Application of an Empty-Diagonal Square

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Abstract

We apply a solution for an empty-diagonal square to produce a design comprising 4 pairs, 6 triples, and 87 quadruples that covers every pair from a 17-element set exactly 4 times and is a minimal such covering using blocks of size not exceeding 4.

1 Introduction

In a recent Bulletin (cf. [2]), I proposed a game using a nine-by-nine square in which the diagonal cells were empty and the other 72 cells were filled by unordered number pairs (i, j) where i and j ranged from 1 to 8 $(i \neq j)$, each pair occurred 3 times, each row and column contained exactly two of each of the 8 integers from 1 to 8, and the pairs (1,2), (3,4), (5,6), (7,8) were not permitted. Thus, there are exactly 24 number pairs used, each occurring thrice, and each row and column contains the set $\{1, 2, \ldots, 8\}$ exactly twice.

Several people have asked me if it is possible to fill in the complete square, subject to these conditions (obviously, one can start out by just filling in pairs, and get quite a long way without violating the conditions). In this note, I will give one solution for the complete square and then give an application of the solution.

2 Display of a Solution

For simplicity, we omit commas in the number pairs and write ij rather than (i, j). This causes no ambiguity since $1 \leq i \leq 8$, $1 \leq j \leq 8$. Also, we designate the nine rows and columns of the square by the letters a, b, c, d, e, f, g, h, i.

	a	b	c	d	e	f	g	h	i
a		17	38	26	68	45	25	47	13
b	38		17	45	26	68	13	25	47
c	17	38		68	45	26	47	13	25
d	27	58	46		14	37	16	28	35
e	46	27	58	37		14	35	16	28
f	58	46	27	14	37		28	35	16
g	15	36	24	18	57	23		48	67
h	24	15	36	23	18	57	67		48
i	36	24	15	57	23	18	48	67	

From the square, we can now read off 72 quadruples consisting of two letters and two numbers. Thus, the first row produces $ab17$, $ac38$, $ad26$, $ae68$, $af45$, $ag25$, $ah47$, $ai13$.

The second row produces $ab38$, $bc17$, $bd45$, $be26$, $bf68$, $bg13$, $bh25$, $bi47$, and this process continues. This gives 72 quadruples in which the letter pairs (L_1, L_2) each occur twice, the number pairs (N_1, N_2) each occur thrice.

3 Two supplementary designs

We now write down a design made up of 4 pairs and 6 quadruples. These are (12), (34), (56), (78) and (1234), (1256), (1278), (3456), (3478), (5678). It is easily verified that these four pairs and six quadruples, along with the 72 mixed letter-number quadruples from the last section, cover all 28 distinct number pairs from 1 to 8 exactly 4 times each.

We also write down a balanced incomplete block design, with parameters (10,15,6,4,2) on the ten letters a, b, c, d, e, f, g, h, i, x, and then delete element x. This leaves us with a design on the symbols a, b, c, d, e, f, g, h, i, comprising 6 triples and 9 quadruples; each letter occurs twice in the triples, and each letter pair occurs twice in the 15 sets. So these 6 triples and nine quadruples, along with the 72 mixed letter-number quadruples formed from the square in Section 2, cover all letter pairs exactly four times.

4 Conclusion

We now have a total of 4 pairs (on the 8 number symbols), 6 triples (on the 9 letter symbols), and $6+72+9 = 87$ quadruples comprising 6 quadruples of the form $(NNNN)$, 72 quadruples of the form $(LLNN)$, and 9 quadruples of the form $(LLLL)$. This gives a design in 97 blocks on 17 symbols in which each of the 136 possible pairs occurs 4 times each. This is a solution for the case $v = 17$, $\lambda = 4$, of the minimization problem discussed in [1].

References

[1] T.S. Griggs, M.J. Grannell, and R.G. Stanton, On λ-fold coverings with maximum block size four for $\lambda = 3$, 4, and 5, (to appear).

[2] R.G. Stanton, Some Design Theory Games, *Bulletin of the ICA* **41** (2004), 61-63.