Two generalizations of the metamorphosis definition

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Abstract

Let $B(k, \lambda)$ be the spectrum of integers n such that there exists a $S_\lambda(2, k, n)$, a balanced incomplete block design of order n, block size k and index λ. Lindner and Rosa [6] introduced the definition of a $S_\lambda(2, 4, n)$ having a metamorphosis into a $S_\lambda(2, 3, n)$ and proved that the necessary condition $n \in B(3, \lambda) \cap B(4, \lambda)$ is also sufficient.

The aim of this paper is to present two different generalizations of Lindner and Rosa's idea in order to consider metamorphoses of $S_\lambda(2, 4, n)$ for $n \in B(4, \lambda)$ and $n \notin B(3, \lambda)$.

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1 Introduction

A balanced incomplete block design $S_\lambda(2, k, n)$ is a pair $(X, \mathcal{B})$, where X is a n-set and $\mathcal{B}$ is a collection of k-subsets of X (blocks) such that any 2-

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subset of X is contained in exactly λ blocks. For $\lambda = 1$ we write $S(2,k,n)$ instead of $S_1(2,k,n)$.

A maximum packing of triples $MPT(n,\lambda)$ is a pair $(X,\mathcal{C})$, where X is an n-set and $\mathcal{C}$ is a collection of 3-subsets of X (blocks) such that: (i) each 2-subset of X is contained in at most λ blocks of $\mathcal{C}$, (ii) if $\mathcal{D}$ is any collection of 3-subsets of X satisfying (i), then $|\mathcal{C}| \geq |\mathcal{D}|$.

Let $(X,\mathcal{C})$ be a $MPT(n,\lambda)$; the leave of $(X,\mathcal{C})$ is a multigraph $(X,\mathcal{E})$ where an edge $\{x,y\} \in \mathcal{E}$ has multiplicity m if and only if the corresponding 2-subset $\{x,y\}$ is contained in exactly $\lambda - m$ blocks of $\mathcal{C}$.

Let $(X,\mathcal{B})$ be a $S_\lambda(2,4,n)$. If a star is removed from each block of $\mathcal{B}$ the resulting collection of triangles $P(\mathcal{B})$ is a partial $S_\lambda(2,3,n)$ $(X,P(\mathcal{B}))$. If the edges belonging to the deleted stars can be arranged into a collection of triangles $T(\mathcal{B})$, then $(X,P(\mathcal{B})\cup T(\mathcal{B}))$ is a $S_\lambda(2,3,n)$, called a metamorphosis of the $S_\lambda(2,4,n)$ $(X,\mathcal{B})$. Lindner and Rosa [6] posed the following *spectrum problem*: "For every positive integer λ, determine the spectrum of integers n such that there exists a $S_\lambda(2,4,n)$ having a metamorphosis into a $S_\lambda(2,3,n)$". The necessary condition for the existence of a $S_\lambda(2,4,n)$ having a metamorphosis into a $S_\lambda(2,3,n)$ is $n \in B(3,\lambda) \cap B(4,\lambda)$, where $B(k,\lambda)$ is the set of the integers n such that there is a $S_\lambda(2,k,n)$. Lindner and Rosa [6] proved that these necessary conditions are also sufficient. Table 1 summarizes Lindner and Rosa's results.

Table 1	
λ (mod 6)	spectrum of $S_\lambda(2,4,n)$ having a metamorphosis into $S_\lambda(2,3,n)$
0	$n \geq 4$
1, 5	$n \equiv 1 \pmod{12}$
2, 4	$n \equiv 1 \pmod{3}$
3	$n \equiv 1 \pmod{4}$

For $n \in B(4,\lambda)$ and $n \notin B(3,\lambda)$, the following question is natural: How can we generalize the metamorphosis definition in order to construct a $S_\lambda(2,4,n)$ having a metamorphosis into *some design as close as possible to a* $S_\lambda(2,3,n)$? The aim of this paper is to present two different answers.

Metamorphosis of a $\mathbf{S_\lambda(2,4,n)}$ into a minimum $\mathbf{S_\lambda(2,3,v)}$. Let $(X,\mathcal{B})$ be a $S_\lambda(2,4,n)$. Let v be the minimum integer such that $v \geq n$ and $v \in B(3,\lambda)$, and let $V = X \cup Y$ where $|Y| = v - n$. If a star is removed from each block of $\mathcal{B}$ the resulting collection of triangles $P(\mathcal{B})$ is a partial $S_\lambda(2,3,n)$ $(X,P(\mathcal{B}))$. If the edges belonging to the deleted stars and to graphs K_Y and $K_{X,Y}$, can be arranged into a collection of triples $T(\mathcal{B})$, then $(X,P(\mathcal{B})\cup T(\mathcal{B}))$ is a $S_\lambda(2,3,v)$, called a metamorphosis of the $S_\lambda(2,4,n)$ $(X,\mathcal{B})$ into the minimum $S_\lambda(2,3,v)$.

Metamorphosis of a $S_\lambda(2,4,n)$ into a $MPT(n,\lambda)$. Let $(X,\mathcal{B})$ be a $S_\lambda(2,4,n)$. If a star is removed from each block of $\mathcal{B}$ the resulting collection of triangles $P(\mathcal{B})$ is a partial $S_\lambda(2,3,n)$ $(X,P(\mathcal{B}))$. If the edges belonging to the deleted stars can be arranged into a collection of triangles $T(\mathcal{B})$ and a collection of edges $\mathcal{E}$ such that $(X,P(\mathcal{B})\cup T(\mathcal{B}))$ is a $MPT(n,\lambda)$ with leave $(X,\mathcal{E})$, then $(X,P(\mathcal{B})\cup T(\mathcal{B}))$ is called a metamorphosis of the $S_\lambda(2,4,n)$ $(X,\mathcal{B})$ into a $MPT(n,\lambda)$.

It is straightforward to see that both these definitions coincide with Lindner and Rosa's metamorphosis whenever $n\in B(3,\lambda)$.

In this paper we solve the spectrum problems related to above definitions, leaving a few open cases in the case of the metamorphosis of a $S_\lambda(2,4,n)$ into a $MPT(n,\lambda)$.

2 Metamorphosis of a $S_\lambda(2,4,n)$ into a minimum $S_\lambda(2,3,v)$

In Table 2 we show the sets $B(k,\lambda)$ of integers n for which there exists a $S_\lambda(2,k,n)$ for $k=3,4$ [11].

Table 2		
$\lambda \pmod 6$	$B(4,\lambda)$	$B(3,\lambda)$
0	$n\geq 4$	$n\geq 3$
1, 5	$n\equiv 1,4 \pmod{12}$	$n\equiv 1,3 \pmod 6$
2, 4	$n\equiv 1 \pmod 3$	$n\equiv 0,1 \pmod 3$
3	$n\equiv 0,1 \pmod 4$	$n\equiv 1 \pmod 2$

Pairing Tables 1 and 2, we get the necessary conditions for the existence of a $S_\lambda(2,4,n)$ having a metamorphosis into a minimum $S_\lambda(2,3,v)$ (see Table 3). The sufficiency for $v=n$ is proved in [6]. In this section we prove the sufficiency for $v>n$.

Table 3		
$\lambda \pmod 6$	n	$v-n$
0	$n\geq 4$	0
1, 5	1 (mod 12)	0
1, 5	4 (mod 12)	3
2, 4	1 (mod 3)	0
3	1 (mod 4)	0
3	0 (mod 4)	1

A (K,λ)-GDD (group divisible design of index λ, block sizes in K and order v) is a triple $(V,\mathcal{G},\mathcal{B})$, where V is a v-set, $\mathcal{G}=\{G_1,G_2,\ldots,G_m\}$ is a partition of V into subsets (called groups), and $\mathcal{B}$ is a collection of subsets (blocks) of V which satisfy the properties:

1. If $B\in\mathcal{B}$ then $|B|\in K$.
2. Every pair of distinct elements of V occurs in exactly λ blocks or one group, but not both.
3. $|\mathcal{G}|>1$.

We say that the (K,λ)-GDD is of type $v_1^{h_1}v_2^{h_2}\ldots v_t^{h_t}$, if there are h_i groups of size v_i, $i=1,2,\ldots,t$. We write (k,λ)-GDD instead of $(\{k\},\lambda)$-GDD.

Let $(V,\mathcal{G},\mathcal{B})$ be a $(4,\lambda)$-GDD. If a star is removed from each block of $\mathcal{B}$ the resulting collection of triangles $P(\mathcal{B})$ is a partial $(3,\lambda)$-GDD $(V,\mathcal{G},P(\mathcal{B}))$. If the edges belonging to the deleted stars can be arranged into a collection of triangles $T(\mathcal{B})$, then $(V,\mathcal{G},P(\mathcal{B})\cup T(\mathcal{B}))$ is a $(3,\lambda)$-GDD, called a metamorphosis of the $(4,\lambda)$-GDD $(V,\mathcal{G},\mathcal{B})$.

The following result is given by Lindner and Rosa [6].

Lemma 2.1. *For every integer $h\geq 5$, there is a $(4,1)$-GDD of type 12^h having a metamorphosis into a $(3,1)$-GDD of type 12^h.*

Obviously, only the cases $\lambda=1,3$ must be considered. Starting cases are collected in the following lemma. See [9] for a proof.

Lemma 2.2. *1. A $S(2,4,n)$ having a metamorphosis into a $S(2,3,n+3)$ exists for $n=4,16,28,40,52$.*

2. A $S_3(2,4,n)$ having a metamorphosis into a $S_3(2,3,n+1)$ exists for $n=4,8,12,16,28,32$.

3. There exists a $S(2,4,16)$ with one hole H of size 4 having a metamorphosis into a partial $S(2,3,16)$ whose leave is given by three 1-factors on vertex set $X\setminus H$.

Theorem 2.3. *A $S(2,4,n)$ having a metamorphosis into a $S(2,3,n+3)$ exists for every integer $n\equiv 4 \pmod{12}$, $n\geq 4$.*

Proof For $n=4,16,28,40,52$, see Lemma 2.2. Let $n=4+12h$, $h\geq 5$. By Lemma 2.1, there is a $(4,1)$-GDD of type 12^h having a metamorphosis into a $(3,1)$-GDD of type 12^h. Denote the groups by G_i, $i=1,2,\ldots,h$. Let $H=\{\infty_1,\infty_2,\infty_3,\infty_4\}$. Produce a $S(2,4,16)$ on vertex set $G_1\cup H$ having a metamorphosis into a $S(2,3,19)$ on vertex set $G_1\cup H\cup\{a_1,a_2,a_3\}$. For every $i=2,3,\ldots,h$, produce a copy of the $S(2,4,16)$, given in 3 of Lemma

2.2, on vertex set $G_i \cup H$ and hole H. This design has a metamorphosis into a partial $S(2,3,16)$ with leave F_j^i, $j = 1,2,3$. Form the triples $\{a_j, x, y\}$, $\{x,y\} \in F_j^i$. □

Theorem 2.4. *A $S_3(2,4,n)$ having a metamorphosis into a $S_3(2,3,n+1)$ exists for every integer $n \equiv 0 \pmod 4$, $n \geq 4$.*

Proof For $n = 4, 8, 12, 16, 28, 32$, see Lemma 2.2. A PBD of order m with block sizes 5, 9 and 13 exists for all $m \equiv 1 \pmod 4$ except for $m = 17, 29, 33$ [11]. Remove a point to obtain a $(\{5,9,13\},1)$-GDD of order $m-1$ with groups whose sizes lie in $\{4,8,12\}$. Place a solution on each block (see [6]) and on each group.

3 Metamorphosis of a $S_\lambda(2,4,n)$ into a $MPT(n,\lambda)$

Let $(X, \mathcal{C})$ be a $MPT(n,\lambda)$ with leave $(X, \mathcal{E})$. If $\mathcal{E} = \emptyset$, then $(X, \mathcal{C})$ is a $S_\lambda(2,3,n)$ and Lindner and Rosa's metamorphosis works. So we have to find a solution for $n \equiv 4 \pmod{12}$, if $\lambda \equiv 1,5 \pmod 6$, and for $n \equiv 0 \pmod 4$, if $\lambda \equiv 3 \pmod 6$. So, only $\lambda = 1,3$ must be considered. Note that there are different graphs which can be leaves of a $MPT(n,3)$, $n \equiv 8 \pmod{12}$ [10]. In this paper we don't consider all possible leaves but only one, as shown in Table 4.

Table 4		
λ	n	leave
$\equiv 1,5 \pmod 6$	$\equiv 4 \pmod{12}$	$1FY$
$\equiv 3 \pmod 6$	$\equiv 0 \pmod{12}$	$1F$
$\equiv 3 \pmod 6$	$\equiv 4 \pmod{12}$	$1FY$
$\equiv 3 \pmod 6$	$\equiv 8 \pmod{12}$	$1F_3$

Here $1F$, $1FY$ and $1F_3$ are the following graphs.

$1F$ — a matching on n vertices;

$1FY$ — a tripole (matching on $n-4$ vertices and a tree on 4 vertices with one vertex of degree 3);

$1F_3$ — a matching on $n-2$ vertices and a triple edge $\{a,b\}$, $\{a,b\}$, $\{a,b\}$.

Starting cases are collected in the following lemma. See [9] for a proof.

Lemma 3.1. *1. There exists a $S(2,4,n)$ having a metamorphosis into a $MPT(16,1)$.*

2. *There exists a $S(2,4,16)$ $(X,\mathcal{B})$ with one hole H of size 4 having a metamorphosis into a partial $S(2,3,16)$ $(X,\mathcal{C})$ whose leave is one 1-factor on vertex set $X \setminus H$.*

3. *A $S_3(2,4,n)$ having a metamorphosis into a $MPT(n,3)$ (with leave shown in Table 4) there is for $n = 8, 12, 20, 24, 32$.*

4. *There esists a $(4,1)$-GDD of type 4^4 having a metamorphosis into a $(3,1)$-GDD of type 4^4.*

5. *There exists a $S_3(2,4,32)$ $(X,\mathcal{B})$ such that: (i) $(X,\mathcal{B})$ embeds a $S_3(2,4,8)$ $(A,\mathcal{A})$ having a metamorphosis into a $MPT(8,3)$ $(A, P(\mathcal{A}) \cup T(\mathcal{A}))$; (ii) $(X,\mathcal{B})$ has a metamorphosis into a $MPT(32,3)$ $(X, P(\mathcal{B}) \cup T(\mathcal{B}))$; (iii) $(A, P(\mathcal{A}) \cup T(\mathcal{A}))$ is embedded into $(X, P(\mathcal{B}) \cup T(\mathcal{B}))$; (iv) the leave of $(A, P(\mathcal{A}) \cup T(\mathcal{A}))$ is a subgraph of the leave of $(X, P(\mathcal{B}) \cup T(\mathcal{B}))$.*

Theorem 3.2. *A $S(2,4,n)$ having a metamorphosis into a $MPT(n,1)$ exists for every integer $n \equiv 4 \pmod{12}$, $n \geq 4$, except possibly for $n = 28, 40, 52$.*

Proof The proof for $n = 4$ is trivial. For $n = 16$, Lemma 3.1 gives a $S(2,4,16)$ having a metamorphosis into a $MPT(16,1)$ which embeds a $MPT(4,1)$. Let $n = 4 + 12h$, $h \geq 5$. By Lemma 2.1, there is a $(4,1)$-GDD of type 12^h having a metamorphosis into a $(3,1)$-GDD of type 12^h. Denote the groups by G_i, $i = 1, 2, \ldots, h$. Let $H = \{\infty_1, \infty_2, \infty_3, \infty_4\}$. As in Lemma 3.1, produce a $S(2,4,16)$ on vertex set $G_1 \cup H$. For $i = 2, 3, \ldots, h$, produce a copy of the $S(2,4,16)$ on vertex set $G_i \cup H$, having the hole H. □

Theorem 3.3. *A $S_3(2,4,n)$ having a metamorphosis into a $MPT(n,3)$ (with leave shown in Table 4) exists for every integer $n \equiv 0 \pmod 4$, $n \geq 4$, except possibly for $n = 28, 36, 40, 44, 48, 52, 56, 68, 80, 92, 104$.*

Proof For $n = 8, 12, 20, 24, 32$, see Lemma 3.1. For $n \equiv 0 \pmod{12}$, $n \geq 60$, place a copy of the $S_3(2,4,12)$, given in Lemma 3.1, into each group of the $(4,1)$-GDD of Lemma 2.1.

For $n \equiv 4 \pmod{12}$, $n \geq 4$ and $n \neq 28, 40, 52$, paste together a solution of $\lambda = 1$ (Theorem 3.2) and a solution of $\lambda = 2$ [6].

Let $n \equiv 8 \pmod{12}$, $n \geq 116$. The Handbook of Combinatorial Designs [11] gives a $(4,1)$-GDD of type 6^u for every $u \geq 5$, and of type $6^u 3$ for every $u \geq 4$. Giving weight 4 to all points, we get a $(4,1)$-GDD $(X, \mathcal{G}_1, \mathcal{B}_1)$ of type 24^u and a $(4,1)$-GDD $(X, \mathcal{G}_2, \mathcal{B}_2)$ of type $24^u 12$, respectively. Let $A = \{a_1, a_2, \ldots, a_8\}$. If $n \equiv 8 \pmod{24}$, then

- On each block of $\mathcal{B}_1$ place a copy of the $(4,1)$-GDD given in 4 of Lemma 3.1.

- For each group $G \in \mathcal{G}_1$, produce a copy of the $S_3(2,4,32)$, given in 5 of Lemma 3.1, having vertex set $G \cup A$ and hole A.
- On the hole A place a $S_3(2,4,8)$ having metamorphosis into a $MPT(8,3)$.

If $n \equiv 20 \pmod{24}$, then

- On each block of $\mathcal{B}_2$ place a copy of the $(4,1)$-GDD given in 4 of Lemma 3.1.
- For each group $G \in \mathcal{G}_2$ such that $|G| = 24$, produce a copy of the $S_3(2,4,32)$, given in 5 of Lemma 3.1, having vertex set $G \cup A$ and hole A.
- On the group of size 20 place a $S_3(2,4,20)$ having metamorphosis into a $MPT(20,3)$. □

4 Open Questions and Remarks

1. Remove the possible exceptions in Theorems 3.2 and 3.3.
2. For $\lambda = 3$ and $n \equiv 8 \pmod{12}$, find a metamorphosis of a $S_\lambda(2,4,n)$ into a $MPT(n,\lambda)$ with any possible leave [10].
3. Let G_1 be a subgraph of G. Then Lindner and Rosa's metamorphosis can be easily generalized in the following way. Let $(X,\mathcal{B})$ be a G-decomposition of the multigraph λK_n [11]. If a graph isomorphic to $G \setminus G_1$ is removed from each G-block of $\mathcal{B}$, the resulting collection of G_1-blocks $P(\mathcal{B})$ is a partial G_1-decomposition of λK_n $(X, P(\mathcal{B}))$. If the edges belonging to the deleted subgraphs can be arranged into a collection of G_1-blocks $T(\mathcal{B})$, then $(X, P(\mathcal{B}) \cup T(\mathcal{B}))$ is a G_1-decomposition of λK_n, called a metamorphosis of the G-decomposition $(X,\mathcal{B})$. The related spectrum problem has been solved for many pairs of graphs G and G_1 [1, 2, 3, 4, 5, 7, 8].

   Extend both metamorphosis definitions, given in this paper for $S_\lambda(2,4,n)$, to G-decompositions of λK_n and solve the related spectrum problems.
4. During the meeting ISGDA (Messina, October 2003) we learnt that the generalization of the metamorphosis definition given in Section 3 is not new. The problem of the metamorphosis of some graph designs of order v and index λ into a $MPT(v,\lambda)$ is considered by other authors and their papers are not published yet. But, as we know, no other paper studies the same problem of Section 3.

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A Partition Problem for Sets of Permutation Matrices

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1 Introduction Consider a set $\mathcal{P}$ of permutation matrices of order n. What is the smallest integer m such that $\mathcal{P}$ can be partitioned into subsets $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ such that

$$\sum\{P : P \in \mathcal{P}_i\}, \quad (i = 1, 2, \ldots, m)$$

are (0,1)-matrices? Let $G(\mathcal{P})$ be a graph with vertex set $\mathcal{P}$ with an edge joining two permutation matrices $P, Q \in \mathcal{P}$ provided P and Q have a 1 in common (that is, a 1 in the same position). The integer m equals the chromatic number $\chi(G(\mathcal{P}))$. Natural sets $\mathcal{P}$ of permutation matrices arise by choosing $A = [a_{ij}]$ to be a $(0,1)$-matrix and

$$\mathcal{P} = \mathcal{P}_A = \{P : P \leq A, P \text{ is a permutation matrix}\}. \tag{1}$$

(Here the inequality $P \leq A$ is interpreted entrywise.) In this case the sets $\mathcal{P}_i$ in the partition must satisfy

$$\sum\{P : P \in \mathcal{P}_i\} \leq A.$$

A more restrictive problem requires that

$$\sum\{P : P \in \mathcal{P}_i\} = A \quad (i = 1, 2, \ldots, m). \tag{2}$$

If (2) holds, then

$$\sum\{P : P \in \mathcal{P}_A\} = mA,$$

and we say that $\mathcal{P}_A$ has a *perfect partition*. The cardinality of the set $\mathcal{P}_A$ equals the *permanent* of A defined, as usual, by:

$$\operatorname{per}(A) = \sum_{(i_1, i_2, \ldots, i_n) \in \mathcal{S}_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n},$$

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where the summation is over the symmetric group $\mathcal{S}_n$ of all permutations of $\{1, 2, \ldots, n\}$.

Suppose that $\mathcal{P}_A$ has a perfect partition. Then there are two consequences for the structure of A. First, there is an integer k such that all row and column sums of A equal k, and this integer k satisfies the equation $\mathrm{per}(A) = mk$. Second, a perfect partition implies that each 1 of A belongs to m permutation matrices $P \leq A$, and hence, where $A(i,j)$ denotes the submatrix of A obtained by deleting row i and column j, that

$$\mathrm{per} A(i,j) = m \text{ if } a_{ij} = 1,$$

that is, the permanental minors of the 1's of A all equal the same constant m.

Let $G_A = G(\mathcal{P}_A)$. Since the chromatic number of G_A equals the minimal number of independent sets into which $\mathcal{P}_A$ can be partitioned, we have

$$\chi(G_A) \geq \frac{\mathrm{per}(A)}{\alpha(G_A)}, \tag{3}$$

where $\alpha(G_A)$ is the maximal size of an independent set of G_A. We can have equality in (3) only if $\alpha(G_A) | \mathrm{per}(A)$. If $\mathcal{P}_A$ has a *perfect partition*, then the integer m in (2) equals $\chi(G_A)$. Since $\chi(G_A)$ is an integer, (3) implies that

$$\chi(G_A) \geq \left\lceil \frac{\mathrm{per}(A)}{\alpha(G_A)} \right\rceil. \tag{4}$$

By a theorem of Folkman and Fulkerson [2] (see also Theorem 6.4.3 in [1]), the independence number $\alpha(G_A)$ equals

$$\min \left\{ \frac{\mathrm{sum}(A_{kl})}{k+l-n} : k+l > n \right\}$$

where the minimum is taken over all pairs of integers k and l with $n < k+l \leq 2n$ and $k \times l$ submatrices A_{kl} of A, and $\mathrm{sum}(A_{kl})$ is the sum of the entries of A_{kl}.

There is a geometrical interpretation of the perfect partition problem. Recall that a necessary condition for the existence of a perfect partition for $\mathcal{P}_A$ is that the sum of matrices in $\mathcal{P}_A$ is a multiple of A. Thus, the average of $\mathcal{P}_A$, which can also be viewed as the centroid of the convex hull of $\mathcal{P}_A$, has the form γA. Clearly, every element in $\mathcal{P}_A$ is an extreme point of the convex hull of $\mathcal{P}_A$. (To see this, note that every element X in $\mathcal{P}_A$ has the same Frobenius norm (trace $XX^t)^{1/2}$ and therefore cannot be written as a convex combination of the others.) If A has row sums and column sums all equal to k, then one needs at least k elements in $\mathcal{P}_A$ whose average (regarded as the centroid of the convex hull of the k elements) is equal to γA; if the desired partition is a partition of $\mathcal{P}_A$ in k-element sets, then each of them has the same average as that of $\mathcal{P}_A$.

In the subsequent discussion, let J_n be the $n \times n$ matrix of all 1's. In the next section we consider perfect partitions of $\mathcal{S}_n = \mathcal{P}_{J_n}$ (where we now regard $\mathcal{S}_n$ as the set of $n \times n$ permutation matrices) and the alternating group $\mathcal{A}_n$ of all $n \times n$ even permutation matrices (permutation matrices with determinant equal to 1). In Section 3, we consider the set $\mathcal{D}_n = \mathcal{P}_{J_n - I_n}$ of $n \times n$ *derangement permutation matrices*; we present some partial results and open problems. Additional open questions are discussed in the final section.

2 Partitioning $\mathcal{S}_n$ and $\mathcal{A}_n$ We have $\alpha(J_n) = n$ and $\mathrm{per}(J_n) = n!$, and it is easy to show that $\sum_{X \in \mathcal{S}_n} X = (n-1)! J_n$. Can we partition $\mathcal{S}_n$ into $(n-1)!$ subsets so that the sum of the matrices in each subset is J_n? The answer is affirmative.

Proposition 2.1 *The set $\mathcal{S}_n = \mathcal{P}_{J_n}$ is a disjoint union of $(n-1)!$ subsets such that the sum of the matrices in each subset is J_n. Hence $\mathcal{S}_n$ has a perfect partition.*

Proof. Let $H = \{I_n, P, \ldots, P^{n-1}\}$ where P is the basic $n \times n$ circulant matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{5}$$

Then H is a cyclic group with n elements whose sum is the matrix J_n. There are $(n-1)!$ cosets of H in $\mathcal{S}_n$. Each coset has the form $QH = \{QP^j : j = 0, \ldots, n-1\}$ for some $Q \in \mathcal{S}_n$. Clearly, the sum of the matrices in each coset is also the matrix J_n. □

Now we consider the group $\mathcal{A}_n$ of even permutation matrices. We have $|\mathcal{A}_n| = n!/2$, and it is not hard to show that $\sum_{X \in \mathcal{A}_n} X = [(n-1)!/2] J_n$ if $n \geq 3$. Can we partition $\mathcal{A}_n$ into $(n-1)!/2$ subsets so that the sum of the matrices in each subset is J_n? We have the following result.

Proposition 2.2 *Suppose $n \geq 3$. The set $\mathcal{A}_n$ can be partitioned into $(n-1)!/2$ subsets so that the sum of the matrices in each subset is J_n.*

Proof. We consider three cases according to n.

Case 1. If $n \geq 3$ is odd, then the basic circulant matrix P is in $\mathcal{A}_n$. Thus $H = \langle P \rangle$ is a subgroup of $\mathcal{A}_n$ with $(n-1)!/2$ cosets, and the sum of the matrices in each coset is J_n.

Case 2. If $n = 4k$ for some positive integer k, we can prove by induction that:

There is a subgroup H in $\mathcal{A}_n$ with n elements whose sum equals J_n, and hence the cosets of the group H will be a desired partition.

When $k = 1$, let H_4 be the subgroup of $\mathcal{A}_4$ containing all the elements of order 2 or 0 (H_4 is the 2-Sylow subgroup of $\mathcal{A}_4$). One can readily check that the the sum of the matrices in H_4 sum up to J_4.

Now, suppose the result is true for $n = 4k$ for some $k \geq 1$. Consider the case when $n = 4(k+1)$. By the induction assumption, there is a group H_{4k} of $\mathcal{A}_{4k}$ such that the sum of the matrices in H_{4k} is J_{4k}. Let $H = \{A \otimes B : A \in H_4,\ B \in H_{4k}\}$, where $X \otimes Y = (x_{ij}Y)$ denotes the usual tensor product of two matrices. Then H is a subgroup of $\mathcal{A}_n$ with $n = 4(k+1)$ elements whose sum is the matrix J_n. By induction, our claim is proved.

Case 3. Let $n = 2m$ for some odd integer m. We consider the subgroup K of $\mathcal{A}_n$ consisting of matrices of the form $A \oplus B$, where A and B are $m \times m$ permutation matrices. There are $(m!)^2/2$ such matrices. To see this, if we allow A and B to be arbitrary matrices in $\mathcal{S}_m$, there will be $(m!)^2$ such matrices in $\mathcal{S}_n$. Since half of them are odd permutations, we see that K has $(m!)^2/2$ elements as asserted.

We claim that K can be partitioned into $m((m-1)!)^2/2$ subsets such that each subset has m elements summing up to $J_m \oplus J_m$. To this end, let $P \in \mathcal{S}_m$ be the basic circulant. Let $G = \langle P \rangle$, and let $Q_1G, \ldots, Q_rG$ be the cosets of G in $\mathcal{S}_m$, where $r = (m-1)!$, $Q_1, \ldots, Q_{r/2} \in \mathcal{A}_m$ and $Q_j \notin \mathcal{A}_m$ for $j > r/2$.

For each $i, j = 1, \ldots, r/2$, consider the following m-element subsets of $\mathcal{A}_n$:

$$\mathcal{S}_{ij1} = \{(Q_i \oplus Q_j)(P \oplus P)^k : k = 0, \ldots, m-1\},$$

$$\mathcal{S}_{ij2} = \{X(I_m \oplus P) : X \in \mathcal{S}_{ij1}\}, \quad \mathcal{S}_{ij3} = \{X(I_m \oplus P^2) : X \in \mathcal{S}_{ij1}\}, \quad \ldots,$$

$$\ldots, \quad \mathcal{S}_{ijm} = \{X(I_m \oplus P^{m-1}) : X \in \mathcal{S}_{ij1}\}.$$

We get $m(r/2)^2$ disjoint m-element subsets of K.

Next, for each $i, j = r/2+1, \ldots, r$, consider

$$\mathcal{S}_{ij1} = \{(Q_i \oplus Q_j)(P \oplus P)^k : k = 0, \ldots, m-1\},$$

$$\mathcal{S}_{ij2} = \{X(I_m \oplus P) : X \in \mathcal{S}_{ij1}\}, \quad \mathcal{S}_{ij3} = \{X(I_m \oplus P^2) : X \in \mathcal{S}_{ij1}\}, \quad \ldots,$$

$$\ldots, \quad \mathcal{S}_{ijm} = \{X(I_m \oplus P^{m-1}) : X \in \mathcal{S}_{ij1}\}.$$

We get another $m(r/2)^2$ disjoint m-element subsets of K.

Consequently, we get $mr^2/2 = m((m-1)!)^2/2$ disjoint m-element subsets of K. Moreover, the matrices in each subset sum up to $J_m \oplus J_m$ as desired.

Now, consider the matrix R obtained by switching the first two rows of $\begin{pmatrix} 0_m & I_m \\ I_m & 0_m \end{pmatrix}$. Then $R \in \mathcal{A}_m$. Let

$$H = K \cup \{RX : X \in K\}.$$

One easily checks that H is the subgroup of $\mathcal{A}_n$ consisting of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}.$$

Moreover, for each set $\mathcal{S}_{ijk}$ defined above, we may construct

$$T_{ijk} = \mathcal{S}_{ijk} \cup \{RX : X \in \mathcal{S}_{ijk}\}.$$

Then each T_{ijk} will have $n = 2m$ elements summing up to J_n, and these T_{ijk} form a partition of the subgroup H.

Now, let $H, W_1H, W_2H, \ldots, W_tH$ be the cosets of H in $\mathcal{A}_n$, where $t+1 = |\mathcal{A}_n|/|H|$. Each coset W_sH is a disjoint union of W_sT_{ijk}'s, and each W_sT_{ijk} has n elements summing up to J_n. □

Corollary 2.3 *The set S_n has a perfect partition in which each part of the partition consists of all even permutation matrices or all odd permutation matrices.*

Proof. As in the proof of Proposition 2.1, the coset of odd permutations also can be partitioned into sets summing to J_n. □

3 Partitioning $\mathcal{D}_n = \mathcal{P}_{J_n - I_n}$: Partial Result Let $L_n = J_n - I_n$. For $n = 2, \ldots, 5$ we show that $\mathcal{D}_n = \mathcal{P}_{L_n}$ can be partitioned into subsets each with $n-1$ matrices that sum to L_n.

In the following discussion, we identify a permutation σ in disjoint cycle representation with the corresponding permutation matrix in $\mathcal{S}_n$. For example, $(1,2)(3,4)$ represents the permutation obtained from the identity matrix by interchanging the first and second rows, and also the third and fourth rows. Then $\sigma \in \mathcal{S}_n$ is a derangement if and only if $\sigma(i) \neq i$ for $i = 1, \ldots, n$. Moreover, the elements in a set of derangements $\{\sigma_1, \ldots, \sigma_{n-1}\} \subseteq \mathcal{D}_n$ sum to L_n if and only if $\sigma_r(i) \neq \sigma_s(i)$ for $r \neq s$ and for all $i = 1, \ldots, n$. We have the following partial result for the partition problem of $\mathcal{P}_{L_n}$.

Proposition 3.1 *The set $\mathcal{D}_n$ has a perfect partition if $n \leq 5$.*

Proof. If $n = 2$, then $\mathcal{D}_n = \{L_n\}$ is a singleton. If $n = 3$, then the members of $\mathcal{D}_n = \{(1,2,3), (1,3,2)\}$ sum to L_n.

For $n = 4$, a permutation belongs to $\mathcal{D}_n$ if and only if it is a 4-cycle or a product of two disjoint transpositions. If

$$F_1 = \{(1,2)(3,4), (1,3,2,4), (1,4,2,3)\},$$

$$F_2 = \{(1,3)(2,4), (1,2,3,4), (1,4,3,2)\},$$

$$F_3 = \{(1,4)(2,3), (1,2,4,3), (1,3,4,2)\},$$

then $\mathcal{D}_4 = \cup_{k=1}^{3} F_k$ and the members of each F_k sum to L_4.

For $n = 5$, a permutation belongs to $\mathcal{D}_n$ if and only if it is of the form $(i_1, i_2, i_3, i_4, i_5)$ or $(i_1, i_2)(i_3, i_4, i_5)$. Let $\mathcal{D}_5' \subset \mathcal{D}_5$ be the set of derangements of the form $(i_1, i_2, i_3, i_4, i_5)$ and let $\mathcal{D}_5'' \subset \mathcal{D}_5$ be the set of derangements of the form $(i_1, i_2)(i_3, i_4, i_5)$. Observe that $|\mathcal{D}_5'| = 24$ and $|\mathcal{D}_5''| = 20$. We show that $\mathcal{D}_5'$ and $\mathcal{D}_5''$ can be partitioned into 6 and 5 subsets, respectively, such that the members of each subset sum to L_5.

Let $\tau_1 = (1,2,3,4,5)$, $\tau_2 = (1,2,3,5,4)$, $\tau_3 = (1,2,4,3,5)$, $\tau_4 = (1,2,4,5,3)$, $\tau_5 = (1,2,5,3,4)$, and $\tau_6 = (1,2,5,4,3)$. If $T_k = \{\tau_k, \tau_k^2, \tau_k^3, \tau_k^4\}$, then the collection of subsets $T_1, \ldots, T_6$ forms a partition of $\mathcal{D}_5'$ such that the members of each T_k sum to L_5. Now, consider the following subsets of $\mathcal{D}_5''$:

$$R_1 = \{(1,2)(3,4,5), (1,3)(2,5,4), (1,4)(2,3,5), (1,5)(2,4,3)\}$$

$$R_2 = \{(2,1)(3,5,4), (2,3)(1,4,5), (2,4)(1,5,3), (2,5)(1,3,4)\}$$

$$R_3 = \{(3,1)(2,4,5), (3,2)(1,5,4), (3,4)(1,2,5), (3,5)(1,4,2)\}$$

$$R_4 = \{(4,1)(2,5,3), (4,2)(1,3,5), (4,3)(1,5,2), (4,5)(1,2,3)\}$$

$$R_5 = \{(5,1)(2,3,4), (5,2)(1,4,3), (5,3)(1,2,4), (5,4)(1,3,2)\}.$$

Then the collection of subsets $R_1, \ldots, R_5$ forms a partition of $\mathcal{D}''_5$ such that the members of each R_k sum to L_5. This completes the partition of $\mathcal{D}_5$. □

The problem of partitioning $\mathcal{D}_n$ with $n \geq 6$ is more difficult. In the following, we describe several different approaches we considered.

First, we divide the set $\mathcal{D}_n$ into subsets according to different cycle decompositions, and we attempt to show that each of these subsets admits a partition into $(n-1)$-element subsets with elements summing to L_n. In particular, when $n = 5$, the partition was done in this way. When we apply this idea to $\mathcal{D}_6$, we get the following subsets:

T_1: the set of length-6 cycles – 120 elements;
T_2: the set of permutations obtained by the product of a 2-cycle and a 4-cycle – 90 elements;
T_3: the set of permutations obtained by the product of two 3-cycles – 40 elements;
T_4: the set of permutations obtained by the product of three 2-cycles – 15 elements.

For each subset, the sum of its elements (say, denoted by X) will be a multiple of L_n because all of the diagonal entries of X are zeroes and $PXP^t = X$ for every permutation matrix P. However, this approach to partitioning $\mathcal{D}_n$ fails when $n = 6$. One can check that the set T_4 cannot be partitioned into three 5-element subsets such that the elements in each subset sum up to L_6.

An alternative idea is to select 15 elements $\tau_1, \ldots, \tau_{15}$ from T_1 and construct disjoint subsets

$$U_i = \{\tau_i^j : j = 1, \ldots, 5\}$$

so that each of them has elements summing up to L_6. Note that each U_i will have two elements in T_1, two elements in T_3, and one element in T_4. If this is done, then we are left with 90 elements in T_1, the entire set T_2, and 10 elements in T_3.

Another scheme is to select one element in T_4 and four elements in T_1 of the form $\tau_1, \tau_1^{-1}, \tau_2, \tau_2^{-1}$ to form a set whose elements sum up to L_6. Here is an example:

$$(1,2)(3,4)(5,6), (1,3,5,2,6,4), (1,4,6,2,5,3), (1,6,3,2,4,5), (1,5,4,2,3,6).$$

In fact, one can construct 15 sets of such form and use up the 15 elements in T_4 together with 60 elements in T_1.

It is also possible to use two elements in T_3 and three elements in T_1 to form a set whose elements sum up to L_6. Here is an example:

$$(1,2,3)(4,5,6), (1,3,2)(4,6,5), (1,4,2,5,3,6), (1,5,2,6,3,4), (1,6,2,4,3,5).$$

One can actually construct 20 subsets of this form and use up the 40 elements in T_3 together with 60 elements in T_1.

One may want to use the two schemes in the last two paragraphs to exhaust the elements in T_1, T_3, and T_4, but this strategy seems to be impossible. Of course, even if it can be done, one must still partition the elements in T_2 into 18 sets, each of which has elements summing up to L_6. Here is an example of such a set:

$$(1,2)(3,4,5,6), (1,3)(2,4,6,5), (1,4)(2,5,3,6), (1,5)(2,6,4,3), (1,6)(2,3,5,4).$$

It is unclear whether one can construct 18 disjoint subsets of T_2 with the desired property.

Thus, the problem of finding a perfect partition for $\mathcal{P}_{L_n}$ seems difficult. We close this section with a statement of the problem and some related questions:

Problem 3.2 For $n \geq 6$, is there a perfect partition for $\mathcal{P}_{L_n}$ or $\mathcal{P}_{L_n} \cap \mathcal{A}_n$?

Problem 3.3 For $n \geq 6$, is there a perfect partition for the collection of permutations in $\mathcal{P}_{L_n}$ with some specific cycle decomposition?

For example, can the set of permutations obtained by the product of a 2-cycle and an $(n-2)$-cycle be partitioned into subsets such that the elements of each subset sum up to L_n? The answer is no for $n = 4$, yes for $n = 5$, and unknown for $n \geq 6$.

4 Additional Problems We continue to use P to denote the basic circulant as defined in (5). Note that $J_n = \sum_{k=0}^{n-1} P^k$ and $L_n = \sum_{k=1}^{n-1} P^k$. For any subsets $K \subseteq \{0, 1, \ldots, n-1\}$, let

$$P_K = \sum_{k \in K} P^k.$$

A general question is:

Problem 4.1 Determine $K \subseteq \{0, 1, \ldots, n-1\}$ so that $\mathcal{P}_{P_K}$ (respectively, $\mathcal{P}_{P_K} \cap \mathcal{A}_n$) has a perfect partition.

By the results in the previous sections, we see that both problems in Problem 4.1 have affirmative answers if $|K| = n$. If $|K| = 1$, then both problems also have affirmative answers trivially. If $|K| = 2$, then we have the following proposition.

Proposition 4.2 *Let $A = I_n + P^k$ with $0 < k < n$. Then $\mathcal{P}_A$ admits a perfect partition.*

Proof. Write P^k in disjoint cycle notation. There are two cases.

Case 1. If (n, k) is relatively prime, then P^k is just one long cycle, and I_n and P^k are the only two elements in $\mathcal{P}_A$, which admits a trivial perfect partition.

Case 2. If $d > 1$ is the greatest common divisor of n and k, and $m = n/d$, then P^k is the product of d cycles of length m. Now, we can rewrite $A = I_n + P^k$ as the direct sum of d $m \times m$ matrices, each of which is $I_m + Q$, where Q is the $m \times m$ basic circulant. In this form, one readily checks that $X \in \mathcal{P}_A$ if and only if $X = X_1 \oplus \cdots \oplus X_d$ such that $X_j \in \{I_m, Q\}$. Thus, there are 2^d matrices in $\mathcal{P}_A$. Moreover, $\mathcal{P}_A$ has a perfect partition consisting of sets of the form $\{X, A - X\}$ with $X \in \mathcal{P}_A$. □

If $|K| = n - 1$, we basically have the $\mathcal{P}_{L_n}$ problem, and we only have partial results. If $|K| = 3$, even the necessary condition for a perfect partition may not hold. Here is an example which can be verified readily.

Example 4.3 *For $n = 5$ there are 13 matrices in $\mathcal{P}_A$ for $A = I_n + P + P^2$ or $A = I_n + P^2 + P^3$. In either case, a perfect partition is impossible.*

Note that in general, if $|K| = n-2$, then $P_K = J_n - P^r - P^s$. Replacing P_K by $P^j P_K$ for a suitable $j \in \{0, \ldots, n-1\}$, we may assume that $(r, s) = (-l, l)$ with $1 \leq l \leq n/2$, or $(r, s) = (0, 1)$. For example, for $n = 5$, we only need to consider the cases in Example 4.3.

If n is even and $(r, s) = (-l, l)$ with $1 \leq l \leq n/2$, then up to a permutation equivalence, i.e., replace A by RAS for some suitable $R, S \in \mathcal{S}_n$, we can assume that $P_K = J_n - (I_{n/2} \otimes J_2)$, which can be viewed as a generalization of L_n. In general, if $n = km$, we consider $L_{n,k} = J_n - (I_m \otimes J_k)$. We have the following.

Proposition 4.4 *Suppose $n \geq 4$ and $n = km$. Then $\operatorname{per}(L_{n,k})$ is a multiple of $(n-k)$. Moreover, if $n > k \geq 2$, then $\left|\mathcal{P}_{L_{n,k}} \cap \mathcal{A}_n\right| = \operatorname{per}(L_{n,k})/2$ is also a multiple of $(n-k)$.*

Proof. Use Laplace expansion about the first row of $L_{n,k}$. Note that all of the submatrices of $L_{n,k}$ obtained by deleting the first row and jth column with $k < j \leq n$ are permutationally equivalent and have the same permanent, say, r. Thus, $\text{per}(L_{n,k}) = (n-k)r$.

Next, suppose $n > k \geq 2$. Then for each $\sigma \in \mathcal{P}_{L_{n,k}}$, we have $(1,2)\sigma \in \mathcal{P}_{L_{n,k}}$, and either σ or $(1,2)\sigma$ is an even permutation. Thus, half of the elements in $\mathcal{P}_{L_{n,k}}$ belong to $\mathcal{A}_n$. Next, consider the Laplace expansion of $\text{per}(L_{n,k})$ as in the first paragraph of the proof. We claim that r is even. To this end, suppose A is obtained from $L_{n,k}$ by deleting its first row and $(k+1)$st column. Note that for any permutation $\sigma \in \mathcal{P}_A$, we have $\sigma(1,2) \in \mathcal{P}_A$, and either σ or $\sigma(1,2)$ is an even permutation (in $\mathcal{S}_{n-1}$). Thus, $|\mathcal{P}_A| = r$ is even. Consequently, $|\mathcal{P}_{L_{n,k}} \cap \mathcal{A}_n| = \text{per}(L_{n,k})/2 = (n-k)(r/2)$ is also a multiple of $(n-k)$. □

Note that the second assertion of the above proposition is not valid for $\mathcal{P}_{L_n}$. As shown in Section 3, the number of even and odd permutations in $\mathcal{P}_{L_n}$ may be different:

n :	3	4	5	6
$\lvert\mathcal{P}_{L_n}\rvert$:	2	9	44	265
$\lvert\mathcal{P}_{L_n} \cap \mathcal{A}_n\rvert$:	2	3	24	130

Nevertheless, for $(n,k) = (3,1),(4,1),(5,1)$ it is not hard to find a perfect partition for $\mathcal{P}_{L_n} \cap \mathcal{A}_n$; see the results in the last section. In general, we have the following.

Problem 4.5 Determine whether there is a perfect partition for $\mathcal{P}_{L_{n,k}}$ (respectively, $\mathcal{P}_{L_{n,k}} \cap \mathcal{A}_n$).

Notice that finding a perfect partition for $\mathcal{P}_{L_{n,n/2}}$ is the same as finding a perfect partition for $\mathcal{P}_A$ with $A = J_{n/2} \oplus J_{n/2}$. Examining Case 3 in the proof of Proposition 2.2, we have the following.

Proposition 4.6 *Suppose n is even. There is always a perfect partition for $\mathcal{P}_{L_{n,n/2}}$ (respectively, $\mathcal{P}_{L_{n,n/2}} \cap \mathcal{A}_n$).*

Answering Problem 4.5 for other values of (n,k) is not so easy. For $(n,k) = (6,2)$, we have an affirmative answer.

Proposition 4.7 *There is a perfect partition for $\mathcal{P}_{L_{6,2}}$.*

Proof. Let $L_{6,2} = (A_{ij})_{1 \leq i,j \leq 3}$, where $A_{ii} = 0_2$ for $i = 1,2,3$, and $A_{ij} = J_2$ for $i \neq j$. We first show that $|\mathcal{P}_{L_{6,2}}| = 80$. Every permutation matrix in $\mathcal{P}_{L_{6,2}}$ is determined by selecting exactly one nonzero entry from each row and column of $L_{6,2}$. Consider the number of ways to construct a matrix $X \in \mathcal{P}_{L_{6,2}}$ if the 1 in the $(1,3)$ position of $L_{6,2}$ is selected to be in X. In the following discussion, a nonzero entry of $L_{6,2}$ is said to be *available* if no other nonzero entry in its row or column has been selected to be in X. We consider two cases, depending on the nonzero entry selected from the second row of $L_{6,2}$.

Case 1. If the $(2,4)$ entry of $L_{6,2}$ is selected to be in X, then the remaining four nonzero entries of X must be obtained by selecting two nonzero entries each from the A_{23} and A_{31} submatrices of $L_{6,2}$. The nonzero entries from each of these two submatrices can be selected in one of two ways: either entirely on the submatrix diagonal or entirely off of the submatrix diagonal. Thus, there are $2 \times 2 = 4$ possible ways to construct X in this case.

Case 2. If the $(2,4)$ entry of $L_{6,2}$ is not selected, then there are two available nonzero entries in the second row of $L_{6,2}$ that can be selected; both choices lie in A_{13}. Each choice sequentially forces the selection of one of two available nonzero entries each from the A_{23}, A_{21}, and A_{31} submatrices, thereby determining the final selection of the only available nonzero entry from the A_{32} submatrix. Thus, there are $2^4 = 16$ ways to construct X in this case.

Combining the two preceding cases, there are $4 + 16 = 20$ matrices in $\mathcal{P}_{L_{6,2}}$ with a 1 in the $(1,3)$ position. By analogous arguments, one can show that there are 20 matrices in $\mathcal{P}_{L_{6,2}}$ with a 1 in the $(1,k)$ position for $k = 4,5,6$. Thus, $|\mathcal{P}_{L_{6,2}}| = 4 \times 20 = 80$.

Next, we show that $P_{L_{6,2}}$ admits a perfect partition. Let $T_2 \in \mathcal{S}_2$ correspond to the permutation $(1,2)$, and let $R_3 \in \mathcal{S}_3$ correspond to the permutation $(1,3,2)$. Let

$$W = \{R_3 \otimes I_2, R_3 \otimes T_2, R_3^t \otimes I_2, R_3^t \otimes T_2\}.$$

Then W, $(3,4)W$, $(5,6)W$, and $(3,4)(5,6)W$ are disjoint subsets of $\mathcal{P}_{L_{6,2}}$ such that the matrices in each subset sum up to $L_{6,2}$, and each of the four subsets contains exactly one matrix from Case 1 above.

The remaining 64 matrices in $\mathcal{P}_{L_{6,2}}$ can be partitioned as follows. Recall that each of the 16 matrices $X_1, \ldots, X_{16}$ from Case 2 above has exactly one nonzero entry from every A_{ij} in $L_{6,2}$ with $i \neq j$. Now, we associate each matrix X_r from Case 2 with three other matrices $X_{r,2}$, $X_{r,3}$, and $X_{r,4}$ in $\mathcal{P}_{L_{6,2}}$ as determined in the following manner:

> $X_{r,2}$: From each nonzero A_{ij}, select the entry horizontally adjacent to the entry that was selected to be in X_r.

$X_{r,3}$: From each nonzero A_{ij}, select the entry vertically adjacent to the entry that was selected to be in X_r.

$X_{r,4}$: From each nonzero A_{ij}, select the entry diagonal to the entry that was selected to be in X_r.

Then we have 16 disjoint sets of the form $\{X_r, X_{r,2}, X_{r,3}, X_{r,4}\}$ such the matrices in each set sum up to $L_{6,2}$. For example, the following four matrices in $\mathcal{P}_{L_{6,2}}$ constitute a set in the partition:

$$X_r = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{r,2} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_{r,3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad X_{r,4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

None of the 64 matrices partitioned into sets of the form $\{X_r, X_{r,2}, X_{r,3}, X_{r,4}\}$ were previously used up in sets of the form σW, because all matrices belonging to sets of the form σW in the partition have either two or zero entries from each nonzero A_{ij} in $L_{6,2}$. We thus have a perfect partition of $\mathcal{P}_{L_{6,2}}$. □

We close the paper with some general questions.

Problem 4.8 For which $n \times n$ (0,1)-matrices A does $\mathcal{P}_A$ have a perfect partition?

Note that such matrices A must be regular, and if k is the constant row and column sum, k must be a factor of the permanent of A. In addition, the permanental minors of the 1's of A are constant.

Problem 4.9 Determine a good upper bound on the chromatic number $\chi(G_A)$ of the permutation graph of a regular matrix A. More specifically, find a constant c_n such that

$$\chi(G_A) \leq c_n \left\lceil \frac{\mathrm{per}(A)}{k} \right\rceil.$$

An even more general problem is the following.

Problem 4.10 Let $\mathcal{P} = \{P_i : i \in I\}$ be a set of permutation matrices of order n. Let $\mathcal{A}$ be a multiset of $(0, 1)$-matrices of order n. When is there a partition of I into sets $I_1, I_2, \ldots, I_m$ such that the matrices $\sum\{P_j : j \in I_i\}$, $(i = 1, 2, \ldots, m)$, are the matrices in $\mathcal{A}$, including multiplicities?

The problem discussed in this paper concerns sets of permutation matrices $\mathcal{P}_A$ where A is a $(0, 1)$-matrix and $\mathcal{A}$ is the multiset consisting of A with a certain multiplicity.

References

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