Augmenting Trees to have Two Disjoint Total Dominating Sets

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Abstract

A total dominating set of a graph $G = (V, E)$ is a set S of vertices such that every vertex is adjacent to a vertex in S. Define $td(G)$ as the minimum number of edges that must be added to G to ensure a partition of V into two total dominating sets of the resulting graph. We show that if G is a tree, then $\ell(G)/2 \leq td(G) \leq \ell(G)/2 + 1$, where $\ell(G)$ is the number of leaves of G.

1 Introduction

We generally use the definitions and terminology of [2]. Let $G = (V, E)$ be a graph. For $v \in V$, the (open) *neighborhood of* v, denoted by $N(v)$, is defined by $\{ u \in V \mid uv \in E \}$. A set $S \subseteq V$ is a *dominating set* of G if for every vertex $u \in V - S$, there exists a vertex $v \in S$ such that $uv \in E$. A dominating set S is a *total dominating set* if every vertex in S is adjacent to another vertex of S and a *restrained dominating set* if every vertex in $V - S$ is adjacent to another vertex of $V - S$. Note that every graph has

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a restrained dominating set and every graph without isolates has a total dominating set, since $S = V$ is such a set.

Total domination in graphs was introduced by Cockayne et al. [3] and restrained domination by Domke et al. [4]. Both total and restrained domination are now well studied in graph theory (see, for example, [6, 7]).

A classical result in domination theory is that if S is a minimal dominating set of a graph G without isolates, then $V - S$ is also a dominating set of G. Thus, the vertex set of every graph without any isolates can be partitioned into two dominating sets. However, it is not the case that the vertex set of every graph can be partitioned into two total dominating sets. For example, the vertex set of C_5 cannot be partitioned into two total dominating sets.

For a given graph G, Heggernes and Telle [8] established that the decision problem whether there is a partition of $V(G)$ into two total dominating sets is NP-complete, even if G is bipartite.

A partition of the vertex set can also be thought of as a coloring. In particular, a partition into two total dominating sets is a 2-coloring of the graph such that no vertex has a monochromatic (open) neighborhood. As an example of such a 2-coloring in K_n with $n \geq 4$, take any 2-coloring with at least two vertices of each color, while in $K_{m,n}$ with $m, n \geq 2$ take any 2-coloring where neither partite set is monochromatic.

Zelinka [9, 10] showed that no minimum degree is sufficient to guarantee the existence of two total dominating sets. Consider the bipartite graph G_n^k formed by taking as one partite set a set A of n elements, and as the other partite set all the k-element subsets of A, and joining each element of A to those subsets it is a member of. Then G_n^k has minimum degree k. As observed in [9], if $n \geq 2k - 1$ then in any 2-coloring of A at least k vertices must receive the same color, and these k are the neighborhood of some vertex.

In contrast, Calkin and Dankelmann [1] and Feige et al. [5] have shown that if the maximum degree is not too large relative to the minimum degree, then sufficiently large minimum degree guarantees arbitrarily many disjoint dominating sets, and hence taking union of pairs, arbitrarily many disjoint total dominating sets.

In this paper we consider the question of how many edges must be added to G to ensure the partition of V into two total dominating sets in the resulting graph. We denote this minimum number by $td(G)$. The answer is the same for restrained domination: in fact, a partition of V into two total dominating sets is exactly the same as a partition of V into two restrained dominating sets. It is clear that $td(G)$ can only exist for graphs with at least four vertices (we do not allow loops). As observed earlier, $td(K_n) = 0$ with $n \geq 4$ and $td(K_{m,n}) = 0$ with $m, n \geq 2$.

We calculate $td(G)$ when G is a cycle C_n or a path P_n on $n \geq 4$ vertices. We show that if T is a tree with ℓ leaves, then $\ell/2 \leq td(T) \leq \ell/2 + 1$.

2 Cycles and Paths

In this section we calculate $td(G)$ when G is a cycle C_n or a path P_n on $n \geq 4$ vertices.

A useful extension in a path is a *stepwise coloring* which is a 2-coloring of the vertices such that no vertex of degree 2 has a monochromatic neighborhood. If one specifies the color of two consecutive vertices, then there is a unique stepwise coloring which extends this. The same is true if one has a path with an even number of vertices, and one specifies the colors of the two end-vertices.

Lemma 1 (*a*) *For the cycle C_n with $n \geq 4$, $td(C_n) = 0$ if $4|n$ and 1 otherwise.*
(*b*) *For the path P_n on $n \geq 4$ vertices, $td(P_n) = 1$ if $4|n$ and 2 otherwise.*

PROOF. (a) Suppose C_n is the cycle $v_1, v_2, \ldots, v_n, v_1$. If $td(C_n) = 0$, then vertices at distance two apart must have opposite colors in the associated 2-coloring, and so n is a multiple of 4. Conversely if n is multiple of 4, color all vertices v_j with $j \equiv 0, 1 \pmod 4$ red and all the remaining vertices blue to produce a 2-coloring in which no vertex has a monochromatic neighborhood. Hence, $td(C_n) = 0$ if and only if $4|n$. It remains to show that $td(C_n) \leq 1$.

If n is odd, let v_1 be colored red, then color the vertices in order v_3, v_5, …, v_{n-2}, v_2, …, v_{n-1} with the colors blue and red such that each vertex receives the opposite color to the previously colored vertex. Then both v_1 and v_{n-1} are colored red and v_n is the only vertex with a monochromatic neighborhood. Since v_3 is colored blue, no vertex will have a monochromatic neighborhood in $G + v_3v_n$. Hence, $td(C_n) \leq 1$.

Suppose $n \equiv 2 \pmod 4$. Color all vertices v_j with $j \equiv 0, 1 \pmod 4$ red and all the remaining vertices blue. Adding the edge $v_1 v_{n-2}$ and recoloring v_{n-1} with the color blue produces a 2-coloring in which no vertex has a monochromatic neighborhood. Hence, $td(C_n) \le 1$.

(b) Since a graph with two disjoint total dominating sets has minimum degree at least 2, $td(P_n) = 1$ if and only if $td(C_n) = 0$. Hence, by part (a), $td(P_n) = 1$ if and only if $4|n$. Since $td(P_n) \le td(C_n) + 1$, the desired result follows from part (a). □

3 Trees

Since a graph with two total dominating sets has minimum degree 2, in general for T a tree with ℓ leaves, $td(T) \ge \ell/2$.

Theorem 1 *Let T be a tree of order at least 4 with ℓ leaves. Then, $td(T) \le \ell/2 + 1$.*

PROOF. We proceed by induction on the number ℓ of leaves. The base case will be trees with at most three leaves. By Lemma 1, $td(P_n) \le 2$ and so the result holds when $\ell = 2$.

Suppose $\ell = 3$. Then there is a unique vertex v of degree more than 2 in T. Define a ***limb*** of a tree as a maximal path starting at a leaf and not containing a vertex of degree 3 or more.

By the Pigeonhole Principle, there is a pair of limbs whose total number of vertices is even. Call such a pair an ***even pair***. Adding an edge e which joins the end-vertices of the even pair to T produces an odd cycle, say $C: v_1, v_2, \ldots, v_k, v_1$. Thus k is odd. We may assume $v = v_k$. Let $v_k, v_{k+1}, \ldots, v_n$ denote the path from v to the end-vertex of $T + e$. Then, $v_1, v_2, \ldots, v_n$ is a hamiltonian path of $T + e$.

If $k \equiv 1 \pmod 4$, then color all vertices v_j with $j \equiv 0, 1 \pmod 4$ red and all the remaining vertices blue, while if $k \equiv 3 \pmod 4$, then color all vertices v_j with $j \equiv 1, 2 \pmod 4$ red and all the remaining vertices blue. In both cases none of $v_2, \ldots, v_{n-1}$ has a monochromatic neighborhood. If $k \equiv 1 \pmod 4$, then v_2 is blue and v_k is red, while if $k \equiv 3 \pmod 4$, then v_2 is red and v_k is blue. Hence neither does v_1 have a monochromatic neighborhood. Thus in both cases we can achieve a 2-coloring in which no vertex has a monochromatic neighborhood by joining v_n to a vertex of the desired color. That is, $td(T) \leq td(T+e)+1 \leq 2$. This completes the base case.

Let $\ell \geq 4$ and suppose then that for all trees T' with ℓ' leaves, where $\ell' < \ell$, that $td(T') \leq \ell'/2+1$. Let T be a tree with ℓ leaves.

Assume first that $\ell \geq 5$. By the Pigeonhole Principle, there are two limbs with a total of an even number of vertices, that is, an even pair.

We claim one can choose an even pair whose removal does not create an end-vertex. If the removal of an even pair would create an end-vertex, then the pair meet at a vertex of degree 3. So if the first even pair found is not usable, the pair meets at a vertex of degree 3. Out of the remaining limbs of which there are at least three, there must be another even pair. If they too are not usable, then they meet at a vertex of degree 3. But then consider three limbs: one from the first pair, one from the second pair, and a limb from neither the first nor the second pair. There is an even pair among these three, and the removal of that pair cannot create an end-vertex.

Now, delete the chosen two limbs to yield a tree T' with $\ell-2$ leaves. By the inductive hypothesis, one can add $(\ell-2)/2+1$ edges to T' and then 2-color it such that no vertex has a monochromatic neighborhood.

Add the two limbs back to T' and join their end-vertices. This is equivalent to introducing a path $v_1, v_2, \ldots, v_{2m}$ of even order and then identifying each of v_1 and v_{2m} with a vertex of T' (possibly the same one). Then extend the 2-coloring of T' to a stepwise coloring of the path. The end-vertices of the path do not have monochromatic neighborhoods even in T'; by the construction of stepwise coloring, no interior vertex of the path has a monochromatic neighborhood. Thus, $td(T) \leq td(T')+1 \leq \ell/2+1$.

Finally, assume T has four limbs. Then as before there exists an even pair. Define T' by removing an even pair from T; this may create an end-vertex. But still T' has at most three leaves. If T' has at least 4 vertices, then by the inductive hypothesis $td(T') \leq 2$; the two limbs can be reinserted as before, and so $td(T) \leq 2+1$, as required.

Suppose then that T' has at most three vertices. In that case, T' has only two leaves and so T must have four limbs and a vertex v of degree 4 adjacent to at least two leaves. One could choose as limbs the two leaves adjacent to v; so the only way one can be forced to a T' of order 3 is if T is a star on four edges. But then coloring the central vertex and one leaf red and the remaining three leaves blue, and then joining one of the blue leaves to each of other three leaves, produces a 2-coloring such that no vertex has a monochromatic neighborhood, so that $td(T) \leq 3$, as required. □

In particular, if a tree T has an odd number ℓ of leaves, then $td(T) = (\ell+1)/2$. There are many examples of trees T with an even number ℓ of leaves for which $td(T) = \ell/2+1$ (for example stars), and many examples with $td(T) = \ell/2$ (for example, the corona of any nontrivial tree, i.e, the tree obtained by adding a pendant edge to each vertex of a nontrivial tree). There does not appear to be an easy characterization of trees T with an even number ℓ of leaves satisfying either $td(T) = \ell/2$ or $td(T) = \ell/2+1$.

As an immediate consequence of Theorem 1 we have the following result.

Corollary 1 *Let F be a forest with nontrivial components and with ℓ leaves. Then, $td(F) \leq \ell/2+1$.*

PROOF. Let $T_1, \ldots, T_k$ denote the components of F. If $k = 1$, then F is a tree and the result follows from Theorem 1. Hence we may assume $k \geq 2$. For $i = 1, 2, \ldots, k$, let u_i and v_i be two distinct leaves in T_i. Let T be the tree obtained from F by adding the $k-1$ edges $u_i v_{i+1}$ for $i = 1, \ldots, k-1$. Then T has $\ell - 2(k-1)$ leaves, and so by Theorem 1, $td(T) \leq \ell/2 - k + 2$. Hence, $td(F) \leq k - 1 + td(T) \leq \ell/2+1$. □

As a consequence of Corollary 1, if F is a forest with nontrivial components and with ℓ leaves, then $\ell/2 \leq td(F) \leq \ell/2+1$. That there exist such forests F with $td(F) = \ell/2+1$ may be seen by considering, for example, the forest $F = mP_k$ where $m \geq 1$ and $k \geq 3$ are both odd integers. Then F has $\ell = 2m$ leaves. If $td(F) = m$, then there exists a set E_F of m edges joining the $2m$ leaves of F such that $td(F + E_F) = 0$. However $F + E_F$ is the disjoint union of cycles at least one of which is odd (since m and k are both odd), and so, by Lemma 1, $td(F + E_F) > 0$, a contradiction. Hence, $td(F) = \ell/2+1$.

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