On the Enumeration of Spanning Trees of the Complete Multipartite Graph

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Abstract. We give a formula for the polynomial

$$p_{K_{n_1,\ldots,n_k}}(x_1,\ldots,x_n) = \sum_{T \text{ spanning tree of } K_{n_1,\ldots,n_k}} x_1^{d_T(1)-1} \cdots x_n^{d_T(n)-1}$$

where $K_{n_1,\ldots,n_k}$ is the complete multipartite graph on $[n]$. Among the consequences is a formula for the number of spanning trees of $K_{n_1,\ldots,n_k}$ with given degree sequence.

Introduction

Austin [1] found a formula for the number of spanning trees of $K_{n_1,\ldots,n_k}$ using the matrix-tree theorem. Good [4] used a multivariate generating function and generalized Lagrange inversion to give another derivation. Eğecioğlu and Remmel [3] found a bijective proof of this formula. A different bijective proof of the formula was recently given by Lewis [5]. In this paper, we give a formula for the polynomial

$$p_{K_{n_1,\ldots,n_k}}(x_1,\ldots,x_n) = \sum_{T \text{ spanning tree of } K_{n_1,\ldots,n_k}} x_1^{d_T(1)-1} \cdots x_n^{d_T(n)-1}$$

where $K_{n_1,\ldots,n_k}$ is the complete multipartite graph on $[n]$. Immediate consequences of this result include the aforementioned results in [1], [3], [4] and [5]; a result of Rényi [7] concerning the complete graph; and a formula for the number of spanning trees of $K_{n_1,\ldots,n_k}$ with given degree sequence.

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The *complete k-partite graph* G with partition $\{A_1, \dots, A_k\}$ is the graph with vertex set $A_1 \cup \cdots \cup A_k$ where uv is an edge of G if and only if $u \in A_i$, $v \in A_j$ with $1 \le i \ne j \le k$. The complete k-partite graph with partite sets of cardinality $n_1, \dots, n_k \in \mathbb{P}$ is denoted $K_{n_1,\dots,n_k}$ and the complete graph with $n \in \mathbb{P}$ vertices is denoted K_n. The leaves of a tree T are denoted $L(T)$ and the edges of T are denoted $E(T)$. The degree of a vertex v in a tree T is denoted $d_T(v)$. For a proper subset L of the vertices V of a graph G, $G - L$ denotes the subgraph of G induced by $V - L$.

The nonnegative integers are denoted by $\mathbb{N}$ and the positive integers by $\mathbb{P}$. For $n \in \mathbb{P}$, $[n] = \{1, \dots, n\}$.

Throughout this paper we work in the commutative ring $\mathbb{Z}[x_1, \dots, x_n]$ of polynomials in the variables $x_1, \dots, x_n$ with integer coefficients. For reference we recall the multinomial theorem in $\mathbb{Z}[x_1, \dots, x_n]$ (see [2; p. 28]). For $\ell \in \mathbb{N}$, $m, n \in \mathbb{P}$ with $m \le n$ and distinct $i_1, \dots, i_m \in [n]$,

$$(x_{i_1} + \cdots + x_{i_m})^\ell = \sum_{\substack{(e_1,\dots,e_m) \in \mathbb{N}^m \\ e_1 + \cdots + e_m = \ell}} \binom{\ell}{e_1, \dots, e_m} x_{i_1}^{e_1} \cdots x_{i_m}^{e_m}$$

where

$$\binom{\ell}{e_1, \dots, e_m} = \frac{\ell!}{e_1! \cdots e_m!}.$$

(As usual $x_i^0 = 1$ and $0! = 1$.)

Our notation and terminology may be found in Comtet [2] and West [8].

Results

For $n_1, \dots, n_k \in \mathbb{P}$ with $k \ge 2$, let $s_0 = 0$, $s_j = s_{j-1} + n_j$ $(1 \le j \le k)$ and $s_k = n = n_1 + \cdots + n_k$. Let $A_1 = [s_1] = \{1, \dots, n_1\}$ and $A_j = [s_j] - [s_{j-1}] = \{s_{j-1} + 1, \dots, s_j\}$ $(2 \le j \le k)$.

Associate $i \in [n]$ with the variable x_i and let $X = X_n = x_1 + \cdots + x_n$ and $X_D = X - \sum_{i \in D} x_i$ for $D \subseteq [n]$. Then, $X_\phi = X$ and $X_{[n]} = 0$. Here $X_D^0 = 1$ for all $D \subseteq [n]$.

Lemma 1. We have,

$$\sum_{\substack{(M_1,\dots,M_k) \\ M_j \subseteq A_j (1 \le j \le k) \\ M = M_1 \cup \cdots \cup M_k}} (-1)^{|M|} X_{M \cup A_1}^{n_1 - 1} \cdots X_{M \cup A_k}^{n_k - 1} X_M^{k-2} = 0.$$

Proof. Our result is immediately seen to be true for $n = 2$ (so $n_1 = n_2 = 1$, $k = 2$) and we assume $n \geq 3$.

Observe that $X_{M\cup A_1}^{n_1-1} \cdots X_{M\cup A_k}^{n_k-1} X_M^{k-2} = 0$ if and only if $n_j \geq 2$ and $M \cup A_j = [n]$ for some $1 \leq j \leq k$, or $k \geq 3$ and $M = [n]$; and 1 if and only if $n = k = 2$ and $n_1 = n_2 = 1$ (which does not occur since $n \geq 3$). Otherwise, after expansion, $X_{M\cup A_1}^{n_1-1} \cdots X_{M\cup A_k}^{n_k-1} X_M^{k-2}$ is a nonempty sum of terms of the form a nonzero integer multiplied by

$$\tau = \prod_{j\in J} \prod_{i\in B_j} x_i^{e_i},$$

where $\phi \neq J \subseteq [k]$, $\phi \neq B_j \subseteq A_j$ $(j \in J)$, $e_i \in \mathbb{P}$ $(j \in J,\ i \in B_j)$ and $\sum_{j\in J} \sum_{i\in B_j} e_i = n - 2$ (≥ 1). For notational convenience, we assume $J = \{1, \ldots, t\}$ (where $1 \leq t \leq k$) and $B_j = \{s_{j-1} + 1, \ldots, s_{j-1} + b_j\}$ $(1 \leq j \leq t)$ so that

$$\tau = x_1^{e_1} \cdots x_{b_1}^{e_{b_1}} x_{s_1+1}^{e_{s_1+1}} \cdots x_{s_1+b_2}^{e_{s_1+b_2}} \cdots x_{s_{t-1}+1}^{e_{s_{t-1}+1}} \cdots x_{s_{t-1}+b_t}^{e_{s_{t-1}+b_t}}$$

where $1 \leq b_j \leq n_j$ $(1 \leq j \leq t)$ and the exponents are positive integers with sum $n-2$. Let $B = B_1 \cup \cdots \cup B_t = \{c_1, \ldots, c_b\}$ written in increasing order, $|B| = b = b_1 + \cdots + b_t$, $Y_j = \sum_{i\in B_j} x_i$ $(1 \leq j \leq t)$ and $Z = Y_1 + \cdots + Y_t$.

Suppose $L_j \subseteq A_j - B_j$ $(1 \leq j \leq t)$, $L_j \subseteq A_j$ $(t < j \leq k$; possibly empty) and $L = L_1 \cup \cdots \cup L_k$. Let $C_j = A_j - B_j - L_j$ $(1 \leq j \leq t)$, $C_j = A_j - L_j$ $(t < j \leq k)$ and $C = C_1 \cup \cdots \cup C_k$. Then, (empty sum is 0)

$$\begin{aligned}
X_{L\cup A_j} &= Z - Y_j + \sum\{x_i : i \in C - C_j\} \quad (1 \leq j \leq t)\,,\\
X_{L\cup A_j} &= Z + \sum\{x_i : i \in C - C_j\} \quad (t < j \leq k)\,,\\
X_L &= Z + \sum\{x_i : i \in C\}\,.
\end{aligned}$$

For $t \geq 2$, the multinomial theorem in $\mathbb{Z}[x_1, \ldots, x_n]$ implies that the coefficient (possibly 0) of τ in $X_{L\cup A_1}^{n_1-1} \cdots X_{L\cup A_k}^{n_k-1} X_L^{k-2}$ is (empty sum is 0)

$$\sum \left\{ \prod_{j=1}^{k} \binom{n_j - 1}{d(j,1), \ldots, d(j,b)} \right\} \binom{k-2}{d(k+1,1), \ldots, d(k+1,b)} \tag{1}$$

where $d(j,i) \geq 0$ is the number of times x_{c_i} is chosen from one of the $n_j - 1 \geq 0$ factors of $X_{L\cup A_j}^{n_j-1}$ $(1 \leq j \leq k,\ 1 \leq i \leq b)$ or from one of the $k - 2 \geq 0$ factors of X_L^{k-2} $(j = k+1,\ 1 \leq i \leq b)$. Here the sum is over all $(k+1) \times b$ matrices $D = [d(j,i)]$ of nonnegative integers where $d(j,1)+\cdots+d(j,b)$ equals $n_j - 1$ $(1 \le j \le k)$ and $k-2$ $(j = k+1)$; $d(j,i) = 0$ $(j = 1$ and $1 \le i \le b_1$, or $2 \le j \le t$ and $b_1+\cdots+b_{j-1}+1 \le i \le b_1+\cdots+b_j)$; and $d(1,i) + \cdots + d(k+1,i) = e_{c_i}$ $(1 \le i \le b)$, should they exist. For $t = 1$, $n_1 \ge 2$, the coefficient of τ in $X_{L\cup A_1}^{n_1-1} \cdots X_{L\cup A_k}^{n_k-1} X_L^{k-2}$ is 0 while no such D exists, and, for $t = 1 = n_1$, use $X_{L\cup A_1}^0 = 1$. In either case, (1) is correct here also. Moreover, (1) implies that the coefficient of τ in $X_{L\cup A_1}^{n_1-1} \cdots X_{L\cup A_k}^{n_k-1} X_L^{k-2}$ is the same for all such $(L_1, \ldots, L_k)$ (this is obvious since only variables in Z can be selected).

Suppose $L_j \subseteq A_j$ $(1 \le j \le k)$, $L = L_1 \cup \cdots \cup L_k$ and τ appears with nonzero coefficient in $X_{L\cup A_1}^{n_1-1} \cdots X_{L\cup A_k}^{n_k-1} X_L^{k-2}$. If $m \in B_j \cap L_j$ $(1 \le j \le t)$, then x_m appears in no $X_{L\cup A_i}$ $(1 \le i \le k)$ nor in X_L; a contradiction. Hence, $L_j \subseteq A_j - B_j$ $(1 \le j \le t)$.

Hence, τ appears with nonzero coefficient in $X_{L\cup A_1}^{n_1-1} \cdots X_{L\cup A_k}^{n_k-1} X_L^{k-2}$ only if $L_j \subseteq A_j - B_j$ $(1 \le j \le t)$. Moreover, the coefficient of τ (given in (1)) is the same in each $X_{L\cup A_1}^{n_1-1} \cdots X_{L\cup A_k}^{n_k-1} X_L^{k-2}$ with $L_j \subseteq A_j - B_j$ $(1 \le j \le t)$. Now,

$$\begin{aligned}
&\sum_{\substack{(L_1,\ldots,L_k) \\ L_j \subseteq A_j - B_j \ (1\le j\le t) \\ L_j \subseteq A_j \ (t<j\le k)}} (-1)^{|L_1|+\cdots+|L_k|} \\
&= \Big(\sum_{L_1 \subseteq A_1 - B_1} (-1)^{|L_1|}\Big) \cdots \Big(\sum_{L_t \subseteq A_t - B_t} (-1)^{|L_t|}\Big)\Big(\sum_{L_{t+1} \subseteq A_{t+1}} (-1)^{|L_{t+1}|}\Big) \\
&\qquad \cdots \Big(\sum_{L_k \subseteq A_k} (-1)^{|L_k|}\Big) \\
&= \begin{cases} 0 \ , & (A_1 - B_1, \ldots, A_t - B_t, A_{t+1}, \ldots, A_k) \ne (\phi, \ldots, \phi), \\ 1 \ , & \text{otherwise,} \end{cases} \\
&= 0\,, \qquad (2)
\end{aligned}$$

since $B_j = A_j$ for $1 \le j \le t = k$ implies that the sum of the exponents in τ is at least n; a contradiction. Consequently, (2; inclusion-exclusion) gives

$$\sum_{\substack{(M_1,\ldots,M_k) \\ M_j \subseteq A_j \ (1\le j\le k) \\ M = M_1 \cup \cdots \cup M_k}} (-1)^{|M|} X_{M\cup A_1}^{n_1-1} \cdots X_{M\cup A_k}^{n_k-1} X_M^{k-2} = 0\,. \ \blacksquare$$

For a connected graph G on $V = \{i_1, \ldots, i_n\} \subseteq [N]$ with $N \in \mathbb{P}$, let

$$p_G(x_{i_1}, \ldots, x_{i_n}) = \sum_{T \text{ spanning tree of } G} x_{i_1}^{d_T(i_1)-1} \cdots x_{i_n}^{d_T(i_n)-1} .$$

Note that this polynomial is independent of the order of the vertices of V in $\mathbb{Z}[x_1, \ldots, x_N]$. For a connected graph G' on $V' = \{i'_1, \ldots, i'_n\} \subseteq [N]$ where $G \cong G'$ by $i_j \leftrightarrow i'_j$ $(1 \le j \le n)$, we have

$$p_{G'}(x_{i'_1}, \ldots, x_{i'_n}) = p_G(x_{i'_1}, \ldots, x_{i'_n})$$

Let $X(G) = x_{i_1} + \cdots + x_{i_n}$ and $X(G)_D = X(G) - \sum_{i \in D} x_i$ for $D \subseteq V$.

We now give our main result which we prove by induction.

Theorem 2. For the complete k-partite graph G on $\{i_1, \ldots, i_n\} \subseteq \mathbb{P}$ with partition $\{D_1, \ldots, D_k\}$ where $|D_j| = n_j \in \mathbb{P}$ $(1 \le j \le k)$ and $k \ge 2$, we have

$$p_G(x_{i_1}, \ldots, x_{i_n}) = X(G)_{D_1}^{n_1-1} \cdots X(G)_{D_k}^{n_k-1} X(G)^{k-2} .$$

Proof. (Induction on n) Our result is immediately seen to be true for $n = 2$ (as $p_G(x_{i_1}, x_{i_2}) = 1 = x_{i_1}^0 x_{i_2}^0 (x_{i_1} + x_{i_2})^0$) and we assume $n \ge 3$. In view of our comment above (where $D_j \leftrightarrow D'_j = \{i' : i \in D_j\}$), we need only find $p_G(x_1, \ldots, x_n)$ when $G = G(A_1, \ldots, A_k)$ is the complete k-partite graph on $[n]$ with partition $\{A_1, \ldots, A_k\}$. Here $X(G) = X$ of Lemma 1.

Let $L_j \subseteq A_j$ $(1 \le j \le k)$, $|L_j| = \ell_j$ $(1 \le j \le k)$, $L = L_1 \cup \cdots \cup L_k$ and $\overrightarrow{L} = (L_1, \ldots, L_k)$. Write $\mathcal{S}_G(\overrightarrow{L})$ for the set of spanning trees T of G with $L \subseteq L(T)$. Then, $\mathcal{S}_G(\overrightarrow{L}) \ne \phi$ if and only if $L = [n] - A_j$ with $n_j = 1$ for some $1 \le j \le k$ or L omits vertices from at least two of $A_1, \ldots, A_k$. Let $\mathcal{L} = \{\overrightarrow{L} : \mathcal{S}_G(\overrightarrow{L}) \ne \phi\}$ so $(A_1, \ldots, A_k) \notin \mathcal{L}$. Then, $[n] - L - A_j \ne \phi$ if $L_j \ne \phi$ $(1 \le j \le k)$ whenever $\overrightarrow{L} \in \mathcal{L}$. For $\overrightarrow{L} \in \mathcal{L}$, let $\mathcal{B}(\overrightarrow{L}) = \{(B_1, \ldots, B_k) : B_j \in ([n] - L - A_j)^{\ell_j} \ (1 \le j \le k)\}$ where B_j is the empty tuple when $\ell_j = 0$. Given $\overrightarrow{L} = (L_1, \ldots, L_k) \in \mathcal{L}$ and $\overrightarrow{B} = (B_1, \ldots, B_k) \in \mathcal{B}(\overrightarrow{L})$, let $L_j = \{a(j,1), \ldots, a(j,\ell_j)\}$ and $B_j = (b(j,1), \ldots, b(j,\ell_j))$ where the elements of L_j are written in increasing order $(1 \le j \le k)$. Write $\mathcal{S}_G(\overrightarrow{L}, \overrightarrow{B})$ for the set of spanning trees T of G with $L \subseteq L(T)$ and $a(j,i)b(j,i) \in E(T)$ $(1 \le j \le k,\ 1 \le i \le \ell_j)$. Then $\mathcal{S}_G(\overrightarrow{L}, \overrightarrow{B}) \ne \phi$ for $\overrightarrow{L} \in \mathcal{L}$ and $\overrightarrow{B} \in \mathcal{B}(\overrightarrow{L})$. Also, $\mathcal{S}_G(\overrightarrow{L}, \overrightarrow{B}) \cap \mathcal{S}_G(\overrightarrow{L}, \overrightarrow{C}) = \phi$ for $\overrightarrow{L} \in \mathcal{L}$ and distinct $\overrightarrow{B}, \overrightarrow{C} \in \mathcal{B}(\overrightarrow{L})$.

Clearly, $T \in \mathcal{S}_G(\overrightarrow{L})$ belongs to precisely one $\mathcal{S}_G(\overrightarrow{L}, \overrightarrow{B})$ with $\overrightarrow{B} \in \mathcal{B}(\overrightarrow{L})$ when $\overrightarrow{L} \in \mathcal{L}$. Hence, $\{\mathcal{S}_G(\overrightarrow{L}, \overrightarrow{B}) : \overrightarrow{B} \in \mathcal{B}(\overrightarrow{L})\}$ partitions $\mathcal{S}_G(\overrightarrow{L})$ when $\overrightarrow{L} \in \mathcal{L}$. Write S_{G-L} for the set of spanning trees of $G-L$. Here, $S_{G-[n]} = \phi$. Then, $S_{G-L} \neq \phi$ if and only if $\overrightarrow{L} \in \mathcal{L}$. It is immediately seen that $\varphi : \mathcal{S}_G(\overrightarrow{L}, \overrightarrow{B}) \to \mathcal{S}_{G-L}$ by $\varphi(T) = T - L$ is a bijection when $\overrightarrow{L} \in \mathcal{L}$. Hence, for $\overrightarrow{L} \in \mathcal{L}$ (empty product equals 1 and $x_i^0 = 1$),

$$\sum_{T \in \mathcal{S}_G(\overrightarrow{L}, \overrightarrow{B})} x_1^{d_T(1)-1} \cdots x_n^{d_T(n)-1}$$

$$= \prod_{j=1}^{k} \left(x_{b(j,1)} \cdots x_{b(j,\ell_j)}\right) \sum_{S \in \mathcal{S}_{G-L}} \prod_{i \in [n]-L} x_i^{d_S(i)-1}$$

and, consequently ($X_{L \cup A_j}^{\ell_j} = 1$ for $\ell_j = 0$),

$$\sum_{T \in \mathcal{S}_G(\overrightarrow{L})} x_1^{d_T(1)-1} \cdots x_n^{d_T(n)-1}$$

$$= X_{L \cup A_1}^{\ell_1} \cdots X_{L \cup A_k}^{\ell_k} \sum_{S \in \mathcal{S}_{G-L}} \prod_{i \in [n]-L} x_i^{d_S(i)-1}. \tag{3}$$

We distinguish three cases for $\overrightarrow{L} \neq \overrightarrow{\phi}$, $\overrightarrow{A} = (A_1, \ldots, A_k)$.

Exactly $k-1$ of the $L_j = A_j$. Assume $L_1 = A_1, \ldots, L_{k-1} = A_{k-1}$ with no loss of generality. Necessarily, $L_k = \phi$ and $n_k = 1$ or $\overrightarrow{L} \notin \mathcal{L}$. Here, ($X_{[n]}^0 = 1$) we calculate

$$\sum_{T \in \mathcal{S}_G(\overrightarrow{L})} x_1^{d_T(1)-1} \cdots x_n^{d_T(n)-1} = x_n^{n-2} = X_{L \cup A_1}^{n_1 - 1} \cdots X_{L \cup A_k}^{n_k - 1} X_L^{k-2}, \tag{4}$$

since $\mathcal{S}_G(\overrightarrow{L})$ contains only one tree.

Exactly t of the $L_j = A_j$; $1 \leq t \leq k-2$. Here $\overrightarrow{L} \in \mathcal{L}$. Assume $L_1 = A_1, \ldots, L_t = A_t$, $L_{t+1} \neq A_{t+1}, \ldots, L_k \neq A_k$ with no loss of generality. By (3) together with induction ($G - L$ is a complete $k-t$ (≥ 2)-partite graph on $[n] - L$ with partition $\{A_{t+1} - L_{t+1}, \ldots, A_k - L_k\}$ and order

$2 \leq n - |L| \leq n-1$), we have

$$\sum_{T \in \mathcal{S}_G(\overrightarrow{L})} x_1^{d_T(1)-1} \cdots x_n^{d_T(n)-1}$$

$$= X_{L\cup A_1}^{\ell_1} \cdots X_{L\cup A_k}^{\ell_k} X(G-L)_{A_{t+1}-L_{t+1}}^{n_{t+1}-\ell_{t+1}-1} \cdots$$

$$\cdots X(G-L)_{A_k-L_k}^{n_k-\ell_k-1} X(G-L)^{k-t-2}$$

$$= X_{L\cup A_1}^{\ell_1} \cdots X_{L\cup A_k}^{\ell_k} X_{L\cup A_{t+1}}^{n_{t+1}-\ell_{t+1}-1} \cdots X_{L\cup A_k}^{n_k-\ell_k-1} X_L^{k-t-2}$$

$$= X_{L\cup A_1}^{n_1-1} \cdots X_{L\cup A_t}^{n_t-1} X_{L\cup A_{t+1}}^{n_{t+1}-1} \cdots X_{L\cup A_k}^{n_k-1} X_L^{k-2}, \tag{5}$$

since $L \cup A_j = L$ $(1 \leq j \leq t)$.

No $L_j = A_j$; not all $L_j = \phi$. Here $\overrightarrow{L} \in \mathcal{L}$. By (3) together with induction ($G - L$ is a complete k (≥ 2)-partite graph on $[n] - L$ with partition $\{A_1 - L_1, \ldots, A_k - L_k\}$ and order $2 \leq n - |L| \leq n-1$), we have

$$\sum_{T \in \mathcal{S}_G(\overrightarrow{L})} x_1^{d_T(1)-1} \cdots x_n^{d_T(n)-1}$$

$$= X_{L\cup A_1}^{\ell_1} \cdots X_{L\cup A_k}^{\ell_k} X(G-L)_{A_1-L_1}^{n_1-\ell_1-1} \cdots X(G-L)_{A_k-L_k}^{n_k-\ell_k-1} X(G-L)^{k-2}$$

$$= X_{L\cup A_1}^{\ell_1} \cdots X_{L\cup A_k}^{\ell_k} X_{L\cup A_1}^{n_1-\ell_1-1} \cdots X_{L\cup A_k}^{n_k-\ell_k-1} X_L^{k-2}$$

$$= X_{L\cup A_1}^{n_1-1} \cdots X_{L\cup A_k}^{n_k-1} X_L^{k-2}. \tag{6}$$

We note that $\overrightarrow{L} \notin \mathcal{L}$ implies $\overrightarrow{L} = \overrightarrow{A}$ or $L \supseteq [n] - A_j$ with $n_j \geq 2$ for some $1 \leq j \leq k$. In either case, $X_{L\cup A_1}^{n_1-1} \cdots X_{L\cup A_k}^{n_k-1} X_L^{k-2} = 0$ as $n \geq 3$. Conversely (see the second paragraph in the proof of Lemma 1), $X_{L\cup A_1}^{n_1-1} \cdots X_{L\cup A_k}^{n_k-1} X_L^{k-2} = 0$ implies $\overrightarrow{L} = \overrightarrow{A}$ or $L \supseteq [n] - A_j$ with $n_j \geq 2$ for some $1 \leq j \leq k$. In either case, $\overrightarrow{L} \notin \mathcal{L}$.

Fix a spanning tree T of G with leaves $M_1 \cup \cdots \cup M_k$ where $M_j \subseteq A_j$ $(1 \leq j \leq k)$. Then, $T \in \mathcal{S}_G\big((L_1, \ldots, L_k)\big)$ if and only if $L_j \subseteq M_j$ $(1 \leq j \leq k)$. Now,

$$\sum_{\substack{(L_1,\ldots,L_k) \\ L_j \subseteq M_j (1 \leq j \leq k)}} (-1)^{|L_1|+\cdots+|L_k|} = \Big(\sum_{L_1 \subseteq M_1} (-1)^{|L_1|}\Big) \cdots \Big(\sum_{L_k \subseteq M_k} (-1)^{|L_k|}\Big)$$

$$= 0, \tag{7}$$

since $(M_1, \ldots, M_k) \neq (\phi, \ldots, \phi)$. Hence (7; inclusion-exclusion) gives

$$\sum_{\substack{\vec{M}=(M_1,\ldots,M_k) \\ M_j \subseteq A_j\,(1\le j\le k) \\ M=M_1\cup\cdots\cup M_k}} (-1)^{|M|} \sum_{T\in\mathcal{S}_G(\vec{M})} x_1^{d_T(1)-1}\cdots x_n^{d_T(n)-1} = 0\,. \qquad (8)$$

Consequently, (8), (4–6), our comments regarding $\mathcal{L}$ and Lemma 1 give

$$\begin{aligned} p_G(x_1,\ldots,x_n) &= \sum_{T\in\mathcal{S}_G(\vec{\phi})} x_1^{d_T(1)-1}\cdots x_n^{d_T(n)-1} \\ &= \sum_{\substack{\vec{M}=(M_1,\ldots,M_k)\neq\vec{\phi} \\ M_j \subseteq A_j\,(1\le j\le k) \\ M=M_1\cup\cdots\cup M_k}} (-1)^{|M|+1} \sum_{T\in\mathcal{S}_G(\vec{M})} x_1^{d_T(1)-1}\cdots x_n^{d_T(n)-1} \\ &= \sum_{\substack{\vec{M}=(M_1,\ldots,M_k)\neq\vec{\phi} \\ M_j \subseteq A_j\,(1\le j\le k) \\ M=M_1\cup\cdots\cup M_k}} (-1)^{|M|+1}\, X_{M\cup A_1}^{n_1-1}\cdots X_{M\cup A_k}^{n_k-1} X_M^{k-2} \\ &= X_{A_1}^{n_1-1}\cdots X_{A_k}^{n_k-1} X^{k-2}\,. \quad \blacksquare \end{aligned}$$

Remark. Alternatively, the referee has suggested that it may be possible to give a bijective proof of Theorem 2 using a sign-reversing involution/pairing on the underlying k-tuples.

Let $\tau(G)$ denote the number of distinct spanning trees of a connected graph G. An immediate consequence of Theorem 2 is the formula for $\tau(K_{n_1,\ldots,n_k})$ given in Austin [1], Good [4], Eğecioğlu and Remmel [3] and Lewis [5].

Corollary 3. ([1], [3], [4], [5]) For $n = n_1 + \cdots + n_k$ where $n_1, \ldots, n_k \in \mathbb{P}$ and $k \geq 2$,

$$\tau(K_{n_1,\ldots,n_k}) = n^{k-2} \prod_{j=1}^{k} (n - n_j)^{n_j - 1}\,.$$

Proof. Clearly,

$$\tau(K_{n_1,\ldots,n_k}) = p_G(1,\ldots,1) = n^{k-2}\prod_{j=1}^{k}(n-n_j)^{n_j-1}$$

where $G = G(A_1,\ldots,A_k) \cong K_{n_1,\ldots,n_k}$ is the graph in Theorem 2. ■

In particular, we have Cayley's theorem for $\tau(K_n)$.

Corollary 4. For $n \geq 2$,

$$\tau(K_n) = n^{n-2}.$$

Proof. Clearly,

$$\tau(K_n) = \tau(K_{\underbrace{1,\ldots,1}_{n}}) = n^{n-2}. \quad ■$$

Another immediate consequence of Theorem 2 is the formula for $p_G(x_1,\ldots,x_n)$ given in Rényi [7] when G is the complete graph on $[n]$. This is also a direct consequence of the encoding in Prüfer [6].

Corollary 5. ([7]) For $n \geq 2$,

$$p_{K_n}(x_1,\ldots,x_n) = (x_1+\cdots+x_n)^{n-2}.$$

Proof. Theorem 2 when $G = G\big(\{1\},\ldots,\{n\}\big)$ is the complete graph on $[n]$ gives

$$p_{K_n}(x_1,\ldots,x_n) = (x_1+\cdots+x_n)^{n-2}. \quad ■$$

For a tree T on $[n]$, let

$$w(T) = \sum_{i=1}^{n} i\, d_T(i)$$

and for a connected graph G on $[n]$, let

$$q_G(x) = \sum_{T \text{ spanning tree of } G} x^{w(T)}.$$

A further consequence of Theorem 2 is a formula for $q_G(x)$, similar to one given in Eğecioğlu and Remmel [3], when G is the graph in Theorem 2. Here $[0]_x = 0$ and $[s]_x = 1 + x + \cdots + x^{s-1}$ for $s \in \mathbb{P}$.

Corollary 6. For the graph G in Theorem 2,

$$q_G(x) = x^{\binom{n+1}{2}+n-2}\,[n]_x^{k-2} \prod_{j=1}^{k} \left\{[s_{j-1}]_x + x^{s_j}[n-s_j]_x\right\}^{n_j-1}.$$

Proof. Clearly,

$$\begin{aligned} q_G(x) &= x^{\binom{n+1}{2}} p_G(x^1, x^2, \ldots, x^n) \\ &= x^{\binom{n+1}{2}+n-2}\,[n]_x^{k-2} \prod_{j=1}^{k} \left\{[s_{j-1}]_x + x^{s_j}[n-s_j]_x\right\}^{n_j-1}. \quad \blacksquare \end{aligned}$$

Formulas for

$$\widetilde{q}_G(x) = \sum_{T \text{ spanning tree of } G} x^{\widetilde{w}(T)}$$

where

$$\widetilde{w}(T) = \sum_{i=1}^{n} f(i)\, d_T(i)$$

and $f(i)$ is a (nonnegative integer-valued) function of i can immediately be given by finding $p_G(x^{f(1)}, x^{f(2)}, \ldots, x^{f(n)})$.

Let $\tau(G; d_1, \ldots, d_n)$ denote the number of spanning trees T of a connected graph G with vertex set $[n]$ where $d_T(i) = d_i$ $(1 \leq i \leq n)$. Our final result is a formula for $\tau(K_{n_1,\ldots,n_k}; d_1, \ldots, d_n)$.

Corollary 7. For $n = n_1 + \cdots + n_k$ where $n_1, \ldots, n_k \in \mathbb{P}$ and $k \geq 2$,

$$\begin{aligned} &\tau(K_{n_1,\ldots,n_k}; d_1, \ldots, d_n) \\ &= \sum \left\{ \prod_{j=1}^{k} \binom{n_j - 1}{d(j,1), \ldots, d(j,n)} \right\} \binom{k-2}{d(k+1,1), \ldots, d(k+1,n)} \end{aligned}$$

where the sum is over all $(k+1) \times n$ matrices $D = [d(j,i)]$ of nonnegative integers where $d(j,1) + \cdots + d(j,n)$ equals $n_j - 1$ $(1 \leq j \leq k)$ and $k-2$ $(j = k+1)$; $d(j,i) = 0$ $(1 \leq j \leq k,\ s_{j-1}+1 \leq i \leq s_j)$; and $d(1,i) + \cdots + d(k+1,i) = d_i - 1$ $(1 \leq i \leq n)$.

Proof. For the graph $G = G(A_1, \ldots, A_k) \cong K_{n_1,\ldots,n_k}$ of Theorem 2, we have (as in (1))

$$\begin{aligned}
&\tau(K_{n_1,\ldots,n_k}; d_1, \ldots, d_n) \\
&= \text{coefficient of } x_1^{d_1-1} \cdots x_n^{d_n-1} \text{ in } p_G(x_1, \ldots, x_n) \\
&= \sum \left\{ \prod_{j=1}^{k} \binom{n_j - 1}{d(j,1), \ldots, d(j,n)} \right\} \binom{k-2}{d(k+1,1), \ldots, d(k+1,n)}
\end{aligned}$$

where the sum is over all $(k+1) \times n$ matrices $D = [d(j,i)]$ of nonnegative integers where $d(j,1) + \cdots + d(j,n)$ equals $n_j - 1$ $(1 \le j \le k)$ and $k-2$ $(j = k+1)$; $d(j,i) = 0$ $(1 \le j \le k,\ s_{j-1} + 1 \le i \le s_j)$; and $d(1,i) + \cdots + d(k+1,i) = d_i - 1$ $(1 \le i \le n)$. ■

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