Minimizing the Number of Partial Matchings in Bipartite Graphs

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Submitted: March 8, 2004

Abstract

Suppose that we know the vertex degrees in one part of a bipartite graph G. We compute the smallest number of matchings of size m that G can have (provided there is at least one). In fact, our results also apply to the more general problem of counting matchings in matroids.

1 Introduction

Let G be a bipartite graph with a bipartition $V(G) = X \cup Y$. Let $X := \{x_1, \ldots, x_n\}$. Let $d_i := d(x_i)$ be the degree of x_i. Here we solve the following problem.

Problem 1 *Given* $\mathbf{d} := (d_1, \ldots, d_n)$ *and an integer* $m \leq n$, *what is the smallest number of matchings of size* m *that* G *can have, provided there is at least one* m*-matching?*

Ostrand [4] (see Hwang [2] for another proof) has settled the case $m = n$ when the matchings to count must contain every vertex of X. McCarthy [3] generalized Ostrand's results to the setting where we have a matroid on Y and we count the number of *independent n-matchings*, that is, we additionally require that the set of matched vertices of Y is an independent set. Our bound on partial matchings holds also for matroids, see Section 4.

One motivation behind this study is that sometimes the existence of a certain combinatorial object can be proved by applying Hall's Marriage Theorem (see [1, Chapter VIII.2] for some examples). Thus a lower bound in Problem 1 should give a quantitative strengthening of these results wherein we deduce a lower bound on the number of the constructed objects.

2 Notation and Preliminary Remarks

When dealing with matroids we will follow the terminology in [5]. Given a matroid $\mathcal{M}$ on Y, let $I_m(G, \mathcal{M})$ denote the number of independent m-matchings. Rado's theorem [6] implies that $(G, \mathcal{M})$ has an independent m-matching if and only if

$$\forall A \subseteq X \quad \rho(\Gamma(A)) \geq |A| - n + m, \tag{1}$$

where ρ is the rank function of $\mathcal{M}$ and $\Gamma(A) := \{y : \exists x \in A \ \{x, y\} \in E(G)\}$.

Any set A achieving the bound in (1) is called *critical*. It is easy to see that for any critical A every independent m-matching contains $\rho(\Gamma(A))$ vertices from $\Gamma(A)$ (the largest possible number) as well as all vertices in $X \setminus A$ but does not connect these two sets.

If $\mathcal{M}$ is the free matroid (that is, $\rho(A) = |A|$ for all $A \subseteq Y$), then (1) gives the well-known defect version of Hall's Marriage Theorem.

Note that m-matchings in G can be equivalently considered as systems of m distinct representatives of the set system $\big(\Gamma(x_1), \ldots, \Gamma(x_n)\big)$. However, in this paper we will use the graph version.

3 Construction and Its Properties

First of all, we can assume without loss of generality that each d_i is positive (otherwise we remove x_i) and that $d_1 \leq \cdots \leq d_n$.

To construct our graph $H = H_m(\mathbf{d})$ we have to specify sets $\Gamma(x_i)$. Let us assume that Y is a sufficiently large initial segment of positive integers.

(In fact, our construction gives $|Y| = \max(m, d_n)$, the smallest possible value.) For $i \in [n]$ define

$$\Gamma(x_i) := \begin{cases} [d_i], & \text{if } d_i \geq i-n+m, \\ [d_i - 1] \cup \{i-n+m\}, & \text{otherwise.} \end{cases}$$

Note that H contains a matching of size m: consider the edges $\{x_i, i-n+m\}$ for $i \in [n-m+1, n]$.

Let us state a few properties of H which we will need later. Let $X_i := \{x_1, \ldots, x_i\}$.

Lemma 2 *If we have $d_i \leq i-n+m$ for some i, then $\Gamma(X_i) = [i-n+m]$. (In particular, X_i is critical and H has no matching of size $m+1$.)*

Proof. For any $j \leq i$ we have $d_j \leq d_i \leq i-n+m$, so $\Gamma(x_j) \subseteq [i-n+m]$, which shows that $\Gamma(X_i) \subseteq [i-n+m]$. The converse inclusion follows by observing that $j \in [m]$ is always connected to x_{j+n-m}. ∎

Lemma 2 allows us to compute $f_m(\mathbf{d})$, the number of m-matchings in H. If $d_i \leq i-n+m$ for some i, then

$$f_m(\mathbf{d}) = \frac{1}{(n-m)!} \prod_{i=1}^{n} \max(d_i + n - m - i + 1, 1). \tag{2}$$

Indeed, if we add $n-m$ new vertices to Y which are connected to everything in X, then, in view of Lemma 2, the new graph H' has precisely $(n-m)! \cdot f_m(\mathbf{d})$ matchings of size n. Note that

$$H' \cong H_n(d_1 + n - m, \ldots, d_n + n - m)$$

and for this graph it is easy to compute the number of n-matchings (alternatively, see Ostrand [4]), giving (2).

If $d_i > i-n+m$ for all i, then we have $\Gamma(x_i) \subseteq \Gamma(x_j)$ for any $i < j$ and the number of m-matchings can be expressed as

$$f_m(\mathbf{d}) = \sum_{1 \leq \nu_1 < \cdots < \nu_m \leq n} \prod_{i=1}^{m} \max(d_{\nu_i} - i + 1, 0). \tag{3}$$

It seems that there is no nice formula, like (2), for $f_m(\mathbf{d})$ in this case.

In the remainder of this paper, when we write $f_m(\mathbf{d})$ we will mean that we remove any zeros from $\mathbf{d}$, reorder $\mathbf{d}$ to be non-decreasing and then use the formulas (2) and (3).

Lemma 3 *The function $f_m(\mathbf{d})$ is non-decreasing with respect to each argument d_i.*

Proof. It is enough to prove the claim when we increase some d_i by 1: $d'_i = d_i + 1$ while all other $d'_j = d_j$. We can assume that either $i = n$ or $d_i < d_{i+1}$. When we analyze the corresponding graphs, H and H', we see that H' is obtained from H by adding one more edge. Of course, this cannot decrease the number of m-matchings. ∎

4 Lower Bound

In this section the term 'matching' implicitly means 'an independent matching.'

Theorem 4 *Let G be a bipartite graph with a bipartition $V(G) = X \cup Y$. Let $\mathcal{M}$ be a matroid on Y with rank function ρ. Let $X := \{x_1, \dots, x_n\}$ and $d_i := \rho(\Gamma(x_i))$. Assume $1 \le d_1 \le \cdots \le d_n$.*

If $I_m(G, \mathcal{M}) \ge 1$, then

$$I_m(G, \mathcal{M}) \ge f_m(d_1, \dots, d_n). \tag{4}$$

Proof. We use induction on n with the case $n = 1$ being trivially true. Let $n \ge 2$. The proof splits into two cases. Recall that a set $A \subseteq X$ is called critical if we have equality in (1).

Case 1 There is a critical $A \subseteq X$ (possibly $A = X$).

This means that $(G, \mathcal{M})$ admits no $(m+1)$-matching. Let G' be obtained from G by adding $n - m$ new vertices to Y which are connected to everything in X. Let the matroid $\mathcal{M}'$ be the matroid union of $\mathcal{M}$ and the free matroid on the new vertices; its rank function is

$$\rho'(B) = \rho(B \cap Y) + |B \setminus Y|.$$

Clearly, $I_n(G, \mathcal{M}) = I_m(G', \mathcal{M}')/(n-m)!$. Now, the result of McCarthy [3], when applied to $(G', \mathcal{M}')$, settles this case.

Case 2 There is no critical set.

Let us bound N_1, the number of m-matchings containing x_1. We can choose a non-loop $y \in \Gamma(x_1)$ in at least d_1 possible ways.

Let us show that the pair $(G', \mathcal{M}')$, where $G' := G - x_1 - y$ and $\mathcal{M}' := \mathcal{M}/y$, has an $(m-1)$-matching. If this is not true, then by (1) we can find $A \subseteq X \setminus \{x_1\}$ with

$$\rho'(\Gamma_{G'}(A)) \leq |A| - (n-1) + (m-1) - 1 = |A| - n + m - 1.$$

This implies that A is critical with respect to $(G, \mathcal{M})$, a contradiction.

Clearly, $\rho'(\Gamma_{G'}(x_i)) \geq d_i - 1$. By the monotonicity of f_m and induction on n, we have

$$N_1 \geq d_1 f_{m-1}(d_2 - 1, \ldots, d_n - 1).$$

To bound N_2, the number of m-matchings omitting x_1, let $G' := G - x_1$. Similarly to above, one can show that $(G', \mathcal{M})$ has an m-matching. Thus

$$N_2 \geq f_m(d_2, \ldots, d_n).$$

To complete the proof, it is enough to prove that

$$f_m(d_1, \ldots, d_n) \leq d_1 f_{m-1}(d_2 - 1, \ldots, d_n - 1) + f_m(d_2, \ldots, d_n). \quad (5)$$

If the value d_1 occurs in $\mathbf{d}$ at most $d_1 + n - m$ times, then in $H_m(\mathbf{d})$ we have $\Gamma(x_1) \subseteq \Gamma(x_i)$ for any i. Splitting m-matchings of $H_m(\mathbf{d})$ into two groups according to whether or not they contain x_1 we conclude that (5) holds. (It is an equality, in fact.)

So, suppose that d_1 appears $j > d_1 + n - m$ times in $\mathbf{d}$: $d_1 = \ldots = d_j$. Here we deduce first that

$$f_m(\mathbf{d}) \leq d_1 f_{m-1}(\mathbf{d}') + f_m(d_2, \ldots, d_n), \quad (6)$$

where $\mathbf{d}'$ consists of $d_1 - 1$ repeated $d_1 + n - m - 1$ times, then d_1 repeated $j - d_1 - n + m$ times, followed by $d_{j+1} - 1, \ldots, d_n - 1$. But in $H_{m-1}(\mathbf{d}')$ the vertices of degree $d_1 - 1$ form a critical set by Lemma 2 so they claim the whole of $[d_1 - 1]$ in any $(m-1)$-matching. The graph $H_{m-1}(\mathbf{d}')$ is obtained from $H_{m-1}(d_2 - 1, \ldots, d_n - 1)$ by adding extra edges connecting $[d_1 - 1] \subseteq Y$ to vertices in X of degree $d_1 - 1$. This shows that

$$f_{m-1}(\mathbf{d}') = f_{m-1}(d_2 - 1, \ldots, d_n - 1)$$

and implies (5) by (6), finishing the proof. ∎

5 Concluding Remarks

Observe that Problem 1 can also be solved if we omit the condition that G contains an m-matching. Indeed, it is straightforward to deduce from (1) that the restrictions on $\mathbf{d}, n$ force an m-matching if and only if $d_i \geq i-n+m$ for each $i \in [n]$.

The question of *maximizing* the number of m-matchings is trivial with the extremal construction being the disjoint union of stars K_{1,d_i}. (While for matroids there is no upper bound at all.)

Acknowledgments

The author thanks the anonymous referee for helpful comments and the students of his Spring'04 lecture course, where Ostrand's theorem was presented, for their enthusiasm.

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