M}}})$, which is isomorphic to $\mathcal{M}$ by the definition of isomorphism numbers, is isomorphic to $\mathcal{M}(g)$. Thus $t_{\mathcal{M}} \leq n_{\mathcal{M}}$.

Let $g_{t_{\mathcal{M}}} \equiv 1_M$ modulo $p^{t_{\mathcal{M}}}$. By Proposition 2.3, $1_M[t_{\mathcal{M}}] \in \mathrm{GL}_M(W_{t_{\mathcal{M}}}(k))$ is an isomorphism from $F_{t_{\mathcal{M}}}(\mathcal{M})$ to $F_{t_{\mathcal{M}}}(\mathcal{M}(g_{t_{\mathcal{M}}}))$. By definition of $t_{\mathcal{M}}$, $\mathcal{M}$ is isomorphic to $\mathcal{M}(g_{t_{\mathcal{M}}})$. Thus $n_{\mathcal{M}} \leq t_{\mathcal{M}}$. □

Proposition 2.3 motivates the following definition of a homomorphism modulo p^s between two F-crystals.

Definition 2.6. *A $W_s(k)$-linear map $h[s] : M_1/p^sM_1 \to M_2/p^sM_2$ is a homomorphism from $F_s(\mathcal{M}_1)$ to $F_s(\mathcal{M}_2)$ if a preimage $h \in \mathrm{Hom}_{W(k)}(M_1, M_2)$ of $h[s]$ under the canonical surjection $\mathrm{Hom}_{W(k)}(M_1, M_2) \to \mathrm{Hom}_{W_s(k)}(M_1/p^sM_1, M_2/p^sM_2)$ satisfies $\varphi_2 h \varphi_1^{-1} \equiv h$ modulo p^s. We call h a lift of $h[s]$ and $h[s]$ a homomorphism modulo p^s from $\mathcal{M}_1$ to $\mathcal{M}_2$.*

Remark 2.7. A homomorphism $h[s]$ modulo p^s between $\mathcal{M}_1$ and $\mathcal{M}_2$ implicitly implies that there exists a lift h of $h[s]$ in $\mathrm{Hom}_{W(k)}(M_1, M_2)$ such that $\varphi_2 h \varphi_1^{-1}$ is also an element in $\mathrm{Hom}_{W(k)}(M_1, M_2)$. Note that $h[s]$ is *not* just a $W_s(k)$-linear homomorphism $h[s] : M_1/p^sM_1 \to M_2/p^sM_2$ such that $h\varphi_1 \equiv \varphi_2 h$ modulo p^s, although this is a consequence of the definition but it is not equivalent to the definition.

Remark 2.8. Note that the definition of an isomorphism between two filtered F-crystals modulo p^s requires that the two F-crystals have the same Hodge polygon described in Subsection 2.1. In Proposition 2.3 we also require that the two F-crystals have the same Hodge polygon. On the other hand, in Definition 2.6, we do not require that the two F-crystals have the same Hodge polygon. It is reasonable to ask if $h[s] \in \mathrm{GL}_M(W_s(k))$ and there exists a lift $h \in \mathrm{GL}_M(W(k))$ of $h[s]$ such that $\varphi_2 h \varphi_1^{-1} \equiv h$ modulo p^s, do (M, φ_1) and (M, φ_2) have the same Hodge polygon so that $h[s]$ induces an isomorphism between $F_s(\mathcal{M}_1)$ and $F_s(\mathcal{M}_2)$? The answer is yes because if $\varphi_2 h \varphi_1^{-1} \equiv h$ modulo p^s, then we know that $\varphi_2 h \varphi_1^{-1} \in \mathrm{GL}_M(W(k))$. Thus $\varphi_2 h \varphi_1^{-1}(\varphi_1(M)) = \varphi_2 h(M) = \varphi_2(M)$. As a result, $\varphi_2 h \varphi_1^{-1}$ induces an isomorphism from $M/\varphi_1(M)$ to $M/\varphi_2(M)$ and thus (M, φ_1) and (M, φ_2) have the same Hodge polygon. Therefore if $\mathcal{M}_1$ and $\mathcal{M}_2$ have different Hodge polygons, then $F_s(\mathcal{M}_1)$ and $F_s(\mathcal{M}_2)$ are not isomorphic modulo p^s.

Proposition 2.9. *Let $s \geq 1$ be an integer. A homomorphism $h[s] : M_1/p^sM_1 \to M_2/p^sM_2$ is a homomorphism from $F_s(\mathcal{M}_1)$ to $F_s(\mathcal{M}_2)$ if and only if there exists a lift h of $h[s]$ in $\mathrm{Hom}_{W(k)}(M_1, M_2)$ such that for every $x \in M_1 \backslash pM_1$, if $\varphi_1(x) \in p^iM_1 \backslash p^{i+1}M_1$, then $h\varphi_1(x) \equiv \varphi_2 h(x)$ modulo p^{s+i}. Moreover, if we fix an F-basis $\mathcal{B}_1 = \{v_1, v_2, \dots, v_r\}$ of $\mathcal{M}_1$, then the condition "for every $x \in M_1 \backslash pM_1$" in the prior sentence can be strengthen to "for all $x \in \mathcal{B}_1$".*

Proof. Let $h[s]$ be a homomorphism from $F_s(\mathcal{M}_1)$ to $F_s(\mathcal{M}_2)$, then there exists $h \in \mathrm{Hom}_{W(k)}(M_1, M_2)$ such that $\varphi_2 h \varphi_1^{-1} \equiv h$ modulo p^s. Let $x \in M_1 \backslash pM_1$ be such that $\varphi_1(x) \in p^iM_1 \backslash p^{i+1}M_1$, whence $\frac{1}{p^i}\varphi_1(x) \in M_1 \backslash pM_1$. Plugging $\frac{1}{p^i}\varphi_1(x)$ into $\varphi_2 h \varphi_1^{-1} \equiv h$ modulo p^s gives the desired congruence $h\varphi_1(x) = \varphi_2 h(x)$ modulo p^{s+i}.

Suppose $h[s] \in \mathrm{Hom}(M_1/p^sM_1, M_2/p^sM_2)$ satisfies that for every $x \in M_1 \backslash pM_1$, if $\varphi_1(x) \in p^iM_1 \backslash p^{i+1}M_1$, then $h\varphi_1(x) \equiv \varphi_2 h(x)$ modulo

p^{s+i}. For every $x \in M_1\backslash\{0\}$, there exists $l \geq 0$ such that $x \in p^l M_1\backslash p^{l+1}M_1$. We write $\varphi_1^{-1}(x) = p^j x'$ for some $j \in \mathbb{Z}$ and $x' \in M_1\backslash pM_1$. Therefore $\varphi_1(x') \in p^{l-j}M_1\backslash p^{l-j+1}M_1$. Plugging $x' = p^{-j}\varphi_1^{-1}(x)$ into the congruence $\varphi_2 h \equiv h\varphi_1$ modulo p^{s+l-j}, we get $\varphi_2 h\varphi_1^{-1}(x) \equiv h(x)$ modulo p^s as $l \geq 0$.

To prove the strengthening part, for all $x \in M_1\backslash pM_1$, $x = \sum_{i=1}^r x_i v_i$ for some $x_i \in W(k)$, we have $\varphi_1(x) = \sum_{i=1}^r p^{e_i}\sigma(x_i)w_i$ for some F-basis $\{w_1, w_2, \ldots, w_r\}$ of $\mathcal{M}_2$. Let $i = \min\{e_i + \mathrm{ord}_p(x_i) \mid 1 \leq i \leq r\}$. Then $\varphi_1(x) \in p^i M_1\backslash p^{i+1}M_1$. Suppose for every $1 \leq i \leq r$, $h\varphi_1(v_i) - \varphi_1 h(v_i) = p^{s+e_i}v_i'$ for some $v_i' \in M_2\backslash pM_2$, we conclude the proof by considering the difference

$$\begin{aligned} h\varphi_1(x) - \varphi_1 h(x) &= \sum_{i=1}^r \sigma(x_i)h(\varphi_1(v_i)) - \sum_{i=1}^r \sigma(x_i)\varphi_1(h(v_i)) \\ &= \sum_{i=1}^r p^{s+e_i}\sigma(x_i)v_i' \in p^{s+i}M_2. \end{aligned}$$

□

Corollary 2.10. *Let $\mathcal{M}$ be an F-crystal over k and let $\mathcal{B} = \{v_1, v_2, \ldots, v_r\}$ be an F-basis of $\mathcal{M}$. For all $g, h \in \mathrm{GL}_M(W(k))$, the reduction of h modulo p^s is an isomorphism between $F_s(\mathcal{M})$ and $F_s(\mathcal{M}(g))$ if and only if for all $v_i \in \mathcal{B}$ we have $h\varphi_1(v_i) \equiv \varphi_2 h(v_i)$ modulo p^{s+e_i} where $e_1 \leq e_2 \leq \cdots \leq e_r$ are the Hodge slopes of $\mathcal{M}$.*

We denote by $\mathrm{Hom}_s(\mathcal{M}_1, \mathcal{M}_2)$ the (additive) group of all homomorphisms modulo p^s from $\mathcal{M}_1$ to $\mathcal{M}_2$, that is, all homomorphisms from $F_s(\mathcal{M}_1)$ to $F_s(\mathcal{M}_2)$. For $i = 1, 2$, if $h_i[s] \in \mathrm{GL}_M(W_s(k))$ is an automorphism of $F_s(\mathcal{M})$, and $h_i \in \mathrm{GL}_M(W(k))$ is a lift of $h_i[s]$ such that $\varphi h_i \varphi^{-1} \equiv h_i$ modulo p^s, then $(h_1h_2)[s]$ is also an automorphism of $F_s(\mathcal{M})$ as $h_1h_2 \in \mathrm{GL}_M(W(k))$ is a lift of $(h_1h_2)[s]$ that satisfies

$$\begin{aligned} (h_1h_2)^{-1}\varphi(h_1h_2)\varphi^{-1} &\equiv h_2^{-1}(h_1^{-1}\varphi h_1\varphi^{-1})\varphi h_2\varphi^{-1} \\ &\equiv h_2^{-1}\varphi h_2\varphi^{-1} \equiv 1 \text{ modulo } p^s. \end{aligned}$$

Thus all automorphisms of $F_s(\mathcal{M})$ form an abstract group $\mathrm{Aut}_s(\mathcal{M})$ under composition.

2.3 $\mathbb{W}_s$ functor

For every affine scheme $\mathbf{X}$ over Spec $W(k)$, there is a functor $\mathbb{W}_s(\mathbf{X})$ from the category of affine schemes over k to the category of sets defined as follows: For every affine scheme Spec R,

$$\mathbb{W}_s(\mathbf{X})(\mathrm{Spec}\, R) := \mathbf{X}(W_s(R)).$$

If $\mathbf{X}$ is of finite type over $W(k)$, it is known that this functor is representable by an affine k-scheme of finite type (see [5, p. 639 Corollary 1]), which will be denoted by $\mathbb{W}_s(\mathbf{X})$. If in addition $\mathbf{X}$ is smooth over Spec $W(k)$, then $\mathbb{W}_s(\mathbf{X})$ is smooth. Indeed, for every k-algebra R and an ideal I of R such that $I^2 = 0$, the kernel of $W_s(R) \to W_s(R/I)$ is of square zero. As $\mathbf{X}$ is smooth, we get that

$$\mathbb{W}_s(\mathbf{X})(R) = \mathbf{X}(W_s(R)) \to \mathbf{X}(W_s(R/I)) = \mathbb{W}_s(\mathbf{X})(R/I)$$

is surjective by [1, Ch. 2, Sec. 2, Prop. 6], whence $\mathbb{W}_s(\mathbf{X})$ is smooth by the loc. cit. Suppose $\mathbf{X}$ is a smooth affine group scheme over Spec $W(k)$, then $\mathbb{W}_s(\mathbf{X})$ is a smooth affine group scheme over k. The reduction epimorphism $W_{s+1}(R) \to W_s(R)$ naturally induces a smooth epimorphism of affine group schemes over k

$$\mathrm{Red}_{s+1,\mathbf{X}} : \mathbb{W}_{m+1}(\mathbf{X}) \to \mathbb{W}_m(\mathbf{X}).$$

The kernel of $\mathrm{Red}_{s+1,\mathbf{X}}$ is a unipotent commutative group isomorphic to $\mathbb{G}_a^{\dim(\mathbf{X}_k)}$. Identifying $\mathbb{W}_1(\mathbf{X}) = \mathbf{X}_k$, an inductive argument shows that $\dim(\mathbb{W}_s(\mathbf{X})) = s \cdot \dim(\mathbf{X}_k)$ and $\mathbb{W}_s(\mathbf{X})$ is connected if and only if $\mathbf{X}_k$ is connected.

2.4 Group schemes pertaining to F-truncations modulo p^s

In this subsection, we construct a smooth (additive) group scheme $\mathbf{Hom}_s(\mathcal{M}_1, \mathcal{M}_2)$ of finite type over k such that its group of k-valued points is $\mathrm{Hom}_s(\mathcal{M}_1, \mathcal{M}_2)$, and a smooth (multiplicative) group scheme $\mathbf{Aut}_s(\mathcal{M})$ of finite type over k such that its group of k-valued points is $\mathrm{Aut}_s(\mathcal{M})$.

Fix $s \geq 1$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two F-crystals over k. Let r_1 and r_2 be the ranks of M_1 and M_2 respectively. We fix $W(k)$-bases $\mathcal{B}_1$ of M_1 and $\mathcal{B}_2$ of M_2 (they are not necessarily F-bases.) Thus a $W(k)$-linear homomorphism $h : M_1 \to M_2$ corresponds to an $r_2 \times r_1$ matrix $X = [h]_{\mathcal{B}_1}^{\mathcal{B}_2} = (x_{ij})_{1\leq i\leq r_2, 1\leq j\leq r_1}$ with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$. Here and in all that follows we adopt the following convention: for any $v \in M_1$, $[h(v)]_{\mathcal{B}_2} = X[v]_{\mathcal{B}_1}$. The Frobenius of $\mathcal{M}_1$ corresponds to an $r_1 \times r_1$ matrix $U = [\varphi_1]_{\mathcal{B}_1}^{\mathcal{B}_1} = (u_{ij})_{1\leq i,j\leq r_1}$ with respect to $\mathcal{B}_1$, and the Frobenius of $\mathcal{M}_2$ corresponds to an $r_2 \times r_2$ matrix $V = [\varphi_2]_{\mathcal{B}_2}^{\mathcal{B}_2} = (v_{ij})_{1\leq i,j\leq r_2}$ with respect to $\mathcal{B}_2$.

Let $W = (w_{ij})_{1\leq i,j\leq r_1}$ be the transpose of the cofactor matrix of U. We have $w_{ij} \in W(k)$. The matrix representation of $\varphi_2 h \varphi_1^{-1}$ with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$ is $V\sigma(X)\sigma(W/\det(U))$. We would like to find conditions on X so that the reduction of h modulo p^s, denoted by $h[s]$, is a homomorphism from

$F_s(\mathcal{M}_1)$ to $F_s(\mathcal{M}_2)$. By definition, the condition $\varphi_2 h \varphi_1^{-1} \equiv h$ modulo p^s is equivalent to the system of equations

$$\frac{1}{\sigma(\det(U))} \sum_{m=1}^{r_1} \sum_{n=1}^{r_2} v_{in}\sigma(x_{nm})\sigma(w_{mj}) \equiv x_{ij} \text{ modulo } p^s, \qquad (2.2)$$

for all $1 \leq i \leq r_1$ and $1 \leq j \leq r_2$. Let $l := \mathrm{ord}_p(\det(U))$, and $\det(U)^{-1} = p^{-l}d$ where $d \in W(k)\backslash pW(k)$. Then the system of equations (2.2) is equivalent to

$$\sum_{m=1}^{r_1} \sum_{n=1}^{r_2} \sigma(d)v_{in}\sigma(x_{nm})\sigma(w_{mj}) \equiv p^l x_{ij} \equiv \theta^l(\sigma^l(x_{ij})) \text{ modulo } p^{s+l}. \quad (2.3)$$

If R is a perfect ring, two elements $u = (u^{(0)}, u^{(1)}, \ldots)$ and $w = (w^{(0)}, w^{(1)}, \ldots)$ of $W(R)$ are congruent modulo p^s if and only if $u^{(i)} \equiv w^{(i)}$ for all $0 \leq i \leq s-1$. This is true because $p^s = (\sigma_R\theta_R)^s = \sigma_R^s\theta_R^s$, and σ_R is an automorphism of $W(R)$ when R is perfect. Thus over perfect rings, the system of equations (2.3) is equivalent to

$$\sum_{m=1}^{r_1} \sum_{n=1}^{r_2} \sigma(d)v_{in}\sigma(x_{nm})\sigma(w_{mj}) \equiv \theta^l(\sigma^l(x_{ij})) \text{ modulo } \theta^{s+l}(W(R)). \quad (2.4)$$

Let $x_{nm} = (x_{nm}^{(0)}, x_{nm}^{(1)}, \ldots)$ and $P_{r,q}$ the polynomial with integral coefficients that computes the q-th coordinate of the p-typical Witt vector which is a product of r p-typical Witt vectors. Then the system of equations (2.4) is equivalent to

$$\sum_{m=1}^{r_1} \sum_{n=1}^{r_2} P_{4,q+l}(\sigma(d), v_{in}, \sigma(x_{nm}), \sigma(w_{mj})) - (x_{ij}^{(q)})^{p^l} = 0 \qquad (2.5)$$

for all $1 \leq i \leq r_1$, $1 \leq j \leq r_2$, and $0 \leq q \leq s-1$, and the equations

$$\sum_{m=1}^{r_1} \sum_{n=1}^{r_2} P_{4,q}(\sigma(d), v_{in}, \sigma(x_{nm}), \sigma(w_{mj})) = 0 \qquad (2.6)$$

for all $1 \leq i \leq r_1$, $1 \leq j \leq r_2$, and $0 \leq q \leq l-1$.

For any three non-negative integers n_1, n_2 and n_3, let R_{n_1,n_2,n_3} be the polynomial k-algebra with variables $x_{ij}^{(q)}$ where $1 \leq i \leq n_1$, $1 \leq j \leq n_2$ and $0 \leq q \leq n_3$. Let $\mathfrak{I}$ be the ideal of $R_{r_1,r_2,s+l-1}$ generated by equations (2.5) and (2.6). Let $\mathbf{Y}_s$ be the scheme theoretic closure of $\mathbf{X}_s = \mathrm{Spec}\, R_{r_1,r_2,s+l-1}/\mathfrak{I}$ under the canonical morphism $\mathrm{Spec}\, R_{r_1,r_2,s+l-1} \to \mathrm{Spec}\, R_{r_1,r_2,s-1}$ induced by the natural inclusion $i : R_{r_1,r_2,s-1} \hookrightarrow R_{r_1,r_2,s+l-1}$. Thus $\mathbf{Y}_s$ is affine and is isomorphic to $\mathrm{Spec}\, R_{r_1,r_2,s-1}/i^{-1}(\mathfrak{I}) =: \mathrm{Spec}\, R_s$.

- If $s \leq l$, then $\imath^{-1}(\mathfrak{J})$ is generated by equations (2.6) for all $1 \leq i \leq r_1$, $1 \leq j \leq r_2$ and $0 \leq q \leq s-1$.
- If $s > l$, then $\imath^{-1}(\mathfrak{J})$ is generated by equations (2.6) for all $1 \leq i \leq r_1$, $1 \leq j \leq r_2$ and $0 \leq q \leq l-1$, and also equations (2.5) for all $1 \leq i \leq r_1$, $1 \leq j \leq r_2$ and $0 \leq q \leq s-l-1$.

For each k-algebra R (not necessarily perfect), the set of R-valued points $\mathbf{Y}_s(R)$ is set of all $W_s(R)$-linear maps

$$h[s] : M_1 \otimes_{W_s(k)} W_s(R) \to M_2 \otimes_{W_s(k)} W_s(R)$$

with the property that there exists a lift

$$h : M_1 \otimes_{W(k)} W(R) \to M_2 \otimes_{W(k)} W(R))$$

such that for each $x \in M$, if $\varphi_1(x) \in p^i M \backslash p^{i+1} M$, then we have

$$h \circ (\varphi_1 \otimes_{W(k)} \sigma_R)(x \otimes 1_{W(R)}) \equiv (\varphi_2 \otimes_{W(k)} \sigma_R) \circ h(x \otimes 1_{W(R)})$$

modulo $M \otimes_{W(k)} \theta^{i+s}(W(R))$. It is clear that $\mathbf{Y}_s(R)$ has a functorial group structure under addition, and thus $\mathbf{Y}_s$ is a group scheme. Let $\mathbf{Hom}_s(\mathcal{M}_1, \mathcal{M}_2) := (\mathbf{Y}_s)_{\mathrm{red}}$. If no confusions can occur, we denote $\mathbf{Hom}_s(\mathcal{M}_1, \mathcal{M}_2)$ by $\mathbf{H}_s$. From the construction of $(R_s)_{\mathrm{red}}$, it is clear that $\mathbf{H}_s$ is a smooth group scheme of finite type over k, and $\mathbf{H}_s(k) = \mathrm{Hom}_s(\mathcal{M}_1, \mathcal{M}_2)$.

The definition of $\mathbf{H}_s$ would not be very useful if it would depend on the choices of $\mathcal{B}_1$ and $\mathcal{B}_2$. We now show that $\mathbf{H}_s$ does not depend on the choices of $\mathcal{B}_1$ and $\mathcal{B}_2$. Let $\mathcal{B}_1'$ and $\mathcal{B}_2'$ be other $W(k)$-bases of M_1 and M_2 respectively. Let $T = (t_{ij})$ be the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_1'$ and $T^{-1} = (t_{ij}')$ be its inverse. Let $S = (s_{ij})$ be the change of basis matrix from $\mathcal{B}_2$ to $\mathcal{B}_2'$ and $S^{-1} = (s_{ij}')$ be its inverse. Let $U' = [\varphi_1]_{\mathcal{B}_1'}^{\mathcal{B}_1'}$ and $V' = [\varphi_2]_{\mathcal{B}_2'}^{\mathcal{B}_2'}$ be the matrix representations of φ_1 and φ_2 with respect to $\mathcal{B}_1'$ and $\mathcal{B}_2'$ respectively. We get that $T^{-1}U'\sigma(T) = U$, $S^{-1}V'\sigma(S) = V$ and $TU^{-1}\sigma^{-1}(T^{-1}) = U'^{-1}$. Let W' be the transpose of the cofactor matrix of U', then $W'/\det(U') = U'^{-1}$. Let Y be the $r_2 \times r_1$ matrix $[h]_{\mathcal{B}_1'}^{\mathcal{B}_2'} = (y_{ij})_{1 \leq i \leq r_2, 1 \leq j \leq r_1}$ representing h with respect to $\mathcal{B}_1'$ and $\mathcal{B}_2'$. Therefore we have $X = S^{-1}YT$. By solving $V'\sigma(Y)\sigma(W'/\det(U')) \equiv Y$ modulo p^s, we get a similar system of equations like (2.5) and (2.6), with d replaced by d', v_{in} replaced by v_{in}', x_{nm} replaced by y_{nm}, and w_{mj} replaced by w_{mj}'. They generate an ideal $\mathfrak{J}'$ of a polynomial algebra $R_{r_1,r_2,s+l-1}'$ with variables $y_{ij}^{(q)}$. We now construct an isomorphism $\imath : R_{r_1,r_2,s+l-1} \to R_{r_1,r_2,s+l-1}'$ induced by the equality $X = S^{-1}YT$. More precisely, as the (i,j)-entry of $S^{-1}YT$ is $\sum_{l,m} s_{il}' y_{lm} t_{mj}$, we define $\imath(x_{ij}^{(q)}) = \sum_{l,m} P_{3,q}(s_{il}', y_{lm}, t_{mj})$. It is easy to see that $\imath$ is an isomorphism as its inverse η can be constructed by the equality $Y = SXT^{-1}$

in a similar way. Now we show that ι induces a well-defined homomorphism $\iota : R_{r_1,r_2,s+l-1}/\mathfrak{I} \to R'_{r_1,r_2,s+l-1}/\mathfrak{I}'$. Suppose that $f \in \mathfrak{I}$, then we want to show that $\iota(f) \in \mathfrak{I}'$. This is equivalent to show that if $V\sigma(X)\sigma(U^{-1}) = X$, then $V'\sigma(Y)\sigma(U'^{-1}) = Y$, assuming that $X = S^{-1}YT$. Indeed, we have $TU^{-1}\sigma^{-1}(T^{-1}) = U'^{-1}$, and $S^{-1}V'\sigma(S) = V$, we get

$$\begin{aligned} Y &= SXT^{-1} = SV\sigma(X)\sigma(U^{-1})T^{-1} = S(S^{-1}V'\sigma(S))\sigma(X)\sigma(U^{-1})T^{-1} \\ &= V'\sigma(S)\sigma(X)\sigma(T^{-1}U'^{-1}\sigma^{-1}(T))T^{-1} = V'\sigma(Y)\sigma(U'^{-1})TT^{-1} \\ &= V'\sigma(Y)\sigma(U'^{-1}). \end{aligned}$$

Thus the induced ι is well-defined. By the same token, we can show that the inverse η also induces a well-defined homomorphism at the level of quotient k-algebras. As ι and η are inverses of each other, we know that $R_{r_1,r_2,s+l-1}/\mathfrak{I} \cong R'_{r_1,r_2,s+l-1}/\mathfrak{I}'$. Let $\mathbf{Y}'_s$ be the scheme theoretic closure of Spec $R'_{r_1,r_2,s+l-1}/\mathfrak{I}'$ under the canonical morphism Spec $R'_{r_1,r_2,s+l-1} \to$ Spec $R'_{r_1,r_2,s-1}$ induced by the natural inclusion $i' : R'_{r_1,r_2,s-1} \hookrightarrow R'_{r_1,r_2,s+l-1}$. It is clear that $\mathbf{Y}_s$ is isomorphic to $\mathbf{Y}'_s$ as k-schemes. To see that they are also isomorphic as k-group schemes under addition, it is enough to see that the definition ι and η respect addition because if $X_1 = S^{-1}Y_1T$ and $X_2 = S^{-1}Y_2T$, then $X_1 + X_2 = S^{-1}(Y_1 + Y_2)T$. Thus the definition of $\mathbf{H}_s$ does not depend on the choice of basis.

If $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, then $r_1 = r_2 = r$. In this case, we denote $\mathbf{Hom}_s(\mathcal{M}_1, \mathcal{M}_2)$ by $\mathbf{End}_s(\mathcal{M})$ or for simplicity $\mathbf{E}_s$ if no confusions can occur.

Now we assume that $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$ (thus $r_1 = r_2 = r$) and construct a group scheme $\mathbf{Aut}_s(\mathcal{M})$ whose k-valued points is $\mathrm{Aut}_s(\mathcal{M})$. Here we make use of a simple fact of Witt vectors: for any k-algebra R, an element $x \in W(R)$ is invertible if and only if $x^{(0)}$ is a unit in R. Put

$$T_s := R_{r,r,s-1}\left[\frac{1}{\det(x_{ij}^{(0)})}\right] \Big/ i^{-1}(\mathfrak{I}).$$

Then Spec $T_s(R)$ contains all the multiplicative invertible elements in $\mathbf{Y}_s(R)$. It is the set of all $W_s(R)$-linear automorphisms

$$h[s] : M \otimes_{W_s(k)} W_s(R) \to M \otimes_{W_s(k)} W_s(R)$$

with the property that there exists a lift $h \in \mathrm{GL}_{M\otimes_{W(k)}W(R)}(W(R))$ of $h[s]$ such that for each $x \in M$, if $\varphi(x) \in p^iM \backslash p^{i+1}M$, then we have

$$h \circ (\varphi \otimes_{W(k)} \sigma_R)(x \otimes 1_{W(R)}) \equiv (\varphi \otimes_{W(k)} \sigma_R) \circ h(x \otimes 1_{W(R)})$$

modulo $M \otimes_{W(k)} \theta^{i+s}(W(R))$. If $h_1[s], h_2[s] \in \operatorname{Spec} T_s(R)$, then $(h_1h_2)[s]$ is in $\operatorname{Spec} T_s(R)$. Here $(h_1h_2)[s]$ is h_1h_2 modulo θ^s. It coincides with the notation that $h[s]$ is h modulo p^s when R is perfect. Hence $\operatorname{Spec} T_s(R)$ has a functorial group structure under composition and thus $\operatorname{Spec} T_s$ is a group scheme. Let $\mathbf{A}_s = \mathbf{Aut}_s(\mathcal{M}) := \operatorname{Spec}(T_s)_{\text{red}}$. Then $\mathbf{A}_s(k) = \operatorname{Aut}_s(\mathcal{M})$ is the group under composition of automorphisms of F-truncations modulo p^s of $\mathcal{M}$. From the construction of $(T_s)_{\text{red}}$, it is clear that $\mathbf{A}_s$ is a smooth group scheme of finite type over k and, as a scheme, it is an open subscheme of $\mathbf{E}_s$. We now study an important invariant $\gamma_{\mathcal{M}}(s) := \dim(\mathbf{Aut}_s(\mathcal{M}))$ associated to $\mathcal{M}$. As $\mathbf{E}_s$ is smooth, all connected components of $\mathbf{E}_s$ have the same dimension. Therefore $\gamma_{\mathcal{M}}(s) = \dim(\mathbf{E}_s)$.

Proposition 2.11. *For every $l \geq 0$, the sequence $(\gamma_{\mathcal{M}}(s+l) - \gamma_{\mathcal{M}}(s))_{s\geq 1}$ is a non-increasing sequence of non-negative integers. Therefore, we have a chain of inequalities $0 \leq \gamma_{\mathcal{M}}(1) \leq \gamma_{\mathcal{M}}(2) \leq \cdots$.*

Proof. For each pair of integers $t \geq s$, there is a canonical reduction homomorphism $\pi_{t,s} : \mathbf{E}_t \to \mathbf{E}_s$. For every perfect k-algebra R, and every $h[s] \in \mathbf{E}_s(R)$, there is a lift h of $h[s]$ such that $\varphi h \varphi^{-1} \equiv h$ modulo p^s, then $\varphi p^{t-s} h \varphi^{-1} \equiv p^{t-s} h$ modulo p^t. Hence we get a monomorphism $p^{t-s} : \mathbf{E}_s \to \mathbf{E}_t$ that sends $h[s]$ to $p^{t-s}h[s]$ at the level of R-valued points. For every perfect k-algebra R and every $h[t] \in \mathbf{E}_t(R)$, $h[t] = p^{t-s}(h'[s])$ for some $h'[s] \in \mathbf{E}_s(R)$ if and only if $h[s]$ belongs to the kernel of $\pi_{t,t-s}$. Hence we have an exact sequence on the level of R-valued points

$$0 \longrightarrow \mathbf{E}_s(R) \xrightarrow{p^{t-s}} \mathbf{E}_t(R) \xrightarrow{\pi_{t,t-s}} \mathbf{E}_{t-s}(R).$$

The dimension of $\operatorname{Im}(\pi_{t,t-s})$ is equal to $\gamma_{\mathcal{M}}(t) - \gamma_{\mathcal{M}}(s) \geq 0$. Because $\pi_{s+1+l,l} = \pi_{s+l,l} \circ \pi_{s+1+l,s+l}$, $\operatorname{Im}(\pi_{s+1+l,l})$ is a subgroup scheme of $\operatorname{Im}(\pi_{s+1,l})$. Hence the dimension of $\operatorname{Im}(\pi_{s+1+l,l})$, which is $\gamma_{\mathcal{M}}(s+1+l) - \gamma_{\mathcal{M}}(s+1)$, is less than or equal to the dimension of $\operatorname{Im}(\pi_{s+l,l})$, which is $\gamma_{\mathcal{M}}(s+l) - \gamma_{\mathcal{M}}(s)$. □

Recall an F-crystal $\mathcal{M}$ is ordinary if its Hodge polygon and Newton polygon coincide. It is well known that the ordinary F-crystals over k are precisely those F-crystals over k which are direct sums of F-crystals of rank 1.

Proposition 2.12. *Let $\mathcal{M}$ be an ordinary F-crystal, then $\gamma_{\mathcal{M}}(s) = 0$ for all $s \geq 1$.*

Proof. If $\mathcal{M}$ is ordinary, then $\mathcal{M} = \oplus_{i=1}^{t} \mathcal{M}_i$ where $\mathcal{M}_i$ are isoclinic ordinary F-crystals. Thus there exists an F-basis $\mathcal{B} = \{v_1, v_2, \ldots, v_r\}$ of M such that $\varphi(v_i) = p^{e_i} v_i$ for $1 \leq i \leq r$. The ideal $\mathfrak{I}$ that defines the representing k-algebra of $\mathbf{E}_s(\mathcal{M})$ is generated by equations of the following two types:

(1) $\sigma(x_{ij}^{(r)}) - x_{ij}^{(r)}$ for all r and $i, j \in I_l$ for all $1 \leq l \leq t$;

(2) $x_{ij}^{(r)}$ for all r and i, j that don't belong to the same I_l.

It is clear now that representing k-algebra is finite dimensional over k. Thus $\mathbf{E}_s$ is of dimension zero, so is $\mathbf{A}_s$. □

3. Isomorphism classes of F-truncations

In this section, we follow the ideas of [4] and [14] to define a group action for each $s \geq 1$ whose orbits parametrize the isomorphism classes of $F_s(\mathcal{M}(g))$ for all $g \in \mathrm{GL}_M(W(k))$. We show that the stabilizer of the identity element of this action has the same dimension as $\mathbf{Aut}_s(\mathcal{M})$, which allows us to study the non-decreasing sequence $(\gamma_{\mathcal{M}}(i))_{i\geq 1}$ via the orbits and the stabilizers of the action. The main result of this section is Theorem 3.15, which is a partial generalization of [4, Theorem 1]. It will play an important role in the proof of the Main Theorem in Section 5.

3.1 Group schemes

In this subsection, we will introduce some affine group schemes that are necessary to define the group actions in order to study isomorphism classes of F-truncations.

Let $\mathcal{M} = (M, \varphi)$ be an F-crystal over k. Recall $\mathbf{GL}_M$ is the group scheme over Spec $W(k)$ with the property that for every $W(k)$-algebra S, $\mathbf{GL}_M(S)$ is the group of S-linear automorphism of $M \otimes_{W(k)} S$. Put $V = M \otimes_{W(k)} B(k)$, then we have canonical identifications

$$\mathbf{GL}_V = \mathbf{GL}_M \times_{W(k)} \mathrm{Spec}\, B(k) = \mathbf{GL}_{\varphi^{-1}(M)} \times_{W(k)} \mathrm{Spec}\, B(k).$$

Let $\mathbf{G}$ be the scheme theoretic closure of $\mathbf{GL}_V$ in $\mathbf{GL}_M \times \mathbf{GL}_{\varphi^{-1}(M)}$ embedded via the composite homomorphism

$$\mathbf{GL}_V \xrightarrow{\Delta} \mathbf{GL}_V \times \mathbf{GL}_V \to \mathbf{GL}_M \times \mathbf{GL}_{\varphi^{-1}(M)}.$$

For any flat $W(k)$-algebra S, $\mathbf{G}(S)$ contains all $h \in \mathbf{GL}_{M\otimes_{W(k)}S}(S)$ with the property that $h(\varphi^{-1}(M)\otimes_{W(k)} S) = \varphi^{-1}(M)\otimes_{W(k)} S$. Let $P_{\mathbf{G}} : \mathbf{G} \to \mathbf{GL}_M$ be the composition of the inclusion and the first projection $\mathbf{GL}_M \times \mathbf{GL}_{\varphi^{-1}(M)} \to \mathbf{GL}_M$.

Let $\mathcal{B} = \{v_1, v_2, \ldots, v_r\}$ be an F-basis of $\mathcal{M}$. There are two direct sum decompositions of $M = \bigoplus_{j=1}^t \widetilde{F}_{\mathcal{B}}^j(M) = \bigoplus_{j=1}^t p^{-f_j}\varphi(\widetilde{F}_{\mathcal{B}}^j(M))$, which implies that $\varphi^{-1}(M) = \bigoplus_{j=1}^t p^{-f_j}\widetilde{F}_{\mathcal{B}}^j(M)$. With respect to $\mathcal{B}$, the representing k-algebras of the following affine group schemes are clear:

- $\mathbf{GL}_V = \operatorname{Spec} B(k)[x_{ij} \mid 1 \le i, j \le r][\frac{1}{\det(x_{ij})}]$;
- $\mathbf{GL}_M = \operatorname{Spec} W(k)[x_{ij} \mid 1 \le i, j \le r][\frac{1}{\det(x_{ij})}]$;
- $\mathbf{GL}_{\varphi^{-1}(M)} = \operatorname{Spec} W(k)[p^{\delta_{ij}} x_{ij} \mid 1 \le i, j \le r][\frac{1}{\det(x_{ij})}]$, where $\delta_{ij} = f_l - f_m$ if $i \in I_l$ and $j \in I_m$; see Subsection 2.1 for the definition of I_l and I_m. Note that $\det(p^{\delta_{ij}} x_{ij}) = \det(x_{ij})$ as for each permutation π of $\{1, 2, \ldots, r\}$, we have $\prod_{i=1}^r p^{\delta_{i\pi(i)}} x_{i\pi(i)} = \prod_{i=1}^r x_{i\pi(i)}$.
- $\mathbf{G} = \operatorname{Spec} W(k)[p^{\epsilon_{ij}} x_{ij} \mid 1 \le i, j \le r][\frac{1}{\det(x_{ij})}]$, where $\epsilon_{ij} = \min(0, \delta_{ij})$.

For any affine scheme $\mathbf{H}$, let $R_{\mathbf{H}}$ be the ring such that $\mathbf{H} = \operatorname{Spec} R_{\mathbf{H}}$. Let $\mathcal{K}$ be the kernel of the composition

$$R_{\mathbf{GL}_M} \otimes R_{\mathbf{GL}_{\varphi^{-1}(M)}} \to R_{\mathbf{GL}_V} \otimes R_{\mathbf{GL}_V} \to R_{\mathbf{GL}_V}.$$

Then $R_{\mathbf{G}} \cong R_{\mathbf{GL}_M} \otimes R_{\mathbf{GL}_{\varphi^{-1}(M)}}/\mathcal{K}$. It is easy to see that the natural homomorphism

$$R_{\mathbf{GL}_M} \otimes R_{\mathbf{GL}_{\varphi^{-1}(M)}}/\mathcal{K} \to W(k)[p^{\epsilon_{ij}} x_{ij} \mid 1 \le i, j \le r]\left[\frac{1}{\det(x_{ij})}\right]$$

is an isomorphism of $W(k)$-algebras.

Proposition 3.1. *The scheme* $\mathbf{G}$ *is a connected smooth, affine group scheme over* $\operatorname{Spec} W(k)$ *of relative dimension* r^2.

Proof. As G is a principal open subscheme of the affine space $\operatorname{Spec} W(k)[p^{\epsilon_{ij}} x_{ij} \mid 1 \le i, j \le r]$ over $W(k)$, it is affine, smooth, integral and of relative dimension r^2. From this the lemma follows. □

Fix an F-basis $\mathcal{B}$ of $\mathcal{M}$. If $l \ne m$, let $\mathbf{G}_{l,m}$ be the maximal subgroup scheme of $\mathbf{GL}_M$ that fixes both

$$\widetilde{F}^1_{\mathcal{B}}(M) \oplus \cdots \oplus \widetilde{F}^{m-1}_{\mathcal{B}}(M) \oplus \widetilde{F}^{m+1}_{\mathcal{B}}(M) \oplus \cdots \oplus \widetilde{F}^t_{\mathcal{B}}(M)$$

and

$$\widetilde{F}^l_{\mathcal{B}}(M) \oplus \widetilde{F}^m_{\mathcal{B}}(M)/\widetilde{F}^l_{\mathcal{B}}(M).$$

With respect to the F-basis $\mathcal{B}$, the (multiplicative) group scheme $\mathbf{G}_{l,m}$ is isomorphic to $\operatorname{Spec} W(k)[x_{ij} \mid i \in I_l, j \in I_m]$. If R is a $W(k)$-algebra, then

$$\mathbf{G}_{l,m}(R) = 1_{M \otimes_{W(k)} R} + \operatorname{Hom}(\widetilde{F}^m_{\mathcal{B}}(M), \widetilde{F}^l_{\mathcal{B}}(M)) \otimes_{W(k)} R,$$

and thus $\mathbf{G}_{l,m} \cong \mathbb{G}_a^{h_{f_l} h_{f_m}}$. If $l = m$, let $\mathbf{G}_{l,l}$ be $\mathbf{GL}_{\widetilde{F}^l_{\mathcal{B}}(M)}$. With respect to the F-basis $\mathcal{B}$, $\mathbf{G}_{l,l}$ is isomorphic to Spec $W(k)[x_{ij} \mid i, j \in I_l][\frac{1}{\det(x_{ij})}]$. Put

$$\begin{aligned}
\mathbf{G}_+ &= \prod_{1 \le m < l \le t} \mathbf{G}_{l,m} \\
&= \mathbf{G}_{t,t-1} \times \mathbf{G}_{t,t-2} \times \mathbf{G}_{t-1,t-2} \cdots \times \mathbf{G}_{3,2} \times \mathbf{G}_{t,1} \times \cdots \times \mathbf{G}_{3,1} \times \mathbf{G}_{2,1}, \\
\mathbf{G}_- &= \prod_{1 \le l < m \le t} \mathbf{G}_{l,m} \\
&= \mathbf{G}_{1,2} \times \mathbf{G}_{1,3} \times \cdots \times \mathbf{G}_{1,t} \times \mathbf{G}_{2,3} \times \cdots \mathbf{G}_{t-2,t-1} \times \mathbf{G}_{t-2,t} \times \mathbf{G}_{t-1,t},
\end{aligned}$$

$$\mathbf{G}_0 := \prod_{l=1}^{t} \mathbf{G}_{l,l}, \qquad \text{and} \qquad \widetilde{\mathbf{G}} := \mathbf{G}_+ \times \mathbf{G}_0 \times \mathbf{G}_-.$$

With respect to the F-basis $\mathcal{B}$,

$$\widetilde{\mathbf{G}} = \operatorname{Spec} W(k)[x_{ij} \mid 1 \le i, j \le r]\left[\frac{1}{\prod_{l=1}^{t} \det(x_{ij})_{i,j \in I_l}}\right].$$

Let $P_m : \widetilde{\mathbf{G}} \to \mathbf{GL}_M$ be the natural product morphism, and let $P_{\widetilde{\mathbf{G}}}$ be the composition

$$P_m \circ \left(1_{\mathbf{G}_+} \times 1_{\mathbf{G}_0} \times \prod_{1 \le l < m \le t} (\bullet)^{p^{f_m - f_l}}\right) : \widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}} \to \mathbf{GL}_M.$$

For any morphism $Q : \mathbf{H}_1 \to \mathbf{H}_2$ of affine schemes, let $Q' : R_{\mathbf{H}_2} \to R_{\mathbf{H}_1}$ be the natural homomorphism induced by Q.

Lemma 3.2. *There is a unique morphism $P : \widetilde{\mathbf{G}} \to \mathbf{G}$ such that $P_{\mathbf{G}} \circ P = P_{\widetilde{\mathbf{G}}}$.*

Proof. The morphism $P_{\mathbf{G}} : \mathbf{G} \to \mathbf{GL}_M$ at the level of $W(k)$-algebras

$$\begin{aligned}
P'_{\mathbf{G}} : W(k)[x_{ij} \mid 1 \le i, j \le r]\left[\frac{1}{\det(x_{ij})}\right] \\
\longrightarrow W(k)[p^{\epsilon_{ij}} x_{ij} \mid 1 \le i, j \le r]\left[\frac{1}{\det(x_{ij})}\right]
\end{aligned}$$

is such that $P'_{\mathbf{G}}(x_{ij}) = x_{ij}$; see the coordinate description of $\mathbf{G}$ for the definition of ϵ_{ij}. Note that $\epsilon_{ij} \le 0$. The morphism $P_{\widetilde{\mathbf{G}}} : \widetilde{\mathbf{G}} \to \mathbf{GL}_M$ at the level of

$W(k)$-algebras

$$P'_{\widetilde{\mathbf{G}}} : W(k)[x_{ij} | 1 \leq i, j \leq r]\left[\frac{1}{\det(x_{ij})}\right] \longrightarrow W(k)[x_{ij} | 1 \leq i, j \leq r]\left[\frac{1}{\prod_{l=1}^{t} \det(x_{ij})_{i,j\in I_l}}\right]$$

is such that $P'_{\widetilde{\mathbf{G}}}(x_{ij}) = P'_m(p^{-\epsilon_{ij}} x_{ij})$. It is easy to check (at the level of R-valued points) that $P'_{\widetilde{\mathbf{G}}}(\det(x_{ij})) = \prod_{l=1}^{t} \det(x_{ij})_{i,j\in I_l}$. This forces $P : \widetilde{\mathbf{G}} \to \mathbf{G}$ to satisfy $P'(p^{\epsilon_{ij}} x_{ij}) = P'_m(x_{ij})$ and it indeed defines a $W(k)$-algebra homomorphism

$$P' : W(k)[p^{\epsilon_{ij}} x_{ij} \mid 1 \leq i, j \leq r]\left[\frac{1}{\det(x_{ij})}\right] \to W(k)[x_{ij} \mid 1 \leq i, j \leq r]\left[\frac{1}{\prod_{l=1}^{t} \det(x_{ij})_{i,j\in I_l}}\right],$$

as

$$P'(\det(x_{ij})) = \det(P'(x_{ij})) = \det(P'_m(p^{-\epsilon_{ij}} x_{ij})) = \det(P'_{\widetilde{\mathbf{G}}}(x_{ij})) = \prod_{l=1}^{t} \det(x_{ij})_{i,j\in I_l}. \qquad \square$$

Lemma 3.3. *For every k-algebra R, the morphism P induces a bijection on $W_s(R)$-valued points for all positive integer s.*

Proof. We first show that P induces a bijection on $W(R)$-valued points.

We start by showing that the image of $P_{\widetilde{\mathbf{G}}}(W(R))$ is the same as the image $P_{\mathbf{G}}(W(R))$ in $\mathbf{GL}_M(W(R))$, which is

$$S := \{(p^{-\epsilon_{ij}} r_{ij})_{1\leq i,j\leq r} \mid r_{ij} \in W(R),\ \det(r_{ij}) \in W(R)^*\}.$$

As $t \times t$ block matrices, these are matrices of the type

$$N = \begin{pmatrix} N_{11} & p^{f_2-f_1}N_{12} & p^{f_3-f_1}N_{13} & \cdots & p^{f_t-f_1}N_{1t} \\ N_{21} & N_{22} & p^{f_3-f_2}N_{23} & \cdots & p^{f_t-f_2}N_{2t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N_{t1} & N_{t2} & N_{t3} & \cdots & N_{tt} \end{pmatrix}$$

where N_{lm} is an arbitrary $h_{f_l} \times h_{f_m}$ matrix for $1 \leq l, m \leq t$ with entries in $W(R)$, and $\det(N) \in W(R)^*$. We claim that N_{ii} are invertible for $1 \leq i \leq t$.

After reduction modulo $\theta(W(R))$, the matrix N is a lower triangular block matrix. The determinant of N modulo $\theta(W(R))$ is $\prod_{i=1}^{t}\det(N_{ii})$ modulo $\theta(W(R))$, which is a unit in R, this implies that $\det(N_{ii})$ modulo $\theta(W(R))$ is a unit in R and hence $\det(N_{ii})$ is a unit in $W(R)$.

Let X be an arbitrary $t \times t$ block matrix in $\mathbf{G}_0(W(R))$ so that the diagonal blocks are denoted by X_{ii}. If $l > m$, let Y_{lm} be an arbitrary $t \times t$ block matrix in $\mathbf{G}_{l,m}(W(R))$ with $\widetilde{Y}_{lm}$ at (l,m) block entry and 0 at everywhere else. If $l < m$, let Z_{lm} be an arbitrary $t \times t$ block matrix in $\mathbf{G}_{l,m}(W(R))$ with $p^{f_m - f_l}\widetilde{Z}_{lm}$ at (l,m) block entry and 0 at everywhere else. We need to show that the set

$$\left\{ \prod_{1\le m<l\le t} Y_{lm} X \prod_{1\le l<m\le t} Z_{lm} \mid X, Y_{lm}, Z_{lm} \text{ satisfy the conditions stated above} \right\}$$

is equal to the set S of all $t \times t$ matrices N as described above. Here the order of the product $\prod_{1\le m<l\le t} Y_{lm}$ is the same as the order in the definition of $\mathbf{G}_+$. The order of the product $\prod_{1\le l<m\le t} Z_{lm}$ is the same as the order in the definition of $\mathbf{G}_-$.

We use induction on t. The base case when $t = 1$ is trivial. Suppose it is true for $t-1$. Then

$$\prod_{1\le m<l\le t-1} Y_{lm} X \prod_{1\le l<m\le t-1} Z_{lm} = \begin{pmatrix} X_{11} & p^{f_2-f_1}X_{12} & \cdots & p^{f_{t-1}-f_1}A_{1,t-1} & 0 \\ X_{21} & X_{22} & \cdots & p^{f_{t-1}-f_2}X_{2,t-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{t-1,1} & X_{t-1,2} & \cdots & X_{t-1,t-1} & 0 \\ 0 & 0 & \cdots & 0 & X_{tt} \end{pmatrix}$$

satisfies $\det(X_{ii}) \in W(R)^*$, and each X_{lm} is an arbitrary $h_{f_l} \times h_{f_m}$ matrix if $l \neq m$. We abbreviate this matrix by $\begin{pmatrix} \widetilde{X} & 0 \\ 0 & X_{tt} \end{pmatrix}$, then

$$\prod_{1\le m\le t-1} Y_{tm} \begin{pmatrix} \widetilde{X} & 0 \\ 0 & X_{tt} \end{pmatrix} \prod_{1\le l\le t-1} Z_{lt} = \begin{pmatrix} 1 & 0 \\ \widetilde{Y} & 1 \end{pmatrix}\begin{pmatrix} \widetilde{X} & 0 \\ 0 & X_{tt} \end{pmatrix}\begin{pmatrix} 1 & \widetilde{Z} \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \widetilde{X} & \widetilde{X}\widetilde{Z} \\ \widetilde{Y}\widetilde{X} & X_{tt} + \widetilde{Y}\widetilde{X}\widetilde{Z} \end{pmatrix}.$$

Here the matrix $\widetilde{Y} = (\widetilde{Y}_{t1}, \widetilde{Y}_{t2}, \ldots, \widetilde{Y}_{t,t-1})$ has size $h_{f_t} \times (r - h_{f_t})$ and the matrix $\widetilde{Z} = (p^{f_t-f_1}\widetilde{Z}_{1t}, p^{f_t-f_2}\widetilde{Z}_{2t}, \ldots, p^{f_t-f_{t-1}}\widetilde{Z}_{t-1,t})^T$ has size $(r-h_{f_t}) \times h_{f_t}$. As $\widetilde{X}$ is invertible, the right multiplication of $\widetilde{X}$ induces a bijection from the set of all $h_{f_t} \times (r - h_{f_t})$ matrices to itself. Thus $\widetilde{Y}\widetilde{X}$ can be any $h_{f_t} \times (r-h_{f_t})$ matrix with $\widetilde{X}$ fixed and $\widetilde{Y}$ varied. Multiplying $\widetilde{X}$ and $\widetilde{Z}$, we get

$$\begin{pmatrix} X_{11} & p^{f_2-f_1}X_{12} & \cdots & p^{f_{t-1}-f_1}X_{1,t-1} \\ X_{21} & X_{22} & \cdots & p^{f_{t-1}-f_2}X_{2,t-1} \\ \vdots & \vdots & \ddots & \vdots \\ X_{t-1,1} & X_{t-1,2} & \cdots & X_{t-1,t-1} \end{pmatrix} \begin{pmatrix} p^{f_t-f_1}\widetilde{Z}_{1t} \\ p^{f_t-f_2}\widetilde{Z}_{2t} \\ \vdots \\ p^{f_t-f_{t-1}}\widetilde{Z}_{t-1,t} \end{pmatrix}$$
$$= \begin{pmatrix} p^{f_t-f_1}(X_{11}\widetilde{Z}_{1t} + \cdots + X_{1,t-1}\widetilde{Z}_{t-1,t}) \\ p^{f_t-f_2}(p^{f_2-f_1}X_{21}\widetilde{Z}_{1t} + \cdots + X_{2,t-1}\widetilde{Z}_{t-1,t}) \\ \vdots \\ p^{f_t-f_{t-1}}(p^{f_{t-1}-f_1}X_{t-1,1}\widetilde{Z}_{1t} + \cdots + X_{t-1,t-1}\widetilde{Z}_{t-1,t}) \end{pmatrix}.$$

To show that $\widetilde{X}\widetilde{Z}$ can be any matrix of the type

$$(p^{f_t-f_1}N_{1t}, p^{f_t-f_2}N_{2t}, \ldots, p^{f_t-f_{t-1}}N_{t-1,t})^T$$

with $\widetilde{X}$ fixed and $\widetilde{Z}$ varied, it is enough to show that the matrix

$$\begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1,t-1} \\ p^{f_2-f_1}X_{21} & X_{22} & \cdots & X_{2,t-1} \\ \vdots & \vdots & \ddots & \vdots \\ p^{f_{t-1}-f_1}X_{t-1,1} & p^{f_{t-1}-f_2}X_{t-1,2} & \cdots & X_{t-1,t-1} \end{pmatrix}$$

is invertible. But this is so because X_{ii} are invertible. When $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{Z}$ are fixed, $X_{tt}+\widetilde{Y}\widetilde{X}\widetilde{Z}$ can be an arbitrary invertible $h_t \times h_t$ matrix with X_{tt} varied because $\widetilde{Y}\widetilde{X}\widetilde{Z}$ modulo p is zero. Thus we have shown that $P_{\widetilde{\mathbf{G}}}(W(R))$ and $P_{\mathbf{G}}(W(R))$ are the same in $\mathbf{GL}_M(W(R))$.

To show that $P(W(R))$ is a bijection, it is enough to show that $P_{\widetilde{\mathbf{G}}}(W(R))$ is an injection. If this is true, as $P_{\mathbf{G}}(W(R))$ is an injection and the image of $P_{\widetilde{\mathbf{G}}}(W(R))$ and $P_{\mathbf{G}}(W(R))$ are the same, then $P(W(R))$ is a bijection. Suppose

$$\prod_{1\le m<l\le t} Y_{lm} X \prod_{1\le l<m\le t} Z_{lm} = \prod_{1\le m<l\le t} Y'_{lm} X' \prod_{1\le l<m\le t} Z'_{lm}, \tag{3.1}$$

and we want to show that $Y_{lm} = Y'_{lm}$ for all $1 \le m < l \le t$, $X = X'$ and $Z_{lm} = Z'_{lm}$ for all $1 \le l < m \le t$. By the definition of Y_{lm} and Z_{lm} it suffices to show that $\prod_{1\le m<l\le t} Y_{lm} = \prod_{1\le m<l\le t} Y'_{lm}$ and $\prod_{1\le l<m\le t} Z_{lm} = \prod_{1\le l<m\le t} Z'_{lm}$. Equality (3.1) is equivalent to

$$\left(\prod_{1\le m<l\le t} Y'_{lm}\right)^{-1} \prod_{1\le m<l\le t} Y_{lm} X = X \prod_{1\le l<m\le t} Z'_{lm} \left(\prod_{1\le l<m\le t} Z_{lm}\right)^{-1}. \tag{3.2}$$

Let $(\prod_{1\le m<l\le t} Y'_{lm})^{-1}\prod_{1\le m<l\le t} Y_{lm} = I + Y$ where Y is strictly lower triangular and $\prod_{1\le l<m\le t} Z'_{lm}(\prod_{1\le l<m\le t} Z_{lm})^{-1} = I + Z$ where Z is strictly upper triangular. The equality (3.2) is equivalent to $YX = XZ$. It is easy to see that $Y = Z = 0$ as X is a diagonal block matrix with invertible blocks X_{ii}. This completes the proof that $P(W(R))$ is a bijection.

To show that $P(W_s(R))$ is injective, let $\bar{f}_1, \bar{f}_2 \in \widetilde{\mathbf{G}}(W_s(R))$ with lifts $f_1, f_2 \in \widetilde{\mathbf{G}}(W(R))$ respectively such that $P(W_s(R))(\bar{f}_1) = P(W_s(R))(\bar{f}_2)$. The images of $P(W(R))(f_1)$ and $P(W(R))(f_2)$ under the reduction epimorphism $\mathbf{G}(W(R)) \to \mathbf{G}(W_s(R))$ are the same. Hence $P(W(R))(f_1)$ and $P(W(R))(f_2)$ are congruent modulo θ^s. As $P(W(R))$ is a bijection, f_1 and f_2 are also congruent modulo θ^s as well. Hence $\bar{f}_1 = \bar{f}_2$.

To show that $P(W_s(R))$ is surjective, let $\bar{f} \in \mathbf{G}(W_s(R))$, a lift $f \in \mathbf{G}(W(R))$ of $\bar{f}$ has a preimage $g \in \widetilde{\mathbf{G}}(W(R))$ such that $P(W(R))(g) = f$ because $P(W(R))$ is surjective. Thus the image of g in $\widetilde{\mathbf{G}}(W_s(R))$ is a preimage of $\bar{f}$. This shows that $P(W_s(R))$ is bijective and thus completes the proof the lemma. □

Corollary 3.4. *The morphism $P : \widetilde{\mathbf{G}} \to \mathbf{G}$ induces an isomorphism $P_{W_s(k)} : \widetilde{\mathbf{G}}_{W_s(k)} \to \mathbf{G}_{W_s(k)}$ for each $s \ge 1$.*

Proof. If $s = 1$, then $P_{W_1(k)} = P_k$. It is an isomorphism by Lemma 3.3. Suppose that $s > 1$. As $R_{\mathbf{G}}$ and $R_{\widetilde{\mathbf{G}}}$ are $W_s(k)$-flat algebras, we get that $p^{s-1}R_{\mathbf{G}}/p^sR_{\mathbf{G}} \cong R_{\mathbf{G}}/pR_{\mathbf{G}}$ and $p^{s-1}R_{\widetilde{\mathbf{G}}}/p^sR_{\widetilde{\mathbf{G}}} \cong R_{\widetilde{\mathbf{G}}}/pR_{\widetilde{\mathbf{G}}}$ by the local criteria on flatness. As a result, $p^{s-1}R_{\mathbf{G}}/p^sR_{\mathbf{G}} \cong p^{s-1}R_{\widetilde{\mathbf{G}}}/p^sR_{\widetilde{\mathbf{G}}}$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p^{s-1}R_{\mathbf{G}}/p^sR_{\mathbf{G}} & \longrightarrow & R_{\mathbf{G}}/p^sR_{\mathbf{G}} & \longrightarrow & R_{\mathbf{G}}/p^{s-1}R_{\mathbf{G}} & \longrightarrow & 0\,. \\
 & & \downarrow{\cong} & & \downarrow{P'_{W_s(k)}} & & \downarrow{P'_{W_{s-1}(k)}} & & \\
0 & \longrightarrow & p^{s-1}R_{\widetilde{\mathbf{G}}}/p^sR_{\widetilde{\mathbf{G}}} & \longrightarrow & R_{\widetilde{\mathbf{G}}}/p^sR_{\widetilde{\mathbf{G}}} & \longrightarrow & R_{\widetilde{\mathbf{G}}}/p^{s-1}R_{\widetilde{\mathbf{G}}} & \longrightarrow & 0
\end{array}$$

An easy induction on s using the five lemma concludes the proof of the lemma. □

Let $\mathcal{B}$ be an F-basis of $\mathcal{M}$. The Spec $W(k)$-scheme $\mathbf{GL}_M$ is represented by the $W(k)$-algebra $W(k)[x_{ij} \mid 1 \le i, j \le r][1/\det(x_{ij})]$. We construct the cocharacter $\mu : \mathbb{G}_m \to \mathbf{GL}_M$ (that depends on $\mathcal{B}$) defined by the k-algebra homomorphism

$$\mu' : W(k)[x_{ij} \mid 1 \le i, j \le r][1/\det(x_{ij})] \to W(k)[x, 1/x]$$

with the property that $\mu'(x_{ij}) = 0$ if $i \ne j$ and $\mu'(x_{ii}) = (1/x)^{e_i}$ for $1 \le i \le r$ where $e_1, e_2, \dots, e_r$ are the Hodge slopes of (M, φ). Put

$\sigma_{\mathcal{M}} := \varphi\mu(B(k))(p)$. It is a σ-linear isomorphism of M defined by the rule $\sigma_{\mathcal{M}}(x) = p^{-f_j}\varphi(x)$ for $x \in \widetilde{F}^j_{\mathcal{B}}(M)$. It is well known [3, A.1.2.6] that there is a $\mathbb{Z}_p$-submodule $M_0 = \{x \in M \mid \sigma_{\mathcal{M}}(x) = x\}$ of M, whose rank is the same as the rank of M and such that $M = M_0 \otimes_{\mathbb{Z}_p} W(k)$. Note that the construction of M_0 also depends on $\mathcal{B}$. We fix a $\mathbb{Z}_p$-basis of $\mathcal{B}_0 = \{w_1, w_2, \dots, w_r\}$ of M_0. It induces a $\mathbb{Z}_p$-basis $\mathcal{B}_0^* = \{e_{ij} \mid 1 \le i, j \le r\}$ of $\mathrm{End}_{\mathbb{Z}_p}(M_0)$ such that $e_{ij}(w_j) = w_i$. Note that

$$\mathrm{End}_{W(R)}(M \otimes_{W(k)} W(R)) = \mathrm{End}_{\mathbb{Z}_p}(M_0) \otimes_{\mathbb{Z}_p} W(R).$$

Let $h = \sum_{i,j} a_{ij}e_{ij} \in \mathrm{End}_{W(R)}(M \otimes_{W(k)} W(R))$, $a_{ij} \in W(R)$ for all $1 \le i, j \le r$. Define

$$\bar{\sigma}_{\mathcal{M}} : \mathrm{End}_{W(R)}(M \otimes_{W(k)} W(R)) \to \mathrm{End}_{W(R)}(M \otimes_{W(k)} W(R))$$

by the formula $\bar{\sigma}_{\mathcal{M}}(h) = \sum_{i,j} \sigma_R(a_{ij})e_{ij}$ and similarly, define

$$\bar{\sigma}_{\mathcal{M}} : \mathrm{End}_{W_s(R)}(M \otimes_{W(k)} W_s(R)) \to \mathrm{End}_{W_s(R)}(M \otimes_{W(k)} W_s(R))$$

by the formula $\bar{\sigma}_{\mathcal{M}}(h[s]) = \left(\sum_{i,j} \sigma_R(a_{ij})e_{ij}\right)[s]$, where $h[s]$ is h modulo θ^s (again this does not contradict to the previous convention that $h[s]$ is h modulo p^s when R is perfect.) One can easily show that the definition of $\bar{\sigma}_{\mathcal{M}}$ does not depend on $\mathcal{B}_0$ and $\mathcal{B}_0^*$ but does depend on $\mathcal{B}$. If R is a perfect field, then $\bar{\sigma}_{\mathcal{M}}$ satisfies $\bar{\sigma}_{\mathcal{M}}(h) = \sigma_{\mathcal{M}} h \sigma_{\mathcal{M}}^{-1}$, which is a formula that does not depend on the choice of $\mathcal{B}_0$ or $\mathcal{B}_0^*$ but does depend on $\mathcal{B}$ since $\sigma_{\mathcal{M}}$ does.

For every $h \in \mathrm{End}_{W(R)}(M \otimes_{W(k)} W(R))$, define

$$\varphi(h) = \bar{\sigma}_{\mathcal{M}}(\mu(B(R))(1/p) \circ h \circ \mu(B(R))(p)).$$

A priori, the definition of $\varphi(h)$ depends on the choice of the F-basis $\mathcal{B}$ of $\mathcal{M}$ as $\bar{\sigma}_{\mathcal{M}}$ and μ do. As $h = \sum_{i,j} a_{ij}e_{ij}$, $a_{ij} \in W(R)$, we get

$$\begin{aligned}
\varphi(h) &= \bar{\sigma}_{\mathcal{M}}(\mu(B(R))(1/p) \circ h \circ \mu(B(R))(p)) \\
&= \bar{\sigma}_{\mathcal{M}}\left(\sum_{i,j}(\mu(B(k))(1/p) \circ e_{ij} \circ \mu(B(k))(p)) \otimes a_{ij}\right) \\
&= \sum_{i,j} \bar{\sigma}_{\mathcal{M}}((\mu(B(k))(1/p) \circ e_{ij} \circ \mu(B(k))(p)) \otimes \sigma_R(a_{ij}) \\
&= \sum_{i,j} \sigma_{\mathcal{M}}\mu(B(k))(1/p) \circ e_{ij} \circ \mu(B(k))(p)\sigma_{\mathcal{M}}^{-1} \otimes \sigma_R(a_{ij}) \\
&= \sum_{i,j} (\varphi \circ e_{ij} \circ \varphi^{-1}) \otimes \sigma_R(a_{ij}).
\end{aligned}$$

Thus $\varphi(h)$ is a $B(R)$-linear endomorphism of $M \otimes_{W(k)} B(R)$ defined by the following rule: let $h = \sum_i h_i \otimes c_i$ under the natural identification (basis free)

$$\mathrm{End}_{W(R)}(M \otimes_{W(k)} W(R)) = \mathrm{End}_{W(k)}(M) \otimes_{W(k)} W(R).$$

For any $m \otimes b \in M \otimes_{W(k)} W(R)$, we have $\varphi(h)(m \otimes b) = \sum_i (\varphi \circ h_i \circ \varphi^{-1})(m) \otimes \sigma_R(c_i) b \in (M \otimes_{W(k)} B(k)) \otimes_{B(k)} B(R) = M \otimes_{W(k)} B(R)$. Thus the definition $\varphi(h)$ does not depend on the choice of $\mathcal{B}$. Note that $\varphi(h)$ might not be an element in $\mathrm{End}_{W(R)}(M \otimes_{W(k)} W(R))$ in general, but it is always an element in $\mathrm{End}_{W(R)}(M \otimes_{W(k)} B(R))$.

Lemma 3.5. *For simplicity, set $\mu(B(R))(p) = \mu(p)$ and $\mu(B(R))(1/p) = \mu(1/p)$. For every $g \in \mathbf{G}_{l,m}(W(R))$, the following three formulae hold.*

(1) *If $m < l$, then $\mu(p) g^{p^{f_l - f_m}} \mu(1/p) = g$.*
(2) *If $m = l$, then $\mu(p) g \mu(1/p) = g$.*
(3) *If $m > l$, then $\mu(p) g \mu(1/p) = g^{p^{f_m - f_l}}$.*

Proof. We first prove (1) when $m < l$. By definition, $g \in \mathbf{G}_{l,m}(W(R))$ if and only if $g = 1_{M \otimes W(R)} + e$ where $e \in \mathrm{Hom}(\widetilde{F}^m_{\mathcal{B}}(M), \widetilde{F}^l_{\mathcal{B}}(M)) \otimes_{W(k)} W(R)$. If $m < l$, then

$$\begin{aligned}\mu(p) g^{p^{f_l - f_m}} \mu(1/p) &= \mu(p)(1_{M \otimes_{W(k)} W(R)} + p^{f_l - f_m} e) \mu(1/p) \\ &= 1_{M \otimes_{W(k)} W(R)} + p^{f_l - f_m} \mu(p) e \mu(1/p).\end{aligned}$$

Thus it suffices to show that $p^{f_l - f_m} \mu(p) e \mu(1/p) = e$.

As $e \in \mathrm{Hom}(\widetilde{F}^m_{\mathcal{B}}(M), \widetilde{F}^l_{\mathcal{B}}(M)) \otimes_{W(k)} W(R)$, $\mu(1/p)$ acts on $\widetilde{F}^m_{\mathcal{B}}(M) \otimes_{W(k)} W(R)$ as p^{f_m}, and $\mu(p)$ acts on $\widetilde{F}^l_{\mathcal{B}}(M) \otimes_{W(k)} W(R)$ as p^{-f_l}, we get the desired equality.

The cases when $m = l$ and $m > l$ are similar and are left to the reader. □

Corollary 3.6. *For every $g \in \mathbf{G}_{l,m}(W(R))$, the following three formulae hold.*

(1) *If $m < l$, then $\bar{\sigma}_{\mathcal{M}}(g^{p^{f_l - f_m}}) = \varphi(g)$.*
(2) *If $m = l$, then $\bar{\sigma}_{\mathcal{M}}(g) = \varphi(g)$.*
(3) *If $m > l$, then $\bar{\sigma}_{\mathcal{M}}(g) = \varphi(g^{p^{f_m - f_l}})$.*

3.2 The group action $\mathbb{T}_s$

Set $\mathbf{G}_s = \mathbb{W}_s(\mathbf{G})$ and $\mathbf{D}_s = \mathbb{W}_s(\mathbf{GL}_M)$. As $\mathbf{G}_{W_s(k)} = \widetilde{\mathbf{G}}_{W_s(k)}$, we have $\widetilde{\mathbf{G}}_s := \mathbb{W}_s(\widetilde{\mathbf{G}}) = \mathbf{G}_s$. The group action

$$\mathbb{T}_s : \mathbf{G}_s \times_k \mathbf{D}_s \to \mathbf{D}_s$$

is defined on R-valued points as follows: For every $h[s] \in \mathbf{G}_s(R)$, $g[s] \in \mathbf{D}_s(R)$, let $h \in \mathbf{G}(W(R))$ be a lift of $h[s]$ under the reduction epimorphism $\mathbf{G}(W(R)) \to \mathbf{G}(W_s(R))$ and $g \in \mathbf{GL}_M(W(R))$ be a lift of $g[s]$ under the reduction epimorphism $\mathbf{GL}_M(W(R)) \to \mathbf{GL}_M(W_s(R))$. Define

$$\mathbb{T}_s(R)(h[s], g[s]) := (hg\varphi(h^{-1}))[s].$$

It is clear that the definition does not depend on the choices of lifts of $h[s]$ and $g[s]$ and does not depend on choice of basis.

To see that $(hg\varphi(h^{-1}))[s] \in \mathbf{D}_s(R)$, let us first recall the identification $\mathbf{G}_{W_s(k)} = \widetilde{\mathbf{G}}_{W_s(k)}$ from Corollary 3.4, thus $h[s] \in \mathbf{G}_s(R) = \mathbf{G}(W_s(R)) = \mathbf{G}_{W_s(k)}(W_s(R)$ is an element of $\widetilde{\mathbf{G}}_{W_s(k)}(W_s(R))$. We can g (non-uniquely) $h[s]$ as a product

$$\prod_{1\le m<l\le t} h_{lm}[s]h_0[s] \prod_{1\le l<m\le t} h_{lm}^{p^{f_m-f_l}}[s],$$

where $\prod_{1\le m<l\le t} h_{lm}[s] \in (\mathbf{G}_-)_{W_s(k)}(W_s(R))$, $h_0[s] \in (\mathbf{G}_0)_{W_s(k)}(W_s(R))$, and $\prod_{1\le l<m\le t} h_{lm}[s] \in (\mathbf{G}_+)_{W_s(k)}(W_s(R))$. Therefore $(hg\varphi(h^{-1}))[s]$ is equal to

$$\left(\prod_{1\le m<l\le t} h_{lm}h_0 \prod_{1\le l<m\le t} h_{lm}^{p^{f_m-f_l}} g\varphi\left(\prod_{1\le l<m\le t} h_{lm}^{p^{f_l-f_m}} h_0^{-1} \prod_{1\le m<l\le t} h_{lm}^{-1}\right)\right)[s]$$

$$= \left(\prod_{1\le m<l\le t} h_{lm}h_0 \prod_{1\le l<m\le t} h_{lm}^{p^{f_m-f_l}} g\right.$$

$$\left.\times \prod_{1\le l<m\le t} \varphi(h_{lm}^{p^{f_l-f_m}})\varphi(h_0^{-1}) \prod_{1\le m<l\le t} \varphi(h_{lm}^{-1})\right)[s]$$

$$= \left(\prod_{1\le m<l\le t} h_{lm}h_0 \prod_{1\le l<m\le t} h_{lm}^{p^{f_m-f_l}} g \prod_{1\le l<m\le t} \bar{\sigma}_{\mathcal{M}}(h_{lm}^{-1})\bar{\sigma}_{\mathcal{M}}(h_0^{-1})\right.$$

$$\left.\times \prod_{1\le m<l\le t} \bar{\sigma}_{\mathcal{M}}((h_{lm}^{-1})^{p^{f_l-f_m}})\right)[s]$$

$$= \prod_{1\le m<l\le t} h_{lm}[s]h_0[s] \prod_{1\le l<m\le t} h_{lm}^{p^{f_m-f_l}}[s]g[s]$$

$$\left.\times \prod_{1\le l<m\le t} \bar{\sigma}_{\mathcal{M}}(h_{lm}^{-1}[s])\bar{\sigma}_{\mathcal{M}}(h_0^{-1}[s]) \prod_{1\le m<l\le t} \bar{\sigma}_{\mathcal{M}}((h_{lm}^{-1}[s])^{p^{f_l-f_m}}\right)$$

which is in $\mathbf{D}_s(R)$. The above formula proves that $\mathbb{T}_s$ is a morphism.

For later use, we record the following formula when $R = k$ and $s = 1$.

$$\begin{aligned} &\mathbb{T}_1(k)(h[1], g[1]) \\ &= \prod_{1 \le m < l \le t} h_{lm}[1] h_0[1] g[1] \prod_{1 \le l < m \le t} (\bar{\sigma}_{\mathcal{M}}(h_{lm}^{-1}[1]))(\bar{\sigma}_{\mathcal{M}}(h_0^{-1}[1])) \end{aligned} \tag{3.3}$$

3.3 Orbits and stabilizers of $\mathbb{T}_s$

Let $1_M[s] \in \mathbf{D}_s(k)$. The image of the morphism

$$\Psi := \mathbb{T}_s \circ (1_{\mathbf{G}_s} \times_k 1_M[s]) : \mathbf{G}_s \cong \mathbf{G}_s \times_k \operatorname{Spec} k \to \mathbf{G}_s \times_k \mathbf{D}_s \to \mathbf{D}_s.$$

is the orbit of $1_M[s]$, which we denoted by $\mathbf{O}_s$. Its Zariski closure $\bar{\mathbf{O}}_s$ is a closed integral subscheme of $\mathbf{D}_s$. The orbit $\mathbf{O}_s$ is a smooth connected open subscheme of $\bar{\mathbf{O}}_s$.

Proposition 3.7. *Let $g_1, g_2 \in \mathbf{GL}_M(W(k))$. Then $g_1[s], g_2[s] \in \mathbf{GL}_M(W_s(k)) = \mathbf{D}_s(k)$ belong to the same orbit of the action $\mathbb{T}_s$ if and only if $F_s(\mathcal{M}(g_1))$ is isomorphic to $F_s(\mathcal{M}(g_2))$.*

Proof. We know that $g_1[s]$ and $g_2[s]$ belong to the same orbit of the action $\mathbb{T}_s$ if and only if there exists $h[s] \in \mathbf{G}_s(k)$ such that $\mathbb{T}_s(h[s], g_1[s]) = (hg_1\varphi h^{-1}\varphi^{-1})[s] = g_2[s]$. This implies that $h[s]$ is an isomorphism from $F_s(\mathcal{M}(g_1))$ to $F_s(\mathcal{M}(g_2))$.

If $h[s]$ is an isomorphism from $F_s(\mathcal{M}(g_1))$ to $F_s(\mathcal{M}(g_2))$, then $hg_1\varphi h^{-1}\varphi^{-1} \equiv g_2$ modulo p^s. To conclude the proof, it is enough to show that $h \in \mathbf{G}_s(k)$, but this is clear from the facts that $h(M) = M$ and $h(\varphi^{-1}(M)) \subset \varphi^{-1}(M)$. □

Corollary 3.8. *The set of orbits of the action $\mathbb{T}_s$ is in natural bijection to the set of isomorphism classes of F-truncations modulo p^s of $\mathcal{M}(g)$ for all $g \in \mathbf{GL}_M(W(k))$.*

Let $\mathbf{S}_s$ be the fibre product defined by the following commutative diagram:

$$\begin{array}{ccc} \mathbf{S}_s & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow{\scriptstyle 1_M[s]} \\ \mathbf{G}_s & \xrightarrow{\ \Psi\ } & \mathbf{D}_s \end{array}$$

It is the stabilizer of $1_M[s]$ and is a subgroup scheme of $\mathbf{G}_s$. We denote by $\mathbf{C}_s$ the reduced scheme $(\mathbf{S}_s)_{\mathrm{red}}$, and $\mathbf{C}_s^0$ the identity component of $\mathbf{C}_s$. Clearly,

$$\dim(\mathbf{S}_s) = \dim(\mathbf{C}_s) = \dim(\mathbf{C}_s^0) = \dim(\mathbf{G}_s) - \dim(\mathbf{O}_s). \tag{3.4}$$

Example 3.9. In this example, we follow the ideas of [14, Section 2.3] to discuss $\mathbb{T}_1(k)$. As a result, we will see that $\mathbf{C}_1^0$ is a unipotent group scheme over k. Let (M, φ) be an F-crystal over k such that $e_1 = 0$. By [15, Section 1.8] or [17, Theorem 1.1], there exist an element $g \in \mathbf{GL}_M(W(k))$ with the property that $g \equiv 1_M$ modulo p, an F-basis $\mathcal{B} = \{v_1, v_2, \ldots, v_r\}$ of $\mathcal{M}$, and a permutation π on the set $I = \{1, 2, \ldots, r\}$ that defines a σ-linear monomorphism $\varphi_\pi : M \Rightarrow M$ with the property that $\varphi_\pi(v_i) = p^{e_i} v_{\pi(i)}$ for all $i \in I$, such that $\mathcal{M}$ is isomorphic to $(M, g\varphi_\pi)$. Let μ be the cocharacter defined with respect to $\mathcal{B}$ and let $\bar{\sigma}_{\mathcal{M}}$ be the σ-linear endomorphism of $\mathrm{End}_{W(R)}(M \otimes_{W(k)} W(R))$ defined with respect to μ. Set

$$I_+ = \{(i, j) \in I \times I \mid i \in I_l, j \in I_m, \quad \text{where} \quad m > l\};$$
$$I_0 = \{(i, j) \in I \times I \mid i, j \in I_l \quad \text{for some } l\};$$
$$I_- = \{(i, j) \in I \times I \mid i \in I_l, j \in I_m, \quad \text{where} \quad m < l\}.$$

See Subsection 2.1 for the definition of I_l and I_m. For $1 \leq i, j \leq r$, let $f_{i,j} \in \mathrm{End}(M)$ be such that $f_{i,j}(v_j) = v_i$ and $f_{i,j}(v_l) = 0$ for $l \neq j$. For every $1 + \bar{f}_{i,j} \in \mathbf{GL}_M(k)$, where $\bar{f}_{i,j}$ is $f_{i,j}$ modulo p, $\bar{\sigma}_{\mathcal{M}}(1 + \bar{f}_{i,j}) = \varphi_\pi(1 + \bar{f}_{i,j})\varphi_{\pi^{-1}} = 1 + \bar{f}_{\pi(i),\pi(j)}$. For every $h = h_+ h_0 h_- \in \mathbf{G}(k) = \mathbf{G}^*(k)$, where $h_\dagger \in \mathbf{G}_\dagger$ for $\dagger \in \{+, 0, -\}$, we know that $h[1] \in \mathbf{S}_1(k) = \mathbf{C}_1(k)$ if and only if

$$h_+[1]h_0[1] = \bar{\sigma}_{\mathcal{M}}(h_0[1])\bar{\sigma}_{\mathcal{M}}(h_-[1])$$

by (3.3). This is equivalent to

$$(h_+h_0)[1] = \bar{\sigma}_{\mathcal{M}}((h_0h_-)[1]) \tag{3.5}$$

Let

$$(h_+h_0)[1] = 1_M[1] + \sum_{(i,j)\in I_+\cup I_0} x_{i,j}\bar{e}_{i,j}$$

and

$$(h_0h_-)[1] = 1_M[1] + \sum_{(i,j)\in I_0\cup I_-} x_{i,j}\bar{e}_{i,j}.$$

Then (3.5) can be rewritten as

$$\sum_{(i,j)\in I_+\cup I_0} x_{i,j}\bar{e}_{i,j} = \sum_{(i,j)\in I_0\cup I_-} x_{i,j}^p \bar{e}_{\pi(i),\pi(j)} \tag{3.6}$$

This is equivalent to three types of equalities:

$$x_{\pi(i),\pi(j)} = x_{i,j}^p \quad \text{if } (i, j) \in I_- \cup I_0 \quad \text{and} \quad (\pi(i), \pi(j)) \in I_+ \cup I_0, \tag{3.7}$$
$$x_{\pi(i),\pi(j)} = 0 \quad \text{if } (i, j) \in I_+ \quad \text{and} \quad (\pi(i), \pi(j)) \in I_+ \cup I_0, \tag{3.8}$$
$$x_{i,j}^p = 0 \quad \text{if } (i, j) \in I_- \cup I_0 \quad \text{and} \quad (\pi(i), \pi(j)) \in I_-. \tag{3.9}$$

We decompose the permutation $\pi \times \pi$ on $I \times I$ into a product of disjoint cycles $\prod_u (\pi \times \pi)_u$. To ease language, we say that a pair $(i, j) \in I \times I$ is *in* $(\pi \times \pi)_u$ if $(\pi \times \pi)_u(i, j) \neq (i, j)$. To study the system of equations defined by (3.7) to (3.9) we consider the following three cases:

(1) Consider $(\pi \times \pi)_u$ such that all (i, j) in $(\pi \times \pi)_u$ are in I_0. By (3.7), $x_{i,j} = x_{i,j}^{p^{\operatorname{ord}((\pi\times\pi)_u)}}$. Thus there are finitely many solutions for $x_{i,j}$.
(2) Consider $(\pi \times \pi)_u$ such that all (i, j) in $(\pi \times \pi)_u$ are in $I_0 \cup I_+$ and at least one (i, j) is in I_+. By (3.8), $x_{i,j} = 0$ for all (i, j) in $(\pi \times \pi)_u$.
(3) Consider $(\pi \times \pi)_u$ such that at least one (i, j) in $(\pi \times \pi)_u$ is in I_-. Let $\nu_\pi(i, j)$ be the smallest positive integer such that

$$(\pi^{\nu_\pi(i,j)}(i), \pi^{\nu_\pi(i,j)}(j)) \in I_+ \cup I_-.$$

By (3.7), $x_{\pi^m(i),\pi^m(j)} = x_{i,j}^{p^m}$ for all $1 \le m < \nu_\pi(i, j)$.

- If $(\pi^{\nu_\pi(i,j)}(i), \pi^{\nu_\pi(i,j)}(j)) \in I_-$, then $x_{\pi^m(i),\pi^m(j)} = 0$ for all $0 \le m \le \nu_\pi(i, j)$.
- If $(\pi^{\nu_\pi(i,j)}(i), \pi^{\nu_\pi(i,j)}(j)) \in I_+$, then $x_{\pi^m(i),\pi^m(j)} = x_{i,j}^{p^m}$ for all $1 \le m \le \nu_\pi(i, j)$.

Thus $x_{\pi^m(i),\pi^m(j)}$ for all $1 \le m < \nu_\pi(i, j)$ has finitely many solutions.

- If $(\pi^{\nu_\pi(i,j)+1}(i), \pi^{\nu_\pi(i,j)+1}(j)) \in I_+ \cup I_0$, then $x_{\pi^{\nu_\pi(i,j)+1}(i),\pi^{\nu_\pi(i,j)+1}(j)}$ equals 0 by (3.8).
- If $(\pi^{\nu_\pi(i,j)+1}(i), \pi^{\nu_\pi(i,j)+1}(j)) \in I_-$, then $x_{\pi^{\nu_\pi(i,j)+1}(i),\pi^{\nu_\pi(i,j)+1}(j)}$ is not related to $x_{i,j}$.

Let I_-^π be a subset of I_- that contains pairs (i, j) such that $(\pi^{\nu_\pi(i,j)}(i), \pi^{\nu_\pi(i,j)}(j)) \in I_+$. We conclude that $h[1] \in \mathbf{C}_1^0(k)$ if and only if the following equations hold:

$$h_+[1]h_0[1] = 1_M[1] + \sum_{(i,j)\in I_-^\pi} \sum_{l=1}^{\nu_\pi(i,j)} x_{i,j}^{p^l} \bar{e}_{\pi^l(i),\pi^l(j)},$$

$$h_0[1] = 1_M[1] + \sum_{(i,j)\in I_-^\pi} \sum_{l=1}^{\nu_\pi(i,j)-1} x_{i,j}^{p^l} \bar{e}_{\pi^l(i),\pi^l(j)},$$

$$h_0[1]h_-[1] = 1_M[1] + \sum_{(i,j)\in I_-^\pi} \sum_{l=0}^{\nu_\pi(i,j)-1} x_{i,j}^{p^l} \bar{e}_{\pi^l(i),\pi^l(j)},$$

where $x_{i,j} \in I_-^\pi$ can take independently all values in k such that $h_0[1] \in \mathbf{G}_1(k)$. This shows that $\operatorname{Lie}(\mathbf{C}_1^0) = \bigoplus_{(i,j)\in I_-^\pi} k\bar{e}_{i,j}$, which contains no non-zero semi-simple elements. Thus $\mathbf{C}_1^0$ has no subgroup isomorphic to $\mathbb{G}_m$ and hence it is unipotent. We also get that the dimension of $\mathbf{C}_1^0$ is equal to the cardinality of I_-^{π}. Therefore the dimension of $\mathbf{O}_1$ is equal to the cardinality of the set $I^2 - I_-^{\pi}$.

Proposition 3.10. *For every $s \geq 1$, the connected smooth group scheme $\mathbf{C}_s^0$ is unipotent.*

Proof. We proceed by induction. The base case $s = 1$ is checked in Example 3.9. Suppose $\mathbf{C}_{s-1}^0$ is unipotent. The image of $\mathbf{C}_s^0$ under the reduction map $\mathrm{Red}_{s,\mathbf{G}} : \mathbf{G}_s \to \mathbf{G}_{s-1}$ is in $\mathbf{C}_{s-1}^0$, and thus is unipotent. The kernel of $\mathbf{C}_s^0 \to \mathbf{C}_{s-1}^0$ is in the kernel of $\mathrm{Red}_{s,\mathbf{G}}$, and thus is unipotent. Therefore $\mathbf{C}_s^0$ is an extension of unipotent group schemes, and thus is unipotent; see [2, Exp. XVII, Prop 2.2]. □

We construct a homomorphism $\Lambda_s : \mathbf{C}_s \to \mathbf{A}_s$ as follows. For every k-algebra R, let $h[s] \in \mathbf{C}_s(R)$. Thus $\varphi(h[s]) = h[s]$. Fix a $\mathbb{Z}_p$-basis $\mathcal{B}_0 = \{v_1, \ldots, v_r\}$ of $\mathcal{M}_0$. Let $B_0^* = \{e_{ij}\}$ be the standard $\mathbb{Z}_p$-basis of $\mathrm{End}_{\mathbb{Z}_p}(M_0)$ induced by B_0. If $h = \sum_{i,j} e_{ij} \otimes a_{ij} \in \mathrm{End}_{\mathbb{Z}_p}(M_0) \otimes W(R) = \mathrm{End}_{W(R)}(M \otimes_{W(k)} W(R))$, where $a_{ij} \in W(R)$, then $\varphi(h[s]) = h[s]$ is equivalent to

$$\sum_{i,j} \varphi e_{ij} \varphi^{-1} \otimes \sigma_R(a_{ij}) \equiv \sum_{i,j} e_{ij} \otimes a_{ij} \text{ modulo } \theta^s(W(R)). \tag{3.10}$$

Let $C = (c_{ij})$ be the matrix representation of φ with respect to $\mathcal{B}_0$ and $C^{-1} = (c'_{ij})$ be its inverse. Using the matrix notation, (3.10) can be restated as

$$(c_{ij})(\sigma_R(a_{ij}))(\sigma_R(c'_{ij})) \equiv (a_{ij}) \text{ modulo } \theta^s(W(R)). \tag{3.11}$$

This implies that a lift h of $h[s]$ satisfies the equation that defines $\mathbf{A}_s$. Thus we can define $\Lambda_s(R)(h[s]) = h[s]$.

Lemma 3.11. *The homomorphism $\Lambda_s(k) : \mathbf{C}_s(k) \to \mathbf{A}_s(k)$ is an isomorphism. Therefore, Λ_s is a finite epimorphism and thus* $\dim(\mathbf{C}_s) = \gamma_{\mathcal{M}}(s)$.

Proof. The group $\mathbf{C}_s(k)$ consists of all $h \in \mathbf{G}_s(k)$ such that $h \equiv \varphi h \varphi^{-1}$ modulo p^s, which are exactly all automorphisms of $F_s(\mathcal{M})$. As $\mathbf{A}_s(k)$ is also the group of automorphisms of $F_s(\mathcal{M})$ and $\Lambda_s(k)$ is the identity map, we know that they are isomorphic.

As $\Lambda_s(k)$ is an isomorphism, Λ_s is a finite epimorphism. Therefore $\dim(\mathbf{C}_s) = \dim(\mathbf{C}_s^0) = \dim(\mathbf{A}_s^0) = \dim(\mathbf{A}_s)$, which by definition is $\gamma_{\mathcal{M}}(s)$. □

Let $\mathbf{T}_{s+1}$ be the reduced group of the group subscheme $\mathrm{Red}_{s+1,\mathbf{G}}^{-1}(\mathbf{C}_s)$ of $\mathbf{G}_{s+1}$, and let $\mathbf{T}_{s+1}^0$ be its identity component. We have a short exact sequence

$$1 \to \mathrm{Ker}(\mathrm{Red}_{s+1,\mathbf{G}}) \to \mathbf{T}_{s+1}^0 \to \mathbf{C}_s^0 \to 1. \tag{3.12}$$

Thus $\mathbf{T}^0_{s+1}$ is unipotent as $\mathrm{Ker}(\mathrm{Red}_{s+1,\mathbf{G}})$ and $\mathbf{C}^0_s$ are. We have the following equality

$$\dim(\mathbf{T}^0_{s+1}) = \dim(\mathrm{Ker}(\mathrm{Red}_{s+1,\mathbf{G}})) + \dim(\mathbf{C}^0_s) = r^2 + \dim(\mathbf{C}^0_s). \quad (3.13)$$

By Lemma 3.11 and (3.13), we know that

$$\dim(\mathbf{T}^0_{s+1}) = r^2 + \gamma_{\mathcal{M}}(s). \quad (3.14)$$

By (3.4) and the fact that $\mathrm{Red}_{s+1,\mathbf{G}}$ is an epimorphism whose kernel has dimension r^2, we know that

$$\gamma_{\mathcal{M}}(s+1) - \gamma_{\mathcal{M}}(s) = r^2 - \dim(\mathbf{O}_{s+1}) + \dim(\mathbf{O}_s). \quad (3.15)$$

Let $\mathbf{V}_{s+1}$ be the inverse image of the point $1_M[s] \in \mathbf{D}_s(k)$. It is isomorphic to the kernel of $\mathrm{Red}_{s+1,\mathbf{D}}$ and thus isomorphic to $\mathbb{A}^{r^2}_k$. The inverse image of $\mathbf{O}_s$ under $\mathrm{Red}^{-1}_{s+1,\mathbf{D}}$ in $\mathbf{D}_{s+1}$ is a union of orbits and $\mathbf{O}_{s+1}$ is one of them. Let $\mathscr{O}_{s+1,s}$ be the set of orbits of the action $\mathbb{T}_{s+1}$ that is contained in $\mathrm{Red}^{-1}_{s+1,\mathbf{D}}(\mathbf{O_s})$. Every orbit in $\mathscr{O}_{s+1,s}$ intersects $\mathbf{V}_{s+1}$ nontrivially.

We now give another description of $\mathscr{O}_{s+1,s}$ in terms of F-truncations modulo p^s of F-crystals. Let $\mathscr{I}_s$ be the set of all F-crystals $\mathcal{M}(g)$ with $g \in \mathbf{GL}_M(W(k))$ up to F-truncations modulo p^s isomorphisms. In other words, if $F_s(\mathcal{M}(g_1))$ is isomorphic to $F_s(\mathcal{M}(g_2))$, then we identify $\mathcal{M}(g_1)$ and $\mathcal{M}(g_2)$ in $\mathscr{I}_s$. By Proposition 3.7, we know that there is a bijection between the set of orbits of $\mathbb{T}_s$ and $\mathscr{I}_s$.

Proposition 3.12. *There is a bijection between $\mathscr{O}_{s+1,s}$ and the subset of $\mathscr{I}_{s+1}$ that contains all $\mathcal{M}(g)$ (up to F-truncation modulo p^{s+1} isomorphism) such that $F_s(\mathcal{M}(g))$ is isomorphic to $F_s(\mathcal{M})$. Therefore, $\mathscr{O}_{s+1,s}$ has only one orbit if $s \geq n_{\mathcal{M}}$.*

Proof. The first part of the proposition follows from the following fact: for every $g \in \mathbf{GL}_M(W(k)$, an orbit of $\mathbb{T}_{s+1}$ that contains the F-truncation modulo p^{s+1} of the F-crystal $\mathcal{M}(g)$ is in $\mathrm{Red}^{-1}_{s+1,\mathbf{D}}(\mathbf{O_s})$ if and only if $F_s(\mathcal{M}(g))$ is isomorphic to $F_s(\mathcal{M})$.

If $s \geq n_{\mathcal{M}}$, let $\mathcal{M}(g)$ be an F-crystal such that $F_s(\mathcal{M}(g))$ is isomorphic to $F_s(\mathcal{M})$. By Corollary 2.5, $\mathcal{M}(g)$ is isomorphic to $\mathcal{M}$. Thus $\mathscr{O}_{s+1,s}$ contains only one element by the first part of the proposition. □

3.4 Monotonicity of $\gamma_{\mathcal{M}}(i)$

Lemma 3.13. *The following two statements are equivalent:*

(i) $\dim(\mathbf{O}_{s+1}) = \dim(\mathbf{O}_s) + r^2$;
(ii) $\mathbf{V}_{s+1} \subset \mathbf{O}_{s+1}$.

Proof. As $\text{Red}_{s+1,\mathbf{D}} : \mathbf{O}_{s+1} \to \mathbf{O}_s$ is faithfully flat, the fibers of this morphism are equidimensional. Hence we have

$$\dim(\mathbf{O}_{s+1}) = \dim(\mathbf{O}_s) + \dim(\mathbf{V}_{s+1} \cap \mathbf{O}_{s+1}). \tag{3.16}$$

If (i) holds, as $\dim(\mathbf{V}_{s+1} \cap \mathbf{O}_{s+1}) = \dim(\mathbf{O}_{s+1}) - \dim(\mathbf{O}_s) = r^2 = \dim(\mathbf{V}_{s+1})$, $\mathbf{V}_{s+1} \cap \mathbf{O}_{s+1}$ is open in $\mathbf{V}_{s+1}$.

Consider the action $\mathbb{T}^0_{s+1} : \mathbf{T}^0_{s+1} \times_k \mathbf{V}_{s+1} \to \mathbf{V}_{s+1}$. By [10, Proposition 2.4.14], we know that all the orbits of $\mathbb{T}^0_{s+1}$ are closed. As the orbits of the action $\mathbb{T}_{s+1} : \mathbf{T}_{s+1} \times_k \mathbf{V}_{s+1} \to \mathbf{V}_{s+1}$ is a finite union of the orbits of the action $\mathbb{T}^0_{s+1}$, we know that the orbits of the action $\mathbb{T}_{s+1}$ is also closed. The orbit of $1_M[s+1]$ under the action of $\mathbb{T}_{s+1}$ is $(\mathbf{V}_{s+1} \cap \mathbf{O}_{s+1})_{\text{red}}$. Because it is an open, closed and dense orbit of $\mathbb{T}_{s+1}$, we know that $\mathbf{V}_{s+1} \cap \mathbf{O}_{s+1} = \mathbf{V}_{s+1}$. Hence $\mathbf{V}_{s+1} \subset \mathbf{O}_{s+1}$.

If (ii) holds, as $\dim(\mathbf{V}_{s+1} \cap \mathbf{O}_{s+1}) = \dim(\mathbf{V}_{s+1}) = r^2$, (i) follows from (3.16). □

Corollary 3.14. *$\gamma_{\mathcal{M}}(s+1) = \gamma_{\mathcal{M}}(s)$ if and only if $\mathscr{O}_{s+1,s}$ has only one element.*

Proof. The first part of the Lemma 3.13 is equivalent to $\gamma_{\mathcal{M}}(s+1) = \gamma_{\mathcal{M}}(s)$ and the second part of the Lemma 3.13 is equivalent to $\mathscr{O}_{s+1,s}$ has only one element. □

Theorem 3.15. *For every F-crystal $\mathcal{M}$, we have*

$$0 \leq \gamma_{\mathcal{M}}(1) < \gamma_{\mathcal{M}}(2) < \cdots < \gamma_{\mathcal{M}}(n_{\mathcal{M}}) = \gamma_{\mathcal{M}}(n_{\mathcal{M}}+1) = \cdots$$

Proof. We first show that for every $1 \leq i \leq n_{\mathcal{M}} - 1$, $\gamma_{\mathcal{M}}(i) \neq \gamma_{\mathcal{M}}(i+1)$. Suppose the contrary, then by Proposition 2.11, $\gamma_{\mathcal{M}}(i) = \gamma_{\mathcal{M}}(j)$ for all $j \geq i$. In particular, we have $\gamma_{\mathcal{M}}(n_{\mathcal{M}}) = \gamma_{\mathcal{M}}(n_{\mathcal{M}} - 1)$. By Corollary 3.14, $\mathscr{O}_{n_{\mathcal{M}},n_{\mathcal{M}}-1}$ contains one element. Let $\mathcal{M}(g)$ be an F-crystal such that $F_{n_{\mathcal{M}}-1}(\mathcal{M}(g))$ is isomorphic to $F_{n_{\mathcal{M}}-1}(\mathcal{M})$, by Proposition 3.12, there is a unique $\mathcal{M}(g)$ up to F-truncation modulo $p^{n_{\mathcal{M}}}$ such that $F_{n_{\mathcal{M}}-1}(\mathcal{M}(g))$ is isomorphic to $F_{n_{\mathcal{M}}-1}(\mathcal{M})$, thus $F_{n_{\mathcal{M}}}(\mathcal{M}(g))$ is isomorphic to $F_{n_{\mathcal{M}}}(\mathcal{M})$. By Lemma 2.2, $\mathcal{M}(g)$ is isomorphic to $\mathcal{M}$. Hence we conclude that $n_{\mathcal{M}} - 1$ is the isomorphism number of $\mathcal{M}$, which is a contradiction.

If $s \geq n_{\mathcal{M}}$, then every F-crystal $\mathcal{M}(g)$ such that $F_s(\mathcal{M}(g))$ is isomorphic to $F_s(\mathcal{M})$, is isomorphic to $\mathcal{M}$. Therefore $F_{s+1}(\mathcal{M}(g))$ is isomorphic to $F_{s+1}(\mathcal{M})$, whence $\mathscr{O}_{s+1,s}$ has only one element. By Corollary 3.14, $\gamma_{\mathcal{M}}(s+1) = \gamma_{\mathcal{M}}(s)$ for all $s \geq n_{\mathcal{M}}$. □

We have a converse of Proposition 2.12.

Proposition 3.16. *If there exists an $s \geq 1$ such that $\gamma_{\mathcal{M}}(s) = 0$, then $\mathcal{M}$ is ordinary.*

Proof. For some $s \geq 1$, if $\gamma_{\mathcal{M}}(s) = 0$, we know that $\gamma_{\mathcal{M}}(1) = 0$ by Theorem 3.15. By Lemma 3.11, we can assume that $\dim(\mathbf{C}_1^0) = 0$. Hence $|I_-^\pi| = 0$; see Example 3.9 for the definition of I_-^π.

As (M, φ) is isomorphic to $(M, g\varphi_\pi)$ for some $g \equiv 1$ modulo p and the isomorphism number of an ordinary F-crystal is less than or equal to 1 (for example, see [18, Section 2.3]), in order to show that $\mathcal{M}$ is ordinary, it is enough to show that (M, φ_π) is ordinary. Write π as a product of disjoint cycle π_u, it is clear that $(M, \varphi_\pi) = \bigoplus_u (M, \varphi_{\pi_u})$. To show that $\mathcal{M}$ is ordinary we can assume that π is a cycle and show that (M, φ_π) is isoclinic ordinary. Let r be the rank of M.

As π is an r-cycle, we know that $(\pi \times \pi) = \prod_{u=1}^r (\pi \times \pi)_u$ and each $(\pi \times \pi)_u$ is an r-cycle (as a permutation of $I \times I$). Recall a pair $(i, j) \in I \times I$ is said to be in $(\pi \times \pi)_u$ if and only if $(\pi \times \pi)_u(i, j) \neq (i, j)$. It is easy to see that if there is a pair $(i, j) \in I_+$ in $(\pi \times \pi)_u$, then there is also a pair $(i, j) \in I_-$ in $(\pi \times \pi)_u$, and vice versa. Since I_-^π is an empty set, we know that there is no $(\pi \times \pi)_u$ such that $(\pi \times \pi)_u$ sends an element in I_- to an element in I_+ by an argument used in Example 3.9. This means that if there is an element in I_- (or I_+ respectively) that is also in $(\pi \times \pi)_u$, then there are elements also elements in I_0 and in I_+ (or I_- respectively) that are in $(\pi \times \pi)_u$.

The fact that I_-^π is empty means that for all $(i, j) \in I_-$, if $\nu_\pi(i, j)$ is the smallest positive integer such that $(\pi^{\nu_\pi(i,j)}(i), \pi^{\nu_\pi(i,j)}(j)) \in I_+ \cup I_-$, then it is in I_-. Start with an element $(i, j) \in I_-$, and apply this fact recursively. We can see that for every integer n, $(\pi^n(i), \pi^n(j)) \notin I_+$. This is a contradiction as we know that there must be some element in I_+ that is in $(\pi \times \pi)_u$. Therefore we conclude that every element in $(\pi \times \pi)_u$ is in I_0. This means that all the Hodge slopes of (M, φ_π) are equal and hence (M, φ_π) is isoclinic ordinary. □

Corollary 3.17. *The inequality $\gamma_{\mathcal{M}}(1) \geq 0$ is an equality if and only if $\mathcal{M}$ is ordinary. When the equality holds, we have $\gamma_{\mathcal{M}}(s) = 0$ for all $s \geq 1$.*

4. Invariants

In this section, we introduce several invariants of F-crystals over k. They are the generalizations of the p-divisible groups case introduced in [7]. It will turn out that these invariants are all equal to the isomorphism number. They provide a good source of computing the isomorphism number from different points of view. All the proofs of this section follow closely the ones of [7].

4.1 Notations

Recall that for every F-crystal $\mathcal{M} = (M, \varphi)$ and every field extension $k \subset k'$ with k' perfect, we have an F-crystal over k'

$$\mathcal{M}_{k'} = (M_{k'}, \varphi_{k'}) := (M \otimes_{W(k)} W(k'), \varphi \otimes \sigma_{k'}).$$

We denote by $\mathcal{M}^* = (M^*, \varphi)$ the dual of $\mathcal{M}$, where $M^* = \mathrm{Hom}_{W(k)}(M, W(k))$ and $\varphi(f) = \varphi f \varphi^{-1}$ for $f \in M^*$. Note that the pair (M^*, φ) is not an F-crystal in general, it is just a latticed F-isocrystal (meaning that φ is an isomorphism after tensored with $B(k)$ but $\varphi(M^*) \not\subset M^*$ in general). We denote by $H_\infty = \mathrm{Hom}(\mathcal{M}_1, \mathcal{M}_2)$ the (additive) group of all homomorphisms of F-crystals from $\mathcal{M}_1$ to $\mathcal{M}_2$. It is a finitely generated $\mathbb{Z}_p$-module. For every integer $s \geq 1$, let $H_s = \mathrm{Hom}_s(\mathcal{M}_1, \mathcal{M}_2) = \mathbf{Hom}_s(\mathcal{M}_1, \mathcal{M}_2)(k)$ be the (additive) group of all homomorphisms from $F_s(\mathcal{M}_1)$ to $F_s(\mathcal{M}_2)$. It is a $\mathbb{Z}_p/p^s\mathbb{Z}_p$-module but not necessarily finitely generated in general. We denote by $\pi_{\infty,s} : H_\infty \to H_s$ and $\pi_{t,s} : H_t \to H_s, t \geq s$ the natural projections. We have two exact sequences:

$$0 \longrightarrow H_\infty \xrightarrow{p^s} H_\infty \xrightarrow{\pi_{\infty,s}} H_s,$$

and

$$0 \longrightarrow H_s \xrightarrow{p} H_{s+1} \xrightarrow{\pi_{s+1,1}} H_1.$$

Let r_1 and r_2 be the ranks of M_1 and M_2 respectively.

4.2 The endomorphism number

In this subsection, we generalize the endomorphism number defined in [7, Section 2] for p-divisible groups. The following proposition is a generalization of [7, Lemma 2.1]. For the sake of generality, we will work with the homomorphism version.

Proposition 4.1. *There exists a non-negative integer $e_{\mathcal{M}_1,\mathcal{M}_2}$ which depends only on $\mathcal{M}_1$ and $\mathcal{M}_2$ with the following property: For every positive integer n and every non-negative integer e, we have* $\mathrm{Im}(\pi_{\infty,n}) = \mathrm{Im}(\pi_{n+e,n})$ *if and only if $e \geq e_{\mathcal{M}_1,\mathcal{M}_2}$.*

Proof. We first prove that $e_{\mathcal{M}_1,\mathcal{M}_2}$ exists for each n and then prove that it does not depend on n. Note that $\pi_{\infty,n} = \pi_{n+1,n} \circ \pi_{\infty,n+1}$ and $\pi_{n+e,n} = \pi_{n+1,n} \circ \pi_{n+e,n+1}$. If $\mathrm{Im}(\pi_{\infty,n+1}) = \mathrm{Im}(\pi_{n+e,n+1})$ for all $e - 1 \geq e_{\mathcal{M}_1,\mathcal{M}_2}(n+1)$, then $\mathrm{Im}(\pi_{\infty,n}) = \mathrm{Im}(\pi_{n+e,n})$. Thus $e_{\mathcal{M}_1,\mathcal{M}_2}(n) \leq e_{\mathcal{M}_1,\mathcal{M}_2}(n+1)+1$. Hence to show that $e_{\mathcal{M}_1,\mathcal{M}_2}(n)$ exists for all positive integer n, it is enough to show that $e_{\mathcal{M}_1,\mathcal{M}_2}(n)$ exists for sufficient large n.

Let $H'_n := \mathrm{Hom}_{W_n(k)}((M_1/p^n M_1, \varphi_1), (M_2/p^n M_2, \varphi_2))$. It is the (additive) group of all $W_n(k)$-linear homomorphisms $h : M_1/p^n M_1 \to M_2/p^n M_2$ such that $\varphi_2 h \equiv h\varphi_1$ modulo p^n. Thus H_n is a subgroup of H'_n.

The existence of $e_{\mathcal{M}_1,\mathcal{M}_2}$ for each n relies on the following commutative diagram:

$$\begin{array}{ccccc}
H_\infty & \xrightarrow{\pi_{\infty,n}} & H_n & \xleftarrow{\pi_{n+e,n}} & H_{n+e} \\
 & \searrow^{\pi'_{\infty,n}} & \big\downarrow & & \big\downarrow \\
 & & H'_n & \xleftarrow{\pi'_{n+e,n}} & H'_{n+e}
\end{array}$$

where $\pi'_{\infty,n}$ and $\pi'_{n+e,n}$ are the natural projections.

By [13, Theorem 5.1.1(a)], we know that for any sufficient large n (in fact $n \geq n_{12}$), there exists a positive integer $e_{\mathcal{M}_1,\mathcal{M}_2}(n)$ such that for all $e \geq e_{\mathcal{M}_1,\mathcal{M}_2}(n)$, $\mathrm{Im}(\pi'_{\infty,n}) = \mathrm{Im}(\pi'_{n+e,n})$. Therefore the images of $\mathrm{Im}(\pi_{\infty,n})$ and $\mathrm{Im}(\pi_{n+e,n})$ in H'_n are the same. Thus $\mathrm{Im}(\pi_{n+e,n}) = \mathrm{Im}(\pi_{\infty,n})$ for all $e \geq e_{\mathcal{M}_1,\mathcal{M}_2}(n)$. This proves that $e_{\mathcal{M}_1,\mathcal{M}_2}(n)$ exists for each n.

Now we show that $e_{\mathcal{M}_1,\mathcal{M}_2}(n)$ does not depend on n. The proof relies on the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Im}(\pi_{\infty,n}) & \xrightarrow{p} & \mathrm{Im}(\pi_{\infty,n+1}) & \longrightarrow & \mathrm{Im}(\pi_{\infty,1}) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle i_1} & & \big\downarrow{\scriptstyle i_2} & & \big\downarrow{\scriptstyle i_3} & & \\
0 & \longrightarrow & \mathrm{Im}(\pi_{n+e,n}) & \xrightarrow{p} & \mathrm{Im}(\pi_{n+1+e,n+1}) & \longrightarrow & \mathrm{Im}(\pi_{1+e,1}) & &
\end{array}$$

with horizontal exact sequences and with all vertical maps injective. By the snake lemma, we have an exact sequence

$$0 \to \mathrm{Coker}(i_1) \to \mathrm{Coker}(i_2) \to \mathrm{Coker}(i_3).$$

If we take $e \geq e_{\mathcal{M}_1,\mathcal{M}_2}(n+1)$, then $\mathrm{Coker}(i_2) = 0$. Thus $\mathrm{Coker}(i_1) = 0$ and $e_{\mathcal{M}_1,\mathcal{M}_2}(n+1) \geq e_{\mathcal{M}_1,\mathcal{M}_2}(n)$. If we take $e \geq \max(e_{\mathcal{M}_1,\mathcal{M}_2}(n), e_{\mathcal{M}_1,\mathcal{M}_2}(1))$, then $\mathrm{Coker}(i_2) = 0$. Thus $e_{\mathcal{M}_1,\mathcal{M}_2}(n+1) \leq \max(e_{\mathcal{M}_1,\mathcal{M}_2}(n), e_{\mathcal{M}_1,\mathcal{M}_2}(1))$. An easy induction on $n \geq 1$ using the sequence of inequalities

$$e_{\mathcal{M}_1,\mathcal{M}_2}(n) \leq e_{\mathcal{M}_1,\mathcal{M}_2}(n+1) \leq \max(e_{\mathcal{M}_1,\mathcal{M}_2}(n), e_{\mathcal{M}_1,\mathcal{M}_2}(1))$$

gives that $e_{\mathcal{M}_1,\mathcal{M}_2}(n) = e_{\mathcal{M}_1,\mathcal{M}_2}(1)$ for all n. □

Definition 4.2. *The non-negative integer $e_{\mathcal{M}_1,\mathcal{M}_2}$ of Proposition 4.1 is called the* homomorphism number *of $\mathcal{M}_1$ and $\mathcal{M}_2$. If $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, we denote $e_{\mathcal{M},\mathcal{M}}$ by $e_{\mathcal{M}}$ and call it the* endomorphism number *of $\mathcal{M}$.*

The following lemma is a generalization of [7, Lemma 2.8(c)] and is proved in a similar way.

Lemma 4.3. *Let $k \subset k'$ be an extension of algebraically closed fields. We have $e_{\mathcal{M}_1,\mathcal{M}_2} = e_{\mathcal{M}_{1,k'},\mathcal{M}_{2,k'}}$.*

Proof. When $m \geq n$, let $\pi_{m,n} : \mathbf{H}_m \to \mathbf{H}_n$ be the canonical reduction homomorphism and let $\mathbf{H}_{m,n}$ be the scheme theoretic image of $\pi_{m,n}$, which is of finite type over k, and whose definition is compatible with base change $k \subset k'$. If $l \geq m$, then $\mathbf{H}_{l,n}$ is a subgroup scheme of $\mathbf{H}_{m,n}$. By the definition of $e_{\mathcal{M}_1,\mathcal{M}_2}$, we have $m - n \geq e_{\mathcal{M}_1,\mathcal{M}_2}$ if and only if $\mathbf{H}_{m,n}(k) = \mathbf{H}_{l,n}(k)$. As k and k' are algebraically closed, we have $\mathbf{H}_{m,n}(k) = \mathbf{H}_{l,n}(k)$ if and only if $\mathbf{H}_{m,n}(k') = \mathbf{H}_{l,n}(k')$. This is further equivalent to $(\mathbf{H}_{m,n})_{k'}(k') = (\mathbf{H}_{l,n})_{k'}(k')$, thus $e_{\mathcal{M}_1,\mathcal{M}_2} = e_{\mathcal{M}_{1,k'},\mathcal{M}_{2,k'}}$. □

4.3 Coarse endomorphism number

In this subsection, we generalize the coarse endomorphism number defined in [7, Section 7] for p-divisible groups. The following proposition is a generalization of [7, Lemma 7.1]. Again for the sake of generality, we will work with the homomorphism version.

Lemma 4.4. *There exists a non-negative integer $f_{\mathcal{M}_1,\mathcal{M}_2}$ that depends on $\mathcal{M}_1$ and $\mathcal{M}_2$ such that for positive integers $m \geq n$, the restriction homomorphism $\pi_{m,n} : H_m \to H_n$ has finite image if and only if $m \geq n + f_{\mathcal{M}_1,\mathcal{M}_2}$.*

Proof. As H_∞ is a finitely generated $\mathbb{Z}_p$-module, $\mathrm{Im}(\pi_{\infty,n})$ inside the p^n-torsion $\mathbb{Z}_p$-module H_n is finite. By Proposition 4.1, there exists $f_{\mathcal{M}_1,\mathcal{M}_2}(n)$ such that for all $m \geq n + f_{\mathcal{M}_1,\mathcal{M}_2}(n)$, $\mathrm{Im}(\pi_{m,n}) = \mathrm{Im}(\pi_{\infty,n})$ is finite.

To show that $f_{\mathcal{M}_1,\mathcal{M}_2}(n)$ is independent of n, we consider the exact sequence

$$0 \longrightarrow \mathrm{Im}(\pi_{m,n}) \xrightarrow{\ p\ } \mathrm{Im}(\pi_{m,n+1}) \longrightarrow \mathrm{Im}(\pi_{m,1}) .$$

It implies that

$$f_{\mathcal{M}_1,\mathcal{M}_2}(n) \leq f_{\mathcal{M}_1,\mathcal{M}_2}(n+1) \leq \max(f_{\mathcal{M}_1,\mathcal{M}_2}(n), f_{\mathcal{M}_1,\mathcal{M}_2}(1))$$

An easy induction on $n \geq 1$ shows that $f_{\mathcal{M}_1,\mathcal{M}_2}(n) = f_{\mathcal{M}_1,\mathcal{M}_2}(1)$ for all $n \geq 1$. □

Definition 4.5. *The non-negative integer $f_{\mathcal{M}_1,\mathcal{M}_2}$ of Lemma 4.4 is called the* coarse homomorphism number *of $\mathcal{M}_1$ and $\mathcal{M}_2$. If $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, we denote $f_{\mathcal{M},\mathcal{M}}$ by $f_{\mathcal{M}}$ and call it the* coarse endomorphism number *of $\mathcal{M}$.*

Proposition 4.6. *We have an inequality $f_{\mathcal{M}_1,\mathcal{M}_2} \leq e_{\mathcal{M}_1,\mathcal{M}_2}$.*

Proof. It is clear as $\mathrm{Im}(\pi_{\infty,n})$ is finite. □

Lemma 4.7. *Let $k \subset k'$ be an extension of algebraically closed fields. We have $f_{\mathcal{M}_1,\mathcal{M}_2} = f_{\mathcal{M}_{1,k'},\mathcal{M}_{2,k'}}$.*

Proof. For positive integers $m \geq n$, we have $m - n < f_{\mathcal{M}_1,\mathcal{M}_2}$ if and only if the image of $\pi_{m,n}$ is infinite by definition. It is further equivalent to the image of $\mathbf{H}_m \to \mathbf{H}_n$ having positive dimension. This property is invariant under base change from k to k' and hence the lemma follows. □

4.4 Level torsion

In this subsection, we generalize the level torsion defined in [7, Section 8.1] for p-divisible groups.

Let H_{12} be the set of all $W(k)$-linear homomorphisms from M_1 to M_2. We have a latticed F-isocrystal (H_{12}, φ_{12}) where $\varphi_{12} : H_{12} \otimes_{W(k)} B(k) \to H_{12} \otimes_{W(k)} B(k)$ is a σ-linear isomorphism defined by the rule $\varphi_{12}(h) = \varphi_2 h \varphi_1^{-1}$. By Dieudonné-Manin's classification of F-isocrystals, we have finite direct sum decompositions

$$(M_1 \otimes_{W(k)} B(k), \varphi_1) \cong \bigoplus_{\lambda_1 \in J_1} E_{\lambda_1}^{m_{\lambda_1}}, \quad (M_2 \otimes_{W(k)} B(k), \varphi_2) \cong \bigoplus_{\lambda_2 \in J_2} E_{\lambda_2}^{m_{\lambda_2}}$$

where the simple F-isocrystals E_{λ_1} and E_{λ_2} have Newton slopes equal to λ_1 and λ_2 respectively, the multiplicities $m_{\lambda_1}, m_{\lambda_2} \in \mathbb{Z}_{>0}$ and the finite index sets $J_1, J_2 \subset \mathbb{Q}_{>0}$ are uniquely determined. From these decompositions, we obtain a direct sum decomposition

$$(H_{12} \otimes_{W(k)} B(k), \varphi_{12}) \cong L_{12}^{+} \oplus L_{12}^{0} \oplus L_{12}^{-},$$

where

$$L_{12}^{+} = \bigoplus_{\lambda_1 < \lambda_2} \mathrm{Hom}(E_{\lambda_1}^{m_{\lambda_1}}, E_{\lambda_2}^{m_{\lambda_2}}), \quad L_{12}^{-} = \bigoplus_{\lambda_1 > \lambda_2} \mathrm{Hom}(E_{\lambda_1}^{m_{\lambda_1}}, E_{\lambda_2}^{m_{\lambda_2}}),$$

$$L_{12}^{0} = \bigoplus_{\lambda_1 = \lambda_2} \mathrm{Hom}(E_{\lambda_1}^{m_{\lambda_1}}, E_{\lambda_2}^{m_{\lambda_2}}).$$

Define

$$O_{12}^{+} = \bigcap_{i=0}^{\infty} \varphi_{12}^{-i}(H_{12} \cap L_{12}^{+}), \quad O_{12}^{-} = \bigcap_{i=0}^{\infty} \varphi_{12}^{i}(H_{12} \cap L_{12}^{-}),$$

$$O_{12}^{0} = \bigcap_{i=0}^{\infty} \varphi_{12}^{-i}(H_{12} \cap L_{12}^{0}) = \bigcap_{i=0}^{\infty} \varphi_{12}^{i}(H_{12} \cap L_{12}^{0}).$$

Let $A^0_{12} = \{x \in H_{12} \mid \varphi_{12}(x) = x\}$ be the $\mathbb{Z}_p$-algebra that contains the elements fixed by φ_{12}. For $\dagger \in \{+, 0, -\}$, each $O^{\dagger}_{12}$ is a lattice of $L^{\dagger}_{12}$. We have the following relations:

$$\varphi(O^+_{12}) \subset O^+_{12}, \qquad \varphi^{-1}(O^-_{12}) \subset O^-_{12},$$
$$\varphi(O^0_{12}) = O^0_{12} = A^0_{12} \otimes_{\mathbb{Z}_p} W(k) = \varphi^{-1}(O^0_{12}).$$

Write $O_{12} := O^+_{12} \oplus O^0_{12} \oplus O^-_{12}$; it is a lattice of $H_{12} \otimes_{W(k)} B(k)$ inside H_{12}. The $W(k)$-module O_{12} is called the *level module* of $\mathcal{M}_1$ and $\mathcal{M}_2$.

Definition 4.8. *The* level torsion *of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the smallest non-negative integer $\ell_{\mathcal{M}_1,\mathcal{M}_2}$ such that*

$$p^{\ell_{\mathcal{M}_1,\mathcal{M}_2}} H_{12} \subset O_{12} \subset H_{12}.$$

If $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, then $\ell_{\mathcal{M}_1,\mathcal{M}_2}$ will be denoted by $\ell_{\mathcal{M}}$.

Remark 4.9. The definition of level torsion in this paper is slightly different from the definition in [16]. When $\mathcal{M}$ is a direct sum of two or more isoclinic ordinary F-crystals of different Newton polygons, its isomorphism number is $n_{\mathcal{M}} = 1$. According to the definition in [16], the level torsion $\ell_{\mathcal{M}} = 1$ but the definition in this paper gives $\ell_{\mathcal{M}} = 0$.

For the duals $\mathcal{M}^*_1$ and $\mathcal{M}^*_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, we can define $\ell_{\mathcal{M}^*_1,\mathcal{M}^*_2}$ in a similar way.

Lemma 4.10. *We have $\ell_{\mathcal{M}_1,\mathcal{M}_2} = \ell_{\mathcal{M}_2,\mathcal{M}_1} = \ell_{\mathcal{M}^*_1,\mathcal{M}^*_2}$.*

Proof. Write $H_{21} := \mathrm{Hom}(M_2, M_1) \cong \mathrm{Hom}(H_{12}, W(k)) =: H^*_{12}$. There is a direct sum decomposition

$$H^*_{12} \otimes_{W(k)} B(k) \cong H_{21} \otimes_{W(k)} B(k) = L^+_{21} \oplus L^0_{21} \oplus L^-_{21}.$$

It is easy to see that

$$L^+_{21} \cong \mathrm{Hom}(L^-_{12}, B(k)) =: L^{-*}_{12}, \quad L^-_{21} \cong \mathrm{Hom}(L^+_{12}, B(k)) =: L^{+*}_{12},$$
$$L^0_{21} \cong \mathrm{Hom}(L^0_{12}, B(k)) =: L^{0*}_{12}$$

are isomorphic as $B(k)$-vector spaces. One can define O_{21} in the same way:

$$O_{21} := O^+_{21} \oplus O^0_{21} \oplus O^-_{21} \cong O^{-*}_{12} \oplus O^{0*}_{12} \oplus O^{+*}_{12}$$

and thus $O_{21} \cong O^*_{12}$. Therefore

$$p^{\ell_{\mathcal{M}_2,\mathcal{M}_1}} H_{21} \subset O_{21} \subset H_{21} \text{ if and only if } p^{\ell_{\mathcal{M}_1,\mathcal{M}_2}} H_{12} \subset O_{12} \subset H_{12},$$

whence $\ell_{\mathcal{M}_1,\mathcal{M}_2} = \ell_{\mathcal{M}_2,\mathcal{M}_1}$. As $H^*_{12} \cong H_{21}$, we get $\ell_{\mathcal{M}_2,\mathcal{M}_1} = \ell_{\mathcal{M}^*_1,\mathcal{M}^*_2}$. □

Lemma 4.11. $\ell_{\mathcal{M}_1\oplus\mathcal{M}_2} = \max\{\ell_{\mathcal{M}_1}, \ell_{\mathcal{M}_2}, \ell_{\mathcal{M}_1,\mathcal{M}_2}\}$.

Proof. The direct sum decomposition into $W(k)$-modules of

$$\mathrm{End}(M_1 \oplus M_2) = \mathrm{End}(M_1) \oplus \mathrm{End}(M_2) \oplus \mathrm{Hom}(M_1, M_2) \oplus \mathrm{Hom}(M_2, M_1)$$

gives birth to the direct sum decomposition of the level module of $\mathcal{M}_1 \oplus \mathcal{M}_2$

$$O = O_{11} \oplus O_{22} \oplus O_{12} \oplus O_{21}.$$

Hence $\ell_{\mathcal{M}_1\oplus\mathcal{M}_2} = \max\{\ell_{\mathcal{M}_1}, \ell_{\mathcal{M}_2}, \ell_{\mathcal{M}_1,\mathcal{M}_2}, \ell_{\mathcal{M}_2,\mathcal{M}_1}\} = \max\{\ell_{\mathcal{M}_1}, \ell_{\mathcal{M}_2}, \ell_{\mathcal{M}_1,\mathcal{M}_2}\}$ by Lemma 4.10. □

Lemma 4.12. *Let $k \subset k'$ be an extension of algebraically closed fields. We have $\ell_{\mathcal{M}_1,\mathcal{M}_2} = \ell_{\mathcal{M}_{1,k'},\mathcal{M}_{2,k'}}$.*

Proof. For $\mathcal{M}_{1,k'}$ and $\mathcal{M}_{2,k'}$, we can define H'_{12} and O'_{12} in an analogous manner. One can check that

$$H'_{12} = H_{12} \otimes_{W(k)} W(k'), \quad O'_{12} = O_{12} \otimes_{W(k)} W(k').$$

The lemma follows easily. □

5. Proof of the Main Theorem

The proofs of this section follow closely the ones of [7, Section 8].

5.1 Notations

For this section, we denote by $H := H_{12}$ the group of $W(k)$-linear homomorphisms from M_1 to M_2, and H_s the group of homomorphisms from $F_s(\mathcal{M}_1)$ to $F_s(\mathcal{M}_2)$. For simplicity, we denote O_{12} by O, $O^\dagger_{12}$ by $O^\dagger$ for $\dagger \in \{+, 0, -\}$, and A^0_{12} by A^0.

5.2 The inequality $e_{\mathcal{M}_1,\mathcal{M}_2} \leq \ell_{\mathcal{M}_1,\mathcal{M}_2}$

We will follow the ideas of [7, Section 8.2] and prove that $\mathrm{Im}(\pi_{\infty,1}) = \mathrm{Im}(\pi_{\ell_{\mathcal{M}_1,\mathcal{M}_2}+1,1})$. For any $\bar{h} \in \mathrm{Im}(\pi_{\ell_{\mathcal{M}_1,\mathcal{M}_2}+1,1})$, let $h \in H_{\ell_{\mathcal{M}_1,\mathcal{M}_2}+1}$ be a preimage of $\bar{h}$. Hence $\varphi_2 h \varphi_1^{-1} \equiv h$ modulo $p^{\ell_{\mathcal{M}_1,\mathcal{M}_2}+1}$, that is, $\varphi_2 h \varphi_1^{-1} - h \in p^{\ell_{\mathcal{M}_1,\mathcal{M}_2}+1}\mathrm{Hom}(M_1, M_2) \subset pO$. By Lemma 5.1 below, there exists $h'' \in pO$ such that

$$\varphi_2 h \varphi_1^{-1} - h = \varphi_2 h'' \varphi_1^{-1} - h''.$$

Thus $h' := h - h'' \in H_\infty$ is a homomorphism whose image in H_1 is exactly $\bar{h}$.

Lemma 5.1. *For each $x \in O$, the equation $x = \varphi_{12}(X) - X$ in X has a solution in O that is unique up to the addition of elements in A^0. Moreover, if $x \in p^s O$, then there exists a solution $X \in p^s O$.*

Proof. Writing $x = x^+ + x^0 + x^-$ with $x^\dagger \in O^\dagger$ for $\dagger \in \{+, 0, -\}$, we will find $y^\dagger \in O^\dagger$ such that $x^\dagger = \varphi_{12}(y^\dagger) - y^\dagger$ for each $\dagger \in \{+, 0, -\}$. Therefore $y = y^+ + y^0 + y^-$ is a solution of the given equation.

Let $y^+ = -\sum_{i=0}^{+\infty} \varphi_{12}^i(x^+)$, and $y^- = \sum_{i=1}^{+\infty} \varphi_{12}^{-i}(x^-)$. Because $\varphi_{12}(O^+) \subset O^+$ and $\varphi_{12}^{-1}(O^-) \subset O^-$, we have $y^+ \in O^+$ and $y^- \in O^-$. It is easy to check that $x^+ = \varphi_{12}(y^+) - y^+$ and $x^- = \varphi_{12}(y^-) - y^-$.

Let $\{v_1, v_2, \ldots, v_r\}$ be a $\mathbb{Z}_p$-basis of A^0; it is also a $W(k)$-basis of O^0. We also write $x^0 = \sum_{i=1}^d x_i v_i$. For $1 \le i \le r$, let $z_i \in W(k)$ be a solution of $\sigma(z_i) - z_i = x_i$ and put $y^0 = \sum_{i=1}^d z_i v_i \in O^0$. Using the fact that $\varphi_{12}(v_i) = v_i$ for all $1 \le i \le r$, it is easy to check that $x^0 = \varphi_{12}(y^0) - y^0$.

If $y, y' \in O$ satisfy the equation $x = \varphi_{12}(X) - X$, we have $\varphi_{12}(y) - y = \varphi_{12}(y') - y'$, i.e. $\varphi_{12}(y - y') = y - y'$, whence $y - y' \in A^0$.

If $x = p^s x' \in p^s O$, then $y = p^s y' \in p^s O$ will be a solution of $x = \varphi_{12}(X) - X$ where $y' \in O$ is a solution of $x' = \varphi_{12}(X) - X$. □

5.3 The inequality $\ell_{\mathcal{M}_1,\mathcal{M}_2} \le f_{\mathcal{M}_1,\mathcal{M}_2}$

We follow the ideas of [7, Section 8.3]. By Lemmas 4.7 and 4.12, we can assume that $k \supset k'[[\alpha]] = R$ where $k \supset k'$ is an extension of algebraically closed fields and for $i = 1, 2$, we have

$$(M_i, \varphi_i) \cong (M_i' \otimes_{W(k')} W(k), \varphi_i' \otimes \sigma),$$

where (M_i', φ_i') are F-crystals over k'. Let $\mathfrak{m}$ be the ideal of R generated by α. Let H' and O' be the analogues of H and O obtained from (M_i', φ_i') instead of (M_i, φ_i). It is easy to check that

$$H = H' \otimes_{W(k')} W(k), \quad O = O' \otimes_{W(k')} W(k),$$

and $p^j O \cap O' = p^j O'$ for $j \in \mathbb{Z}_{\ge 0}$.

Let $x = x^+ + x^0 + x^- \in O'$ where $x^\dagger \in O'^\dagger$, $\dagger \in \{+, 0, -\}$.

Lemma 5.2. *For each $\eta \in W(\mathfrak{m})$, the equation $\eta x = \varphi_{12}(X) - X$ has a solution $x_\eta \in O$, that is unique up to the addition of an element of A^0.*

Proof. Put

$$x_\eta^+ = -\sum_{i=0}^{\infty} \varphi_{12}^i(\eta x^+) \in O^+, x_\eta^- = \sum_{i=1}^{\infty} \varphi_{12}^{-i}(\eta x^-) \in O^-,$$

and

$$x_\eta^0 = -\sum_{i=0}^{\infty} \varphi_{12}^i(\eta x^0) \in O^0.$$

The elements $x_\eta^\pm$ are well-defined as $\{\varphi_{12}^i(x^+)\}_{i\in\mathbb{Z}_{\geq 0}}$ and $\{\varphi_{12}^i(x^-)\}_{i\in\mathbb{Z}_{\geq 0}}$ are p-adically convergent in O'^+ and O'^- respectively. As $\{\sigma^i(\eta)\}_{i\in\mathbb{Z}_{\geq 0}}$ is a α-adically convergent in $W(R)$, x_η^0 is convergent in O^0. One can check that

$$x_\eta := x_\eta^+ + x_\eta^0 + x_\eta^- \in O$$

satisfies $\eta x = \varphi_{12}(x_\eta) - x_\eta$.

Suppose $\eta x = \varphi_{12}(x_\eta) - x_\eta$ and $\eta x = \varphi_{12}(x_\eta^\circ) - x_\eta^\circ$, we have $x_\eta - x_\eta^\circ = \varphi_{12}(x_\eta - x_\eta^\circ)$, hence the lemma. □

We define a homomorphism of abelian groups $\Omega_x : W(\mathfrak{m}) \to H/A^0$ by the formula $\Omega_x(\eta) = x_\eta + A^0$ where $x_\eta \in O \subset H$ satisfies $\eta x = \varphi_{12}(x_\eta) - x_\eta$. By Lemma 5.2, it is well-defined.

Let $x \in p^{\ell_{\mathcal{M}_1,\mathcal{M}_2}} H' \backslash pO'$. For all $\eta \in W(\mathfrak{m})$, $\varphi_{12}(x_\eta) - x_\eta = \eta x \in p^{\ell_{\mathcal{M}_1,\mathcal{M}_2}} H'$, thus $\varphi_{12}(x_\eta) \equiv x_\eta$ modulo $p^{\ell_{\mathcal{M}_1,\mathcal{M}_2}}$. This implies that x_η is a homomorphism modulo $p^{\ell_{\mathcal{M}_1,\mathcal{M}_2}}$ from $\mathcal{M}_1$ to $\mathcal{M}_2$. Hence $x_\eta \in H_{\ell_{\mathcal{M}_1,\mathcal{M}_2}}$. Clearly, every homomorphism of F-crystals is a homomorphism modulo powers of p. Hence $A^0 \subset H_{\ell_{\mathcal{M}_1,\mathcal{M}_2}}$. Thus the image of Ω_x is in $H_{\ell_{\mathcal{M}_1,\mathcal{M}_2}}/A^0$.

Suppose $f_{\mathcal{M}_1,\mathcal{M}_2} < \ell_{\mathcal{M}_1,\mathcal{M}_2}$, we will show that $x \in pO'$, which is a contradiction! Let $\bar{\pi}_{\ell_{M_1,M_2},1} : H_{\ell_{\mathcal{M}_1,\mathcal{M}_2}}/A^0 \to H_1/A^0$ be the homomorphism induced by $\pi_{\ell_{M_1,M_2},1}$. The image of

$$\bar{\pi}_{\ell_{M_1,M_2},1} \circ \Omega_x : W(\mathfrak{m}) \to H_{\ell_{\mathcal{M}_1,\mathcal{M}_2}}/A^0 \to H_1/A^0$$

takes only finitely many values H_1/A^0 as $\mathrm{Im}(\pi_{\ell_{\mathcal{M}_1,\mathcal{M}_2},1})$ is finite by the assumption that $f_{\mathcal{M}_1,\mathcal{M}_2} < \ell_{\mathcal{M}_1,\mathcal{M}_2}$. Since $\mathfrak{m}$ is infinite (and thus $W(\mathfrak{m})$ is infinite), the kernel of $\bar{\pi}_{\ell_{M_1,M_2},1} \circ \Omega_x$ is infinite. There exists $\eta = (\eta_0, \eta_1, \dots) \in W(\mathfrak{m})$ with $\eta_0 \neq 0$ such that $x_\eta \in pH$. Thus $x_\eta \in O \cap pH =: N$. Let $N' = O' \cap pH'$, we have $N \cong N' \otimes_{W(k')} W(k)$.

Lemma 5.3. *An element $\bar{z} \in O/pO$ lies in N/pO if and only if for every k'-linear map $\rho : O'/pO' \to k'$ with $\rho(N'/pO') = 0$ we have $(\rho \otimes 1_k)(\bar{z}) = 0$.*

Proof. For every k'-linear map $\rho : O'/pO' \to k'$ with $\rho(N'/pO') = 0$,

$$\mathrm{Ker}(\rho \otimes 1_k) = \mathrm{Ker}(\rho) \otimes_{k'} k \supset N'/pO' \otimes_{k'} k = N/pO.$$

Set $S = \{\rho : O'/pO' \to k \mid \rho(N'/pO') = 0\}$. We have $\bigcap_{\rho\in S} \mathrm{Ker}(\rho \otimes 1_k) = N/pO$. This concludes the proof. □

By Lemma 5.3, for every k'-linear map $\rho : O'/pO' \to k'$ such that $\rho(N'/pO') = 0$, we have $(\rho \otimes 1_k)(\bar{x}_\eta) = 0$. Therefore the following equality holds in R

$$\sum_{i=0}^{\infty}\rho(\varphi_{12}^{i}(\bar{x}^{+}))\eta_0^{p^i}+\sum_{i=0}^{\infty}\rho(\varphi_{12}^{i}(\bar{x}^{0}))\eta_0^{p^i}-\sum_{i=1}^{\infty}\rho(\varphi_{12}^{-i}(\bar{x}^{-}))\eta_0^{p^{-i}}$$
$$=(\rho\otimes 1_k)(\bar{x}_\eta)=0. \tag{5.1}$$

Because the Newton slopes of (O^+,φ_{12}) and (O^-,φ_{12}^{-1}) are positive, there exists a big enough n such that $\varphi_{12}^{i}(x_+)\in pO'^+$ and $\varphi_{12}^{-i}(x_-)\in pO'^-$ for $i>n$. As $\rho(N'/pO')=0$,

$$\rho(\varphi_{12}^{i}(\bar{x}^{+}))=0,\quad \rho(\varphi_{12}^{-i}(\bar{x}^{-}))=0,\quad \forall\, i>n. \tag{5.2}$$

Thus (5.1) is reduced to

$$-\sum_{i=-n}^{-1}\rho(\varphi_{12}^{i}(\bar{x}^{-}))\eta_0^{p^i}+\sum_{i=0}^{n}(\rho(\varphi_{12}^{i}(\bar{x}^{+}))$$
$$+\rho(\varphi_{12}^{i}(\bar{x}^{0})))\eta_0^{p^i}+\sum_{i=n+1}^{\infty}\rho(\varphi_{12}^{i}(\bar{x}^{0}))\eta_0^{p^i}=0. \tag{5.3}$$

Write

$$\Phi(\beta)=-\sum_{i=0}^{n-1}\rho(\varphi_{12}^{i-n}(\bar{x}^{-}))\beta^{p^i}+\sum_{i=n}^{2n}(\rho(\varphi_{12}^{i-n}(\bar{x}^{+}))+\rho(\varphi_{12}^{i-n}(\bar{x}^{0})))\beta^{p^i}$$
$$+\sum_{i=2n+1}^{\infty}\rho(\varphi_{12}^{i-n}(\bar{x}^{0}))\beta^{p^i}\in k'[[\beta]].$$

Then (5.3) is equivalent to $\Phi(\eta_0^{p^{-n}})=0$ where $\eta_0^{p^{-n}}\in\alpha^{p^{-n}}k'[[\alpha^{p^{-n}}]]$. As $\eta_0^{p^{-n}}\neq 0$, we deduce that $\Phi(\beta)=0$ by [7, Lemma 8.9]. Combining (5.2), we get

$$\rho(\varphi_{12}^{-i}(\bar{x}^{-}))=0,\ \forall\, i\geq 1,\quad \rho(\varphi_{12}^{i}(\bar{x}^{0}))=0,\quad \rho(\varphi_{12}^{i}(\bar{x}^{+}))=0,\ \forall\, i>n$$
$$\rho(\varphi_{12}^{i}(\bar{x}^{+}))+\rho(\varphi_{12}^{i}(\bar{x}^{0}))=0,\ \forall\, i=0,\dots,n. \tag{5.4}$$

As φ_{12} is bijective on O'^0 and thus on O'^0/pO'^0, the subspace $V\subset O'^0/pO'^0$ generated by $\{\varphi_{12}^{i}(\bar{x}^0)\mid i\geq 0\}$ satisfies $\varphi_{12}(V)=V$ and thus $\varphi_{12}^{j}(V)=V$ for every $j\geq 0$. This implies that V is generated by $\{\varphi_{12}^{i}(\bar{x}_0)\mid i>n\}$ and hence for $0\leq i\leq n$, $\varphi_{12}^{i}(\bar{x}_0)$ is a linear combination of elements in $\{\varphi_{12}^{i}(\bar{x}_0)\mid i>n\}$ whence $\rho(\varphi_{12}^{i}(\bar{x}_0))=0$ for all $i=0,\dots,n$. This allows us to extend (5.3) to get

$$\rho(\varphi_{12}^{-i}(\bar{x}^{-}))=0,\ \forall\, i\geq 1,\quad \rho(\varphi_{12}^{i}(\bar{x}^{0}))=0,\quad \rho(\varphi_{12}^{i}(\bar{x}^{+}))=0,\ \forall\, i\geq 0. \tag{5.5}$$

Finally, since $x \in p^{\ell_{\mathcal{M}_1,\mathcal{M}_2}} H'$ and $\ell_{\mathcal{M}_1,\mathcal{M}_2} > f_{\mathcal{M}_1,\mathcal{M}_2} \geq 0$, we have $x \in pH'$ and thus $x \in pH' \cap O' =: N'$. As $\rho(N'/pO') = 0$, we have $0 = \rho(\bar{x}) = \rho(\bar{x}^+ + \bar{x}^0 + \bar{x}^-) = \rho(\bar{x}^-)$. Thus (5.5) can be further extended to

$$\rho(\varphi_{12}^i(\bar{x}^+)) = 0, \quad \rho(\varphi_{12}^i(\bar{x}^0)) = 0, \quad \rho(\varphi_{12}^{-i}(\bar{x}^-)) = 0, \quad \forall\, i \geq 0. \quad (5.6)$$

By Lemma 5.3 and (5.6), we have $\varphi_{12}^i(x^+), \varphi_{12}^i(x^0), \varphi_{12}^{-i}(x^-) \in pH$ and thus in pH' for all $i \geq 0$. By the definition of O', we have $x = x^+ + x^0 + x^- \in pO'$. This reaches the desired contradiction.

5.4 The equality $f_{\mathcal{M}} = n_{\mathcal{M}}$

In this subsection, we show that $f_{\mathcal{M}} = n_{\mathcal{M}}$ when $\mathcal{M}$ is not an ordinary F-crystal. Thus in this case, $n_{\mathcal{M}} > 0$. Recall E_s is the set of all endomorphisms of $F_s(\mathcal{M})$ and $\mathbf{E}_s(k) = E_s$. The restriction homomorphism $\pi_{s,1} : E_s \to E_1$ has finite image if and only if the image of $\pi_{s,1} : \mathbf{E}_s \to \mathbf{E}_1$ has zero dimension, if and only if $s \geq 1 + f_{\mathcal{M}}$ by definition. The dimension of $\pi_{s,1}$ is $\gamma_{\mathcal{M}}(s) - \gamma_{\mathcal{M}}(s-1)$. It is zero if and only if $s > n_{\mathcal{M}}$ by Theorem 3.15. As $s \geq 1 + f_{\mathcal{M}}$ if and only if $s > n_{\mathcal{M}}$, we conclude that $f_{\mathcal{M}} = n_{\mathcal{M}}$.

5.5 Conclusion

By Subsections 5.2, 5.3, 5.4 and Proposition 4.6, we have the following two theorems:

Theorem 5.4. *We have equalities* $f_{\mathcal{M}_1,\mathcal{M}_2} = e_{\mathcal{M}_1,\mathcal{M}_2} = \ell_{\mathcal{M}_1,\mathcal{M}_2}$.

Theorem 5.5. *If* $\mathcal{M}$ *is not ordinary, then* $n_{\mathcal{M}} = f_{\mathcal{M}} = e_{\mathcal{M}} = \ell_{\mathcal{M}}$.

Corollary 5.6. *We have equalities* $f_{\mathcal{M}_1,\mathcal{M}_2} = f_{\mathcal{M}_2,\mathcal{M}_1} = f_{\mathcal{M}_1^*,\mathcal{M}_2^*}$ *and* $e_{\mathcal{M}_1,\mathcal{M}_2} = e_{\mathcal{M}_2,\mathcal{M}_1} = e_{\mathcal{M}_1^*,\mathcal{M}_2^*}$.

Proof. This is clear by Theorem 5.4 and Lemma 4.10. □

6. Application to F-crystal of rank 2

In [18, Theorem 1.4], we proved that if $\mathcal{M}$ is a non-isoclinic F-crystal of rank 2, and is not a direct sum of two F-crystals of rank 1, then $n_{\mathcal{M}} \leq 2\lambda_1$ where λ_1 is the smallest Newton slope of $\mathcal{M}$. Now we show that the inequality is in fact an equality. For the sake of completeness, we state the theorem of isomorphism number of rank 2 in all cases.

Theorem 6.1. *Let $\mathcal{M}$ be an F-crystal of rank 2 with Hodge slopes 0 and $e > 0$. Let λ_1 be the smallest Newton slope of $\mathcal{M}$. Then we have the following three disjoint cases:*

(i) *if $\mathcal{M}$ is a direct sum of two F-crystals of rank 1, then $n_{\mathcal{M}} = 1$;*
(ii) *if $\mathcal{M}$ is not a direct sum of two F-crystals of rank 1 and is isoclinic, then $n_{\mathcal{M}} = e$;*
(iii) *if $\mathcal{M}$ is not a direct sum of two F-crystals of rank 1 and is non-isoclinic, then $n_{\mathcal{M}} = 2\lambda_1$.*

Proof. Parts (i) and (ii) are proved in [18, Theorem 1.4 (i) and (ii)]. In the case of Part (iii), [18, Theorem 1.4 (iii)] proves only the inequality $n_{\mathcal{M}} \leq 2\lambda_1$. The proof of [18, Theorem 1.4 (iii)] has a minor mistake that can be easily fixed. In this paper, we will only prove the equality $n_{\mathcal{M}} = 2\lambda_1$ in the case of Part (iii).

We show that $\lambda_1 > 0$ by showing that the assumption that $\lambda_1 = 0$ leads to a contradiction. If $\lambda_1 = 0$, then the Hodge polygon and the Newton polygon of $\mathcal{M}$ coincide. By [6, Theorem 1.6.1], we can decompose $\mathcal{M}$ into a direct sum of two F-crystals of rank 1. Hence $\lambda_1 > 0$. Let λ_2 be the other Newton slope of $\mathcal{M}$. As $\mathcal{M}$ is not isoclinic, $\lambda_1 < \lambda_2$. It is easy to see that λ_1 and λ_2 are two positive integers. Hence there is a $W(k)$-basis $\mathcal{B}_1 = \{x_1, x_2\}$ of M such that $\varphi(x_1) = p^{\lambda_1}x_1$ and $\varphi(x_2) = ux_1 + p^{\lambda_2}x_2$ where $u \in W(k)$. If u is a non-unit and belongs to $pW(k)$, then $\varphi(M) \subset pM$ and thus the smallest Hodge slope of $\mathcal{M}$ must be positive. This contradicts the assumption of the proposition, hence u is a unit. By solving equations of the form $\varphi(z) = p^{\lambda_1}z$ and $\varphi(z) = p^{\lambda_2}z$, we find a $B(k)$-basis $\mathcal{B}_2 = \{y_1 = x_1, y_2 = vx_1 + p^{\lambda_1}x_2\}$ of $M[1/p]$ with v a unit in $W(k)$ such that $\sigma(v) + u = p^{\lambda_2-\lambda_1}v$. It is easy to see that there is a unique v satisfying this equation.

Let $\mathcal{B}_1 \otimes \mathcal{B}_1^*$ be the $W(k)$-basis of $\mathrm{End}(M)$ that contains $x_i \otimes x_j^*$ for all $1 \leq i, j \leq 2$, where $(x_i \otimes x_j^*)(x_j) = x_i$. It is a $B(k)$-basis of $\mathrm{End}(M[1/p])$. We compute the formula of $\varphi : \mathrm{End}(M[1/p]) \to \mathrm{End}(M[1/p])$ with respect to $\mathcal{B}_1$ as follows:

$$\begin{aligned}
\varphi(x_1 \otimes x_1^*) &= x_1 \otimes x_1^* - p^{-\lambda_2}ux_1 \otimes x_2^*,\\
\varphi(x_2 \otimes x_1^*) &= p^{-\lambda_1}ux_1 \otimes x_1^* + p^{\lambda_2-\lambda_1}x_2 \otimes x_1^*\\
&\quad - p^{-\lambda_1-\lambda_2}u^2x_1 \otimes x_2^* - p^{-\lambda_1}ux_2 \otimes x_2^*,\\
\varphi(x_1 \otimes x_2^*) &= p^{\lambda_1-\lambda_2}x_1 \otimes x_2^*,\\
\varphi(x_2 \otimes x_2^*) &= p^{-\lambda_2}ux_1 \otimes x_2^* + x_2 \otimes x_2^*.
\end{aligned}$$

Similarly the set $\mathcal{B}_2 \otimes \mathcal{B}_2^*$ is another $B(k)$-basis of $\mathrm{End}(M[1/p])$. As $\varphi(y_1) = p^{\lambda_1}y_1$ and $\varphi(y_2) = p^{\lambda_2}y_2$, we compute the formula of

$\varphi : \mathrm{End}(M[1/p]) \to \mathrm{End}(M[1/p])$ with respect to $\mathcal{B}_2$ as follows:

$$\varphi(y_2 \otimes y_1^*) = p^{\lambda_2-\lambda_1} y_2 \otimes y_1^*, \quad \varphi(y_1 \otimes y_1^*) = y_1 \otimes y_1^*,$$
$$\varphi(y_2 \otimes y_2^*) = y_2 \otimes y_2^*, \quad \varphi(y_1 \otimes y_2^*) = p^{\lambda_1-\lambda_2} y_1 \otimes y_2^*.$$

Therefore, we have found $B(k)$-bases for

$$L^+ = \langle y_2 \otimes y_1^* \rangle_{B(k)}, \quad L^0 = \langle y_1 \otimes y_1^*, y_2 \otimes y_2^* \rangle_{B(k)}, \quad L^- = \langle y_1 \otimes y_2^* \rangle_{B(k)}.$$

We compute the change of basis matrix from $\mathcal{B}_1 \otimes \mathcal{B}_1^*$ to $\mathcal{B}_2 \otimes \mathcal{B}_2^*$ as follows:

$$\begin{aligned}
y_1 \otimes y_1^* &= x_1 \otimes x_1^* - \frac{v}{p^{\lambda_1}} x_1 \otimes x_2^*,\\
y_2 \otimes y_1^* &= v x_1 \otimes x_1^* + p^{\lambda_1} x_2 \otimes x_1^* - \frac{v^2}{p^{\lambda_1}} x_1 \otimes x_2^* - v x_2 \otimes x_2^*,\\
y_1 \otimes y_2^* &= \frac{1}{p^{\lambda_1}} x_1 \otimes x_2^*,\\
y_2 \otimes y_2^* &= \frac{v}{p^{\lambda_1}} x_1 \otimes x_2^* + x_2 \otimes x_2^*.
\end{aligned}$$

It is easy to see that $p^{\lambda_1} y_i \otimes y_j^* \in \mathrm{End}(M) \backslash p\mathrm{End}(M)$ for $i, j \in \{1, 2\}$. We get that

(a) $O^+ = \langle p^{\lambda_1} y_2 \otimes y_1 \rangle_{W(k)}$;
(b) $N := \langle y_1 \otimes y_1^* + y_2 \otimes y_2^*, p^{\lambda_1} y_2 \otimes y_2^* \rangle_{W(k)} \subset O^0$ is a lattice;
(c) $O^- = \langle p^{\lambda_1} y_1 \otimes y_2^* \rangle_{W(k)}$.

We now show that in fact $N = O^0$. As $O^0 = A^0 \otimes W(k)$, it is enough to show that $A^0 \subset N$. Suppose

$$a x_1 \otimes x_1^* + b x_2 \otimes x_1^* + c x_1 \otimes x_2^* + d x_2 \otimes x_2^* \in A^0,$$

we have

$$\begin{aligned}
&\varphi(a x_1 \otimes x_1^* + b x_2 \otimes x_1^* + c x_1 \otimes x_2^* + d x_2 \otimes x_2^*)\\
&= (\sigma(a) - \sigma(b) p^{-\lambda_1} u) x_1 \otimes x_1^* + \sigma(b) p^{\lambda_2-\lambda_1} x_2 \otimes x_1^*\\
&\quad + (-\sigma(b) p^{-\lambda_1} u + \sigma(d)) x_2 \otimes x_2^*\\
&\quad + (-\sigma(a) p^{-\lambda_2} u - \sigma(b) p^{-\lambda_1-\lambda_2} u^2 + \sigma(c) p^{\lambda_1-\lambda_2} + \sigma(d) p^{-\lambda_2} u) x_1 \otimes x_2^*\\
&= a x_1 \otimes x_1^* + b x_2 \otimes x_1^* + c x_1 \otimes x_2^* + d x_2 \otimes x_2^*.
\end{aligned}$$

Hence

$$a = \sigma(a) - \sigma(b)p^{-\lambda_1}u, \tag{6.1}$$

$$b = \sigma(b)p^{\lambda_2-\lambda_1}, \tag{6.2}$$

$$c = -\sigma(a)p^{-\lambda_2}u - \sigma(b)p^{-\lambda_1-\lambda_2}u^2 + \sigma(c)p^{\lambda_1-\lambda_2} + \sigma(d)p^{-\lambda_2}u, \tag{6.3}$$

$$d = -\sigma(b)p^{-\lambda_1}u + \sigma(d). \tag{6.4}$$

By (6.2), we know that $b = 0$. Hence $a = \sigma(a)$, $d = \sigma(d)$ by (6.1) and (6.4), and

$$c = -ap^{-\lambda_2}u + \sigma(c)p^{\lambda_1-\lambda_2} + dp^{-\lambda_2}u,$$

by (6.3), namely

$$p^{\lambda_1}(p^{\lambda_2-\lambda_1}c - \sigma(c)) = (d-a)u.$$

In order to have a solution for c, we need $d - a \in p^{\lambda_1}W(k)$. Let $d - a = p^{\lambda_1}\alpha$ for some $\alpha \in \mathbb{Z}_p$ as $a, d \in \mathbb{Z}_p$. Then we have a unique solution c such that

$$p^{\lambda_2-\lambda_1}c - \sigma(c) = \alpha u.$$

As $u = p^{\lambda_2-\lambda_1}v - \sigma(v)$, we get $c = \alpha v$. It is now easy to see that

$$\begin{aligned} &ax_1 \otimes x_1^* + bx_2 \otimes x_1^* + cx_1 \otimes x_2^* + dx_2 \otimes x_2^* \\ &= a(x_1 \otimes x_1^* + x_2 \otimes x_2^*) + (\alpha v)x_1 \otimes x_2^* +, (d-a)x_2 \otimes x_2^* \\ &= a(y_1 \otimes y_1^* + y_2 \otimes y_2^*) + \alpha p^{\lambda_1} y_2 \otimes y_2^* \in N. \end{aligned}$$

Hence $N = O^0$.

The change of basis matrix from $\{y_1 \otimes y_1^* + y_2 \otimes y_2^*, p^{\lambda_1}y_2 \otimes y_1^*, p^{\lambda_1}y_1 \otimes y_2^*, p^{\lambda_1}y_2 \otimes y_2^*\}$ to $\mathcal{B}_1 \otimes \mathcal{B}_1^*$ is

$$A = \begin{pmatrix} 1 & p^{\lambda_1}v & 0 & 0 \\ 0 & p^{2\lambda_1} & 0 & 0 \\ 0 & -v^2 & 1 & v \\ 1 & -p^{\lambda_1}v & 0 & p^{\lambda_1} \end{pmatrix}.$$

To find an upper bound of $\ell_{\mathcal{M}}$, we compute the inverse of A:

$$A^{-1} = \frac{1}{p^{2\lambda_1}} \begin{pmatrix} p^{2\lambda_1} & -p^{\lambda_1}v & 0 & 0 \\ 0 & 1 & 0 & 0 \\ p^{\lambda_1}v & -v^2 & p^{2\lambda_1} & -p^{\lambda_1}v \\ -p^{\lambda_1} & 2v & 0 & p^{\lambda_1} \end{pmatrix}.$$

Thus the smallest number ℓ such that all entries of $p^{\ell}A^{-1} \in W(k)$ is $2\lambda_1$. Hence $\ell_{\mathcal{M}} = 2\lambda_1$. By Theorem 1.2, we have $n_{\mathcal{M}} = 2\lambda_1$. □

Acknowledgement

The author would like to thank Adrian Vasiu for several suggestions and conversations and the anonymous referee for numerous suggestions.

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