A lower bound for the size of the largest critical sets in Latin squares*

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Abstract

A critical set in an $n \times n$ array is a set C of given entries, such that there exists a unique extension of C to an $n \times n$ Latin square and no proper subset of C has this property. The cardinality of the largest critical set in any Latin square of order n is denoted by $\mathrm{lcs}(n)$. We give a lower bound for $\mathrm{lcs}(n)$ by showing that $\mathrm{lcs}(n) \geq n^2(1 - \frac{2+\ln 2}{\ln n}) + n(1 + \frac{\ln(8\pi)}{\ln n}) - \frac{\ln 2}{\ln n}$.

1 Introduction

A Latin square of order n is an $n \times n$ array of integers chosen from the set $X = \{1, 2, \ldots, n\}$ such that each element of X occurs exactly once in each row and exactly once in each column. A Latin square can also be written as a set of ordered triples $\{(i, j; k) \mid$ symbol k occurs in cell (i, j) of the array$\}$.

A partial Latin square P of order n is an $n \times n$ array with entries chosen from the set $X = \{1, 2, \ldots, n\}$, such that each element of X occurs at most once in each row and at most once in each column. Hence there are cells in the array that may be empty, but the cells that are filled have been filled

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so as to conform with the Latin property of the array. Let P be a partial Latin square of order n. Then $|P|$ is said to be the size of the partial Latin square and the set of cells $\mathcal{S}_P = \{(i,j) \mid (i,j;k) \in P\}$ is said to determine the shape of P.

A partial Latin square C contained in a Latin square L is said to be uniquely completable if L is the only Latin square of order n with k in the cell (i,j) for every $(i,j;k) \in C$. A critical set C contained in a Latin square L is a partial Latin square that is uniquely completable and no proper subset of C satisfies this requirement. The name "critical set" and the concept were invented by a statistician, John Nelder, about 1977, and his ideas were first published in a note [**4**]. This note posed the problem of giving a formula for the size of the largest and smallest critical sets for a Latin square of a given order. Let $\text{lcs}(n)$ denote the size of the largest critical set in any Latin square of order n. Nelder [**5**] constructed a critical set of size $(n^2-n)/2$ for the $n \times n$ back circulant Latin square. He conjectured that $\text{lcs}(n) = (n^2-n)/2$. This equality was shown to be false in 1978, when Curran and van Rees [**3**], found that $\text{lcs}(4) \geq 7$. The following is an example of a largest critical set of size 11 for a 5×5 Latin square, taken from [**1**], which also contradicts Nelder's conjecture.

2		4	3	
		1	2	
	2	3	1	
3	1	2		

In the following table some known values of $\text{lcs}(n)$ for $n \leq 6$ are listed,

n	1	2	3	4	5	6
$\text{lcs}(n)$	0	1	3	7	11	18

and in the following table some known lower bounds for $\text{lcs}(n)$ are shown for $7 \leq n \leq 10$,

n	7	8	9	10
$\text{lcs(n)} \geq$	25	37	44	57

See [**1**] for the references. Recently Bean and Mahmoodian [**1**] have found the upper bound $\text{lcs}(n) \leq n^2 - 3n + 3$. Nelder's $(n^2-n)/2$ is the best lower bound that is found for $\text{lcs}(n)$ so far. In this note we improve this bound asymptotically for n large enough ($n \geq 195$).

2 A lower bound for $\mathrm{lcs}(n)$

Theorem 1 *For any integer n we have,*

$$\mathrm{lcs}(n) \geq n^2(1 - \frac{2 + \ln 2}{\ln n}) + n(1 + \frac{\ln (8\pi)}{\ln n}) - \frac{\ln 2}{\ln n}.$$

Proof. By Theorem 17.2 in **[7]**, as a result of van der Warden conjecture, we know the following bound for $L(n)$, the number of Latin squares of order n: $L(n) \geq \frac{(n!)^{2n}}{n^{n^2}}$.

If in a partial Latin square all the entries, except the entries of the first row and the first column be given, then it is uniquely completable. So every Latin square has at least one critical set which has no intersection with its first row and first column. And also obviously the number of these critical sets is greater than or equal to $L(n)$. For choosing the shape of such a critical set we have at most $2^{(n-1)^2}$ ways, and for choosing the entries of each given shape we have at most $n^{\mathrm{lcs}(n)}$ different ways. So the number of critical sets is less than or equal to $2^{n^2-2n+1}n^{\mathrm{lcs}(n)}$. Thus the following inequalities hold:

$$\frac{(n!)^{2n}}{n^{n^2}} \leq L(n) \leq 2^{n^2-2n+1}n^{\mathrm{lcs}(n)}.$$

Now by Stirling's approximation formula, (see for example **[2]**), we can replace $n!$ with a smaller value $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. So

$$\frac{(2\pi)^n n^{2n^2+n}}{e^{2n^2}n^{n^2}} \leq 2^{n^2-2n+1}n^{\mathrm{lcs}(n)},$$

or

$$\frac{(2\pi)^n n^{n^2+n}}{e^{2n^2}2^{n^2-2n+1}} \leq n^{\mathrm{lcs}(n)}.$$

Thus, $n\ln(2\pi) + (n^2+n)\ln n - 2n^2 - (n^2-2n+1)\ln 2 \leq \mathrm{lcs}(n)\ln n$. This implies that $n^2(1 - \frac{2+\ln 2}{\ln n}) + n(1 + \frac{2\ln 2+\ln(2\pi)}{\ln n}) - \frac{\ln 2}{\ln n} \leq \mathrm{lcs}(n)$. ■

Note. Stinson and van Rees **[6]** have shown that $\mathrm{lcs}(2^m) \geq 4^m - 3^m$. This lower bound for $n = 2^m$, is better than the bound given in Theorem 1.

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A note on the Grundy number

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Abstract

In 1982, Hedetniemi, Hedetniemi, and Beyer gave a bound on the Grundy number of a tree. This note provides a simpler proof of their result, as well as a bound in the same spirit for general simple graphs.

1 Introduction

The family of *complete minimum broadcast trees* is defined recursively; the first member $\mathcal{T}_1$ consists of a single vertex, and $\mathcal{T}_i$ is derived by adjoining a pendant vertex to every vertex of $\mathcal{T}_{i-1}$ for $i > 1$. An alternative characterisation of the family is as follows: $\mathcal{T}_1$ consists of a single vertex, which is designated as the root of the tree; $\mathcal{T}_i$ consists of copies of each of $\mathcal{T}_1, \ldots, \mathcal{T}_{i-1}$, plus a new root vertex which is adjacent to the root vertices of the smaller trees.

The Grundy number $\Gamma(G)$ of a graph G can be defined as the largest number of colours that can be assigned to the vertices of G using the greedy algorithm with a suitable ordering of the vertices: each vertex v_i will receive the least colour not assigned to any adjacent $v_j, j < i$. This parameter was first defined and studied in [1]; several alternative characterisations are given in [4].)

In [2], it is mentioned that there is a proof in [3] that for any tree T, $\Gamma(T) \leq 1 + \log_2 |V(T)|$. They also show that this bound is sharp by demonstrating that the family of complete minimum broadcast trees satisfies the above with equality, and that the Grundy number of any graph is equal to the Grundy number of the largest such broadcast tree it contains as a subgraph.

This last claim is false; as a simple counterexample, the path P_4 is a minimum broadcast tree with Grundy number 3, but $\Gamma(C_4) = 2$. In fact, the claim is not even true if we restrict our attention to induced subgraphs; all we can say is that $\Gamma(G)$ of any graph G is at least the Grundy number of the largest complete minimum broadcast tree that is an induced subgraph

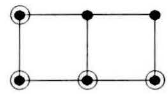

Figure 1: The graph $K_3 \times K_2$; the circled vertices induce P_4

of G. For example, consider the graph $K_3 \times K_2$ depicted in Figure 1; it contains P_4 as an induced subgraph (and is too small to contain the next largest CMBT), but $\Gamma(K_3 \times K_2) = 4$.

2 Confirming the Inequality

The basic result, however, holds true; what follows is an elementary proof.

Theorem 1 *Let T be a tree; then $\Gamma(T) \leq 1 + \log_2 |V(T)|$, with equality if and only if T is a complete minimum broadcast tree.*

Proof. We proceed by induction on the Grundy number. Since the expression on the right-hand side of the inequality is at least 1, the inequality holds for any tree T with $\Gamma(T) = 1$. Note that there is only one such T, namely T_1, for which equality holds.

We now assume that if T is a tree and $\Gamma(T) = j \leq k$ then T has at least 2^{j-1} vertices, and that T is a CMBT if $|V(T)| = 2^{j-1}$. Let H be a tree with Grundy number $k+1$ which is minimal; in other words, which contains no induced subtrees with Grundy number at least $k+1$. Let G_i denote the set of vertices in H receiving the colour i in a Grundy colouring of H. By minimality $|G_{k+1}| = 1$, since otherwise we can easily find an induced subtree with Grundy number $k+1$. Let v be the sole vertex in G_{k+1}; v must have at least one neighbour in each of $G_1, \ldots, G_k$; let v_i be a neighbour of v in G_i. Finally, let H_i denote the component of $H \setminus \{v\}$ containing v_i.

Note that for any i, the graph H_i can have a Grundy number of at most k, since H was minimal; thus we can invoke our induction hypothesis on H_i. Also note that the Grundy number of H_i is at least i, since the ordering of vertices which gave us a colouring of H can be restricted to the vertices of H_i and yield the same colouring locally. And so counting the number of vertices of H gives us:

$$
\begin{aligned}
|V(H)| &\geq |\{v\}| + \sum_{i=1}^{k} |V(H_i)| \\
&\geq 1 + \sum_{i=1}^{k} 2^{i-1} \\
&= 1 + (2^k - 1) \\
&= 2^k
\end{aligned}
$$

Taking logarithms on both sides yields the inequality.

Now suppose that H is a tree with Grundy number k that is not minimal; then we can find an induced subtree with Grundy number k by deleting all but one of the vertices that have been coloured k in a Grundy colouring. Since this new graph satisfies the inequality, the original H must as well.

Finally, let H be a tree with Grundy number $k+1$ and 2^k vertices. Then each $|V(H_i)|$ term in the above equation must be exactly equal to 2^{i-1}; and by the induction hypothesis, this can only happen in a tree with Grundy number i if that tree is the CMBT $\mathcal{T}_i$. The alternative characterisation of CMBTs given above then shows that $H = \mathcal{T}_{k+1}$. □

Theorem 2 *Let T be a tree, and suppose that some ordering of the vertices gives a greedy colouring where the vertex $v \in V(T)$ receives colour j. Then T contains $\mathcal{T}_j$ as an induced subgraph with v as its root.*

Proof. We proceed by induction on j. If $j = 1$ then the result is obvious, since the vertex v itself induces $\mathcal{T}_1$ with v as its root. Now suppose that the theorem is true for all $j < k$, and let v be a vertex coloured k in some greedy colouring. Then we can find vertices $v_1, v_2, \ldots v_{k-1}$ which are adjacent to v such that the vertex v_i receives colour i in the greedy colouring. Let T_i be the component of $T \setminus \{v\}$ which contains v_i; by the induction hypothesis, T_i contains a subset of vertices S_i which induces a copy of $\mathcal{T}_i$ rooted at v_i. Let $S = (\cup_{i=1}^{k-1} S_i) \cup \{v\}$; then $T[S]$ consists of a vertex v adjacent to the root vertices of $\mathcal{T}_1, \ldots \mathcal{T}_{k-1}$; in other words, $T[S] = \mathcal{T}_k$. □

Corollary 1 *The Grundy number of a tree T is equal to the highest order of a CMBT contained in T as an induced subgraph.*

Proof. If G is a graph and H an induced subgraph of G, then $\Gamma(H) \leq \Gamma(G)$; therefore the Grundy number of a tree T is at least the order of the largest CMBT induced by some subset of its vertices. If T has Grundy number k, then some vertex v gets coloured k in a greedy colouring of T; by the previous theorem, T therefore contains an induced copy of $\mathcal{T}_k$ rooted at v. Hence, the Grundy number of T is also at most the order of a largest induced CMBT. □

3 A More General Bound

We can also derive an upper bound for the Grundy number of a general graph, in a similar spirit. It is well known that any proper n-colouring of a graph G is equivalent to a graph homomorphism from G to K_n; it is easily seen that any colouring given by a greedy algorithm has the additional property that every edge in K_n has a preimage in G under the homomorphism. This implies the following:

Theorem 3 *Let G be a graph on m edges. Then* $\Gamma(G) \leq \frac{1+\sqrt{1+8m}}{2}$.

Proof. Let $\Gamma(G) = n$. Then we can find a homomorphism from G to K_n such that each edge in the latter has a preimage. This means that $m \geq \binom{n}{2}$, the number of edges in K_n. Solving for n yields the desired inequality. □

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