Power Series Identities Generated by Two Recent Integer Sequences

A. Frazer Jarvis, Peter J. Larcombe[†] and David R. French[‡]

Department of Pure Mathematics
Hicks Building, University of Sheffield, Sheffield S3 7RH, U.K.
{A.F.Jarvis@sheffield.ac.uk}

[†]Derbyshire Business School
University of Derby, Kedleston Road, Derby DE22 1GB, U.K.
{P.J.Larcombe@derby.ac.uk}

[‡]115, John O'Gaunt's Way, Belper, Derbyshire DE56 0DA, U.K.

Abstract

Generating functions for the Catalan-Larcombe-French sequence and counterpart Fennessey-Larcombe-French sequence, together with properties of the elliptic integrals from which they are formulated, give rise to unusual power series identities. The hypergeometric versions of these identities can be validated independently using known hypergeometric transformations.

1 Introduction

Consider the Catalan-Larcombe-French sequence

$$\{P_0, P_1, P_2, P_3, P_4, \ldots\} = \{1, 8, 80, 896, 10816, \ldots\} \tag{1}$$

arising from a certain transformation of the complete elliptic integral of the first kind

$$K(c) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - c^2 \sin^2(\phi)}} \, d\phi \tag{2}$$

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(where $c \in (-1,1) \setminus \{0\}$), and the counterpart Fennessey-Larcombe-French sequence

$$\{V_0, V_1, V_2, V_3, V_4, \ldots\} = \{1, 8, 144, 2432, 40000, \ldots\} \tag{3}$$

produced from the same transformation applied to the complete elliptic integral of the second kind

$$E(c) = \int_0^{\frac{\pi}{2}} \sqrt{1 - c^2 \sin^2(\phi)}\, d\phi. \tag{4}$$

Formulations of (1),(3) have been presented in some detail in [1,2], respectively, and they are registered on Sloane's On-Line Encyclopaedia of Integer Sequences as Sequence Nos. A053175 and A065409.[1] The sequence (1) is due originally to Eugène Catalan, and was so named in the article by Larcombe *et al.* [3, p.74]. It is well known that the integrals (2),(4) satisfy coupled first order differential equations, namely [1, Results I,II, pp.44-45],

$$\frac{dE}{dc} = \frac{E(c) - K(c)}{c} \tag{5}$$

and

$$\frac{dK}{dc} = \frac{1}{c}\left(\frac{E(c)}{1-c^2} - K(c)\right), \tag{6}$$

and we shall make use of them as appropriate in order to derive interesting power series identities not seen anywhere to date it would appear.

Let $b^2(c) = 1 - c^2 \in (0,1)$, and define

$$P(c) = \sum_{s=0}^{\infty} P_s \left(\frac{1-b(c)}{16}\right)^s, \qquad V(c) = \sum_{s=0}^{\infty} V_s \left(\frac{1-b(c)}{16}\right)^s. \tag{7}$$

In [1, (24), p.39] it was established that

$$K(c) = \frac{\pi}{2} P(c) \tag{8}$$

en route to the identification of the sequence (1). Likewise,

$$E(c) = \frac{\pi}{2} b(c) V(c), \tag{9}$$

see [2, (13), p.182)]. Setting $b(x) = 1 - 16x$, or equivalently $c^2(x) = 1 - b^2(x) = 32x(1-8x)$, it follows from (7)-(9) that

$$K(x) = \frac{\pi}{2} P(x), \qquad E(x) = \frac{\pi}{2} b(x) V(x), \tag{10}$$

[1] See the web site at **http://www.research.att.com/~njas/sequences/**.

where

$$P(x) = P(c(x)) = \sum_{s=0}^{\infty} P_s x^s, \qquad V(x) = V(c(x)) = \sum_{s=0}^{\infty} V_s x^s. \tag{11}$$

This, then, sets out the main theory underpinning the results to be presented here, being centred around the generating functions P, V associated with the two sequences introduced and the elliptic integrals from which they are derived. Aside from some remarks of interest at its conclusion, Section 2 is devoted entirely to the proof of a new form for the general term of the Fennessey-Larcombe-French sequence. An essential component of the proof is the intimate connection between the function $P(x)$ and the celebrated arithmetic-geometric mean (attached to Gauss) discussed in an earlier paper by the authors [4]. The analysis leads logically to surprising, if not remarkable, identities which are each an equality of formal power series and detailed in a final section; they can be recovered independently in hypergeometric form using known hypergeometric transformations.

2 A New Form for V_n

Let $[\nu]$ be, for arbitrary ν real, the greatest integer not exceeding ν. Adopting standard hypergeometric series notation, we have the following:

Theorem 1 *For $n \geq 0$,*

$$\begin{aligned} V_n &= 8^n \sum_{p=0}^{[n/2]} (4p+1) \left[\frac{1}{4^p} \binom{2p}{p} \right]^2 \binom{n}{2p} \\ &= 8^n F\left(\left. \begin{matrix} \frac{5}{4}, \frac{1}{2}, -\frac{1}{2}(n-1), -\frac{1}{2}n \\ \frac{1}{4}, 1, 1 \end{matrix} \right| 1 \right). \end{aligned}$$

Proof Theorem 1 of [4, p.155] states that

$$P(x) = \frac{1}{M(1, 1-16x)}, \tag{12}$$

the function M denoting the arithmetic-geometric mean of Gauss; for real $a, b \geq 0$, $M(a, b)$ is the common limit of two sequences $\{a_n\}_0^\infty, \{b_n\}_0^\infty$ which, from initial values $a_0 = a$, $b_0 = b$, each converge under the constituent iterative scheme (a full explanation is offered in [4] and elsewhere). In turn, we can write

$$P(x) = \sum_{k=0}^{\infty} \binom{2k}{k}^2 \frac{1}{16^k} (8x)^{2k} (1-8x)^{-(2k+1)} \tag{13}$$

from [4, (15),(16), p.156], which is a key result. Noting that $x(b) = \frac{1}{16}(1-b)$, then in terms of b

$$\begin{aligned}
(8x)^{2k}(1-8x)^{-(2k+1)} &= \left(\frac{1-b}{2}\right)^{2k}\left(\frac{1+b}{2}\right)^{-(2k+1)} \\
&= \left(\frac{1-b}{2}\right)^{4k+1}\left(\frac{1-b}{2}\right)^{-(2k+1)}\left(\frac{1+b}{2}\right)^{-(2k+1)} \\
&= \left(\frac{1-b}{2}\right)^{4k+1}\left(\frac{1-b^2}{4}\right)^{-(2k+1)} \\
&= 2(1-b)^{4k+1}(1-b^2)^{-(2k+1)}, \qquad (14)
\end{aligned}$$

and further, as a function of $c = \sqrt{1-b^2}$,

$$(8x)^{2k}(1-8x)^{-(2k+1)} = 2\left(1-\sqrt{1-c^2}\right)^{4k+1} c^{-2(2k+1)}, \qquad (15)$$

so that

$$\begin{aligned}
P(c) &= P(x(b(c))) \\
&= 2\sum_{k=0}^{\infty}\binom{2k}{k}^2 \frac{1}{16^k}\left(1-\sqrt{1-c^2}\right)^{4k+1} c^{-2(2k+1)}, \qquad (16)
\end{aligned}$$

and, by (8),

$$K(c) = \pi\sum_{k=0}^{\infty}\binom{2k}{k}^2\frac{1}{16^k}\, g(k;c), \qquad (17)$$

where $g(k;c) = \left(1-\sqrt{1-c^2}\right)^{4k+1} c^{-2(2k+1)}$. The differential equation (6) gives, when re-arranged,

$$b^2(c)\left(c\frac{dK}{dc} + K(c)\right) = E(c) = \frac{\pi}{2}b(c)V(c) \qquad (18)$$

using (9), whence

$$\begin{aligned}
V(c) &= \frac{2b(c)}{\pi}\left(c\frac{dK}{dc} + K(c)\right) \\
&= 2b(c)\sum_{k=0}^{\infty}\binom{2k}{k}^2\frac{1}{16^k}\left(g(k;c) + c\frac{\partial g}{\partial c}\right) \qquad (19)
\end{aligned}$$

by (17). It is unnecessary to give the precise workings (these are time consuming, though routine), the expression $g(k;c)+c\frac{\partial g}{\partial c}$ being readily found to yield, after some pleasing simplification,

$$\begin{aligned} V(c) &= 2\sum_{k=0}^{\infty}\binom{2k}{k}^2 \frac{(4k+1)}{16^k}\left(1-\sqrt{1-c^2}\right)^{4k+1} c^{-2(2k+1)} \\ &= \sum_{p=0}^{\infty}\binom{2p}{p}^2 \frac{(4p+1)}{16^p}(8x)^{2p}(1-8x)^{-(2p+1)} \\ &= V(x) \end{aligned} \tag{20}$$

with a change in summing index and writing $c = c(x)$. Expanding the series $(1-8x)^{-(2p+1)}$ binomially (reader exercise) as

$$(1-8x)^{-(2p+1)} = (8x)^{-2p}\sum_{n=0}^{\infty}\binom{n}{2p}(8x)^n, \tag{21}$$

Theorem 1 follows quickly on identifying V_n as the coefficient of x^n in the generating function $V(x)$ defined in (11).□

<u>Remark 1</u> In [4] it was established that, for $n \geq 0$,

$$\begin{aligned} P_n &= 8^n \sum_{p=0}^{[n/2]}\left[\frac{1}{4^p}\binom{2p}{p}\right]^2\binom{n}{2p} \\ &= 8^n F\left(\left.\begin{matrix}\frac{1}{2}, -\frac{1}{2}(n-1), -\frac{1}{2}n \\ 1, 1\end{matrix}\right| 1\right), \end{aligned} \tag{22}$$

see Theorem 2 therein (p.157). Comparing the binomial coefficient sum form of P_n here with that of V_n in Theorem 1 above, we observe that the only difference is the appearance of a factor $(4p+1)$ in the summand of the latter. It is clear by inspection, therefore, that $V_0 = P_0$, $V_1 = P_1$, and moreover that $V_n > P_n$ for $n \geq 2$ (when there is always at least one value of $p > 0$ in the sum range $p = 0, \ldots, [\frac{1}{2}n]$ for which $4p + 1 > 1$), a result proven at some length in [2] as Theorem 2.

<u>Remark 2</u> In [2, (14),(27), pp.182,186] it was found that, for $n \geq 0$,

$$\begin{aligned} V_n &= -\frac{1}{n!}\sum_{p+q=n}\frac{(2q+1)}{(2p-1)}\binom{2p}{p}\binom{2q}{q}\frac{(2p)!(2q)!}{p!q!} \\ &= (2n+1)\binom{2n}{n}^2 F\left(\left.\begin{matrix}-n, \frac{1}{2}, -\frac{1}{2} \\ \frac{1}{2}-n, -\frac{1}{2}-n\end{matrix}\right| -1\right). \end{aligned} \tag{23}$$

An attractive hypergeometric identity is immediate from Theorem 1 and (23), holding for integer $n \geq 0$; a formulation based on existing hypergeometric transformations remains, so far, elusive, and if achievable will undoubtedly be a non-trivial one:

$$(2n+1)\binom{2n}{n}^2 F\left(\begin{matrix}-n,\frac{1}{2},-\frac{1}{2}\\ \frac{1}{2}-n,-\frac{1}{2}-n\end{matrix}\,\middle|\,-1\right) = 8^n F\left(\begin{matrix}\frac{5}{4},\frac{1}{2},-\frac{1}{2}(n-1),-\frac{1}{2}n\\ \frac{1}{4},1,1\end{matrix}\,\middle|\,1\right). \tag{24}$$

Remark 3 In [2, (25), p.185] it was shown that $V_n/2n\binom{2n}{n}^2 \to 1$ as $n \to \infty$. Theorem 1 now allows us to write down

$$F\left(\begin{matrix}\frac{5}{4},\frac{1}{2},-\frac{1}{2}(n-1),-\frac{1}{2}n\\ \frac{1}{4},1,1\end{matrix}\,\middle|\,1\right) \sim 2^{1-3n} n \binom{2n}{n}^2 \tag{25}$$

for large n.

Thanks are expressed to Prof. Dr. Wolfram Koepf for verifying (24),(25) using his specialist software package "hsum6.mpl".[2]

3 Power Series Identities

3.1 First Principles Derivations

On the basis of the previous two sections, we give first principles derivations of the aforementioned power series identities.

Theorem 2.1

$$\sum_{k=0}^{\infty}\binom{2k}{k}^2 (2x)^{2k}(1-8x)^{-(2k+1)} = \sum_{k=0}^{\infty}\binom{2k}{k}^2 (2x)^k(1-8x)^k.$$

Proof Based on its integral definition, it is known that, when expanded as a series in c [1, (28), p.40],

$$\begin{aligned} K(c) &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-c^2\sin^2(\phi)}}\, d\phi \\ &= \frac{\pi}{2}\sum_{k=0}^{\infty}\binom{2k}{k}^2 (c^2/16)^k. \end{aligned} \tag{26}$$

[2]Visit the site **http://www.mathematik.uni-kassel.de/~koepf/Publikationen** for more information.

Thus, employing (10),

$$\begin{aligned} P(x) &= \frac{2}{\pi}K(x) \\ &= \sum_{k=0}^{\infty}\binom{2k}{k}^2 (c^2(x)/16)^k \\ &= \sum_{k=0}^{\infty}\binom{2k}{k}^2 (2x)^k(1-8x)^k, \end{aligned} \tag{27}$$

giving Theorem 2.1 on equating $P(x)$ in (27) and (13).□

Theorem 2.2

$$(1-16x)\sum_{k=0}^{\infty}(4k+1)\binom{2k}{k}^2(2x)^{2k}(1-8x)^{-(2k+1)} = \sum_{k=0}^{\infty}\frac{1}{1-2k}\binom{2k}{k}^2(2x)^k(1-8x)^k.$$

Proof In a manner similar to the above, we first note that a binomial expansion of $E(c)$ (4) is [2, (B1), p.189]

$$E(c) = -\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{1}{2k-1}\binom{2k}{k}^2(c^2/16)^k, \tag{28}$$

so that, again via (10),

$$\begin{aligned} b(x)V(x) &= \frac{2}{\pi}E(x) \\ &= \sum_{k=0}^{\infty}\frac{1}{1-2k}\binom{2k}{k}^2(c^2(x)/16)^k \\ &= \sum_{k=0}^{\infty}\frac{1}{1-2k}\binom{2k}{k}^2(2x)^k(1-8x)^k. \end{aligned} \tag{29}$$

On the other hand,

$$b(x)V(x) = (1-16x)\sum_{k=0}^{\infty}(4k+1)\binom{2k}{k}^2(2x)^{2k}(1-8x)^{-(2k+1)} \tag{30}$$

using (20), yielding the identity immediately.□

We obtain, as a bonus, two additional identities by re-visiting (6), into which $K(c)$ (26) may be substituted to give

$$\begin{aligned} E(c) &= b^2(c)\left(c\frac{dK}{dc}+K(c)\right) \\ &= \frac{\pi}{2}b^2(c)\sum_{k=0}^{\infty}\binom{2k}{k}^2\frac{1}{16^k}\left(c\cdot 2kc^{2k-1}+c^{2k}\right) \\ &= \frac{\pi}{2}b^2(c)\sum_{k=0}^{\infty}(2k+1)\binom{2k}{k}^2(c^2/16)^k, \end{aligned} \tag{31}$$

whereupon, by (10) yet once more,

$$\begin{aligned} b(x)V(x) &= \frac{2}{\pi}E(x) \\ &= (1-16x)^2\sum_{k=0}^{\infty}(2k+1)\binom{2k}{k}^2(2x)^k(1-8x)^k; \end{aligned} \tag{32}$$

Theorems 2.3 & 2.4 are written down on reconciling (32) with, respectively, (30) and (29).

<u>**Theorem 2.3**</u>

$$\sum_{k=0}^{\infty}(4k+1)\binom{2k}{k}^2(2x)^{2k}(1-8x)^{-(2k+1)} = (1-16x)\sum_{k=0}^{\infty}(2k+1)\binom{2k}{k}^2(2x)^k(1-8x)^k.$$

<u>**Theorem 2.4**</u>

$$\sum_{k=0}^{\infty}\frac{1}{1-2k}\binom{2k}{k}^2(2x)^k(1-8x)^k = (1-16x)^2\sum_{k=0}^{\infty}(2k+1)\binom{2k}{k}^2(2x)^k(1-8x)^k.$$

Note that taking the series $E(c)$ (28) and substituting it into (5) merely leads to $K(c)$ $(= E(c) - c\frac{dE}{dc})$ as seen in (26), and, therefore, to no further identities. None of Theorems 2.1-2.4 are to be located in Gould's well known listing of results [5], and we are of the opinion that they appear here for the first time.

3.2 Independent Hypergeometric Formulations

Only three of Theorems 2.1-2.4 are independent ones (for clearly any two of Theorems 2.2-2.4 imply the other). Omitting the tedious algebraic steps incurred in conversion, their equivalent (respective) hypergeometric versions are thus:

Theorem 3.1

$$\frac{1}{1-8x}F\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\,\middle|\,\frac{64x^2}{(1-8x)^2}\right)=F\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\,\middle|\,32x(1-8x)\right).$$

Theorem 3.2

$$\frac{1-16x}{1-8x}F\left(\begin{matrix}\frac{5}{4},\frac{1}{2},\frac{1}{2}\\\frac{1}{4},1\end{matrix}\,\middle|\,\frac{64x^2}{(1-8x)^2}\right)=F\left(\begin{matrix}-\frac{1}{2},\frac{1}{2}\\1\end{matrix}\,\middle|\,32x(1-8x)\right).$$

Theorem 3.3

$$\frac{1}{1-8x}F\left(\begin{matrix}\frac{5}{4},\frac{1}{2},\frac{1}{2}\\\frac{1}{4},1\end{matrix}\,\middle|\,\frac{64x^2}{(1-8x)^2}\right)=(1-16x)F\left(\begin{matrix}\frac{3}{2},\frac{1}{2}\\1\end{matrix}\,\middle|\,32x(1-8x)\right).$$

Theorem 3.4

$$F\left(\begin{matrix}-\frac{1}{2},\frac{1}{2}\\1\end{matrix}\,\middle|\,32x(1-8x)\right)=(1-16x)^2F\left(\begin{matrix}\frac{3}{2},\frac{1}{2}\\1\end{matrix}\,\middle|\,32x(1-8x)\right).$$

By way of a check, it is a useful exercise to formulate these through purely hypergeometric means. Theorem 3.1 is a special case of the quadratic transformation (see Identity No. 15.3.27 on p.561 of Abramowitz and Stegun [6] with the positive square root taken in the r.h.s.)

$$F\left(\begin{matrix}a,b\\a-b+1\end{matrix}\,\middle|\,z\right)=(1+\sqrt{z})^{-2a}F\left(\begin{matrix}a,a-b+\frac{1}{2}\\2a-2b+1\end{matrix}\,\middle|\,4\sqrt{z}(1+\sqrt{z})^{-2}\right)\quad(33)$$

for $a=b=\frac{1}{2}$, $z=64x^2/(1-8x)^2$. Theorem 3.4 is itself the particular instance, when $a=-\frac{1}{2}$, $b=\frac{1}{2}$, $c=1$ and $z=32x(1-8x)$, of Euler's well known linear transformation [6, Identity No. 15.3.3, p.559]

$$F\left(\begin{matrix}a,b\\c\end{matrix}\,\middle|\,z\right)=(1-z)^{c-a-b}F\left(\begin{matrix}c-a,c-b\\c\end{matrix}\,\middle|\,z\right).\quad(34)$$

Theorem 3.3, regarding which the authors are grateful to Dr. Axel Riese for his guidance, requires more effort. Referring to the on-line instruction manual accompanying C. Krattenthaler's package "HYP",[3] we first apply transformation T3236—namely

$$F\left(\begin{matrix} a, \frac{1}{2}a+1, b \\ \frac{1}{2}a, a-b+1 \end{matrix} \,\middle|\, z\right) = \frac{1-z}{(1+z)^{a+1}} F\left(\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 \\ a-b+1 \end{matrix} \,\middle|\, \frac{4z}{(1+z)^2}\right) \tag{35}$$

for the very well poised l.h.s. series—with $a = b = \frac{1}{2}$, $z = 64x^2/(1-8x)^2$, which gives

$$F\left(\begin{matrix} \frac{1}{2}, \frac{5}{4}, \frac{1}{2} \\ \frac{1}{4}, 1 \end{matrix} \,\middle|\, \frac{64x^2}{(1-8x)^2}\right) = \frac{(1-16x)(1-8x)}{(1-16x+128x^2)^{\frac{3}{2}}} F\left(\begin{matrix} \frac{3}{4}, \frac{5}{4} \\ 1 \end{matrix} \,\middle|\, \frac{256x^2(1-8x)^2}{(1-16x+128x^2)^2}\right). \tag{36}$$

The r.h.s. series of (36) is then transformed by using T2112 (as stated in the documentation but with $z \to -z$, taking the negative root of the l.h.s. argument throughout the r.h.s.), that is,

$$F\left(\begin{matrix} a, a+\frac{1}{2} \\ b+\frac{1}{2} \end{matrix} \,\middle|\, z^2\right) = (1+z)^{-2a} F\left(\begin{matrix} 2a, b \\ 2b \end{matrix} \,\middle|\, \frac{2z}{1+z}\right), \tag{37}$$

choosing $a = \frac{3}{4}$, $b = \frac{1}{2}$, $z = 16x(1-8x)/(1-16x+128x^2)$, whereupon Theorem 3.3 follows; this, in conjunction with Theorem 3.4, gives Theorem 3.2.

4 Summary

The Catalan-Larcombe-French and Fennessey-Larcombe-French sequences are recent announcements possessing interesting properties which have been reported in [1-4] and elsewhere. In this article, using their theoretical basis, a new form for the general term of the latter sequence has been derived, motivating seemingly novel power series identities that are available independently in hypergeometric guise.

References

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