ORTHOGONAL LABELLINGS OF CATERPILLARS OF SMALL DIAMETER

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Abstract. An orthogonal double cover (ODC) $\mathcal{G}$ of the complete graph K_n is a collection of n spanning subgraphs $G_1, G_2, \ldots, G_n$ of K_n such that

- every edge of K_n belongs to exactly two of the G_i 's and
- every pair of G_i 's intersect in exactly one edge.

If $G_i \cong G$ for all $i \in \{1, 2, \ldots, n\}$, then $\mathcal{G}$ is an ODC of K_n by G. An ODC of K_n is *cyclic* if the cyclic group of order n is a subgroup of its automorphism group. In this paper, we obtain cyclic ODCs of K_n by caterpillars of small diameter.

1 Introduction

In this paper, we consider only finite simple graphs. Our notation and terminology are as in [1]. For an integer $n \geq 2$, let K_n be the complete graph with n vertices. A collection $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ of n spanning subgraphs of K_n is an *orthogonal double cover* (briefly ODC) of K_n if
(i) every edge of K_n belongs to exactly two members of $\mathcal{G}$ and
(ii) any two distinct subgraphs G_i, G_j have exactly one edge in common.
If $G_i \cong G$ for all $i \in \{1, 2, \ldots, n\}$, then $\mathcal{G}$ is an ODC of K_n by G.

If $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ is an ODC of K_n by G, then
$n \mid E(G) \mid = \mid E(G_1) \mid + \mid E(G_2) \mid + \ldots + \mid E(G_n) \mid = 2 \mid E(K_n) \mid = n(n-1)$
and hence G has $n-1$ edges.

Gronau, Mullin and Rosa conjectured the following:

Conjecture [3]. If T is an arbitrary tree with n vertices, $n \geq 2$, other than the path P_4 with 3 edges, then there exists an ODC of K_n by T.

An ODC of K_n is *cyclic* if the cyclic group of order n is a subgroup of its automorphism group.

Let $V(K_n) = \mathbb{Z}_n$ be the vertex set of K_n, where $\mathbb{Z}_n$ is the ring of integers *modulo* n. The *distance* of two vertices $x, y \in \mathbb{Z}_n$ is the circular-distance defined by $d(x,y) = min\{| x - y |, n - | x - y |\}$. The edge xy is said to have *length* $d(x,y)$. In [2], circular-distance is named as $Lee - distance$.

Given two edges $e_1 = u_1v_1$, $e_2 = u_2v_2$ of the same length ℓ, the $rotation - distance$ $r(\ell)$ between e_1 and e_2 is defined by $r(\ell) = min\{r_1, r_2 : (u_1 + r_1)(v_1 + r_1) = e_2, (u_2 + r_2)(v_2 + r_2) = e_1\}$.

Given a graph $G = (V, E)$ with n vertices and $n-1$ edges (isolated vertices are permitted), a labelling $\phi : V \to \mathbb{Z}_n$ is an *orthogonal labelling* of G if
(i) for every $k \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$, G contains exactly *two* edges of length k, and exactly *one* edge of length $\frac{n}{2}$ if n is even,
(ii) $\{r(k) : k \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}\} = \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$.

The following theorem of Gronau, Mullin and Rosa [3] relates cyclic ODCs and orthogonal labellings.

Theorem [3]. A cyclic ODC of K_n by a graph G exists if and only if there exists an orthogonal labelling of G.

Let $n_1, n_2, \ldots, n_r$, $r \geq 1$, be integers, where $n_1, n_r \geq 1$ and $n_i \geq 0$ for $i \in \{2, 3, \ldots, r-1\}$. The *caterpillar* $C(n_1, n_2, \ldots, n_r)$ is the tree obtained from the path $P_r := x_1 x_2 \ldots x_r$ by joining the vertex x_i to n_i new vertices $x_{i,1}, x_{i,2}, \ldots, x_{i,n_i}$ for each i. Clearly, $C(n_1, n_2, \ldots, n_r)$ is of diameter $r+1$.

In this paper, we concentrate on orthogonal labellings of caterpillars of small diameter. Also, in this paper, we define only the labelling and the verification for that to be an orthogonal labelling is omitted.

If T is a caterpillar of diameter 2, then $r = 1$ and hence it is a star. Any bijection $\phi : V(C(n_1)) \to \mathbb{Z}_{n_1+1}$ is an orthogonal labelling of $C(n_1)$. In [5], Leck and Leck showed that the caterpillar $C(n_1, n_2)$ of diameter 3 has an orthogonal labelling if and only if $n_1 + n_2 + 2$ and $n_1 n_2$ are not relatively prime. Gronau, Mullin and Rosa [3], have constructed ODCs (not necessarily cyclic) for all caterpillars of diameter 3, except P_4.

2 Caterpillars of diameter 4

Gronau, Mullin and Rosa [3], obtained an orthogonal labelling for the caterpillar $C(2k, 1, 2k)$, $(k > 0)$. They also observed that the caterpillars $C(2,2,2)$ and $C(2,3,2)$ have no orthogonal labelling. In [6], orthogonal labellings have been given for all $C(k, t, k+t+r)$ with $k > 0$, $t \geq 0$, $r \geq 0$. Also, in [6], Leck and Leck have constructed ODCs (not necessarily cyclic) for all caterpillars of diameter 4. Now we extend this known class of caterpillars.

Theorem 1. For all $k \geq 1$ and $t \geq 0$, the caterpillar $C(1, 2(k+t)+1, 2k)$

has an orthogonal labelling.

Proof. Define a labelling $\phi_1 : V(C(1, 2(k+t)+1, 2k)) \rightarrow \mathbb{Z}_{4k+2t+5}$ by $\phi_1(x_1) = 2k$, $\phi_1(x_2) = 0$, $\phi_1(x_3) = 4k+2t+4$, $\phi_1(x_{1,1}) = 4k+2t+3$,

$$\phi_1(x_{2,i}) = \begin{cases} i & \text{if } 1 \le i \le 2k-1, \\ i+1 & \text{if } 2k \le i \le 2k+2t+1 \end{cases}$$

and $\phi_1(x_{3,i}) = 2k+2t+2+i$ if $1 \le i \le 2k$. ■

Theorem 2. For any $k \ge 1$, the caterpillars $C(4, 4k+1, 4)$ and $C(2k, 4k-3, 2k)$ have orthogonal labellings.

Proof. (i) Define a labelling $\phi_2 : V(C(4, 4k+1, 4)) \rightarrow \mathbb{Z}_{4k+12}$ by $\phi_2(x_1) = 0$, $\phi_2(x_2) = k+3$, $\phi_2(x_3) = 2k+6$, $\phi_2(x_{1,1}) = k+4$, $\phi_2(x_{1,2}) = 2k+5$, $\phi_2(x_{1,3}) = 3k+10$, $\phi_2(x_{1,4}) = 4k+11$,

$$\phi_2(x_{2,i}) = \begin{cases} i+1 & \text{if } 1 \le i \le k, \\ i+4 & \text{if } k+1 \le i \le 2k, \\ i+7 & \text{if } 2k+1 \le i \le 3k, \\ 3k+9 & \text{if } i = 3k+1, \\ i+9 & \text{if } 3k+2 \le i \le 4k+1, \end{cases}$$

$\phi_2(x_{3,1}) = 1$, $\phi_2(x_{3,2}) = k+2$, $\phi_2(x_{3,3}) = 2k+7$ and $\phi_2(x_{3,4}) = 3k+8$.

(ii) Define a labelling $\phi_3 : V(C(2k, 4k-3, 2k)) \rightarrow \mathbb{Z}_{8k}$ by $\phi_3(x_1) = 0$, $\phi_3(x_2) = 2k$, $\phi_3(x_3) = 4k$,

$$\phi_3(x_{1,i}) = \begin{cases} 2k-1+2i & \text{if } 1 \le i \le k, \\ 4k-1+2i & \text{if } k+1 \le i \le 2k, \end{cases}$$

$$\phi_3(x_{2,i}) = \begin{cases} 2i & \text{if } 1 \le i \le k-1, \\ 2i+2 & \text{if } k \le i \le 2k-2, \\ 2i+4 & \text{if } 2k-1 \le i \le 4k-3 \end{cases}$$

and

$$\phi_3(x_{3,i}) = \begin{cases} 2i-1 & \text{if } 1 \le i \le k, \\ 2k-1+2i & \text{if } k+1 \le i \le 2k. \end{cases}$$ ■

3 Caterpillars of diameter 5

Gronau, Mullin and Rosa [3], observed that among the caterpillars of diameter 5 with $n \le 10$ vertices, the caterpillars not admitting an orthogonal labelling is $C(3,0,0,1)$. An orthogonal labelling for the caterpillar $C(3,0,0,2k+1)$ is given in [4]. In this section, we find orthogonal labellings for some caterpillars of diameter 5.

Theorem 3. For all $k \ge 0$ and all $t \ge 2$, the caterpillar $C(1, 2k+1, 4t+3, 1)$ has an orthogonal labelling.

Proof. Case 1. $k = 2r$

Define a labelling $\psi_1 : V(C(1, 4r+1, 4t+3, 1)) \to \mathbb{Z}_{4r+4t+10}$ by $\psi_1(x_1) = 1$, $\psi_1(x_2) = 2r+2t+5$, $\psi_1(x_3) = 0$, $\psi_1(x_4) = 4r+4t+9$, $\psi_1(x_{1,1}) = 4r+4t+7$,

$$\psi_1(x_{2,i}) = \begin{cases} i+4 & \text{if } 1 \le i \le r, \\ 2t+i & \text{if } r+1 \le i \le 2r, \\ 2r+2t+6 & \text{if } i = 2r+1, \\ 2t+8+i & \text{if } 2r+2 \le i \le 3r+1, \\ 4t+4+i & \text{if } 3r+2 \le i \le 4r+1, \end{cases}$$

$$\psi_1(x_{3,i}) = \begin{cases} i+1 & \text{if } 1 \le i \le 3, \\ r+1+i & \text{if } 4 \le i \le 2t-1, \\ 2r+1+i & \text{if } 2t \le i \le 2t+3, \\ 2r+3+i & \text{if } 2t+4 \le i \le 2t+6, \\ 3r+3+i & \text{if } 2t+7 \le i \le 4t+2, \\ 4r+4t+8 & \text{if } i = 4t+3 \end{cases}$$

and $\psi_1(x_{4,1}) = 4r+4t+6$.

Case 2. $k = 2r+1$

Define a labelling $\psi_2 : V(C(1, 4r+3, 4t+3, 1)) \to \mathbb{Z}_{4r+4t+12}$ by $\psi_2(x_1) = 1$, $\psi_2(x_2) = 2r+2t+6$, $\psi_2(x_3) = 0$, $\psi_2(x_4) = 4r+4t+11$, $\psi_2(x_{1,1}) = 4r+4t+9$,

$$\psi_2(x_{2,i}) = \begin{cases} i+4 & \text{if } 1 \le i \le r, \\ r+t+3 & \text{if } i = r+1, \\ 2t+i & \text{if } r+2 \le i \le 2r+1, \\ 2r+2t+7 & \text{if } i = 2r+2, \\ 2t+8+i & \text{if } 2r+3 \le i \le 3r+2, \\ 3r+3t+9 & \text{if } i = 3r+3, \\ 4t+4+i & \text{if } 3r+4 \le i \le 4r+3, \end{cases}$$

$$\psi_2(x_{3,i}) = \begin{cases} i+1 & \text{if } 1 \le i \le 3, \\ r+1+i & \text{if } 4 \le i \le t+1, \\ r+2+i & \text{if } t+2 \le i \le 2t-1, \\ 2r+2+i & \text{if } 2t \le i \le 2t+3, \\ 2r+4+i & \text{if } 2t+4 \le i \le 2t+6, \\ 3r+4+i & \text{if } 2t+7 \le i \le 3t+4, \\ 3r+5+i & \text{if } 3t+5 \le i \le 4t+2, \\ 4r+4t+10 & \text{if } i = 4t+3 \end{cases}$$

and $\psi_2(x_{4,1}) = 4r+4t+8$. ■

For $t = 0$ and 1, the caterpillar $C(1, 2k+1, 4t+3, 1)$ is $C(1, 2k+1, 3, 1)$ and $C(1, 2k+1, 7, 1)$, respectively. Next, we consider $C(1, 2k+1, 7, 1)$ for any k and $C(1, 2k+1, 3, 1)$ for odd k's.

Theorem 4. Let $k \ge 2$ be an integer. The caterpillars $C(1, 4k+3, 3, 1)$ and

$C(1, 2k+1, 7, 1)$ have orthogonal labellings.

Proof. (i) Define a labelling $\psi_3 : V(C(1, 4k+3, 3, 1)) \to \mathbb{Z}_{4k+12}$ by $\psi_3(x_1) = 4k+11$, $\psi_3(x_2) = 0$, $\psi_3(x_3) = 2k+6$, $\psi_3(x_4) = 1$, $\psi_3(x_{1,1}) = 4k+8$,

$$\psi_3(x_{2,i}) = \begin{cases} i+1 & \text{if } 1 \le i \le k+1, \\ i+2 & \text{if } k+2 \le i \le 2k+3, \\ i+4 & \text{if } 2k+4 \le i \le 3k+4, \\ i+5 & \text{if } 3k+5 \le i \le 4k+2, \\ 4k+10 & \text{if } i = 4k+3, \end{cases}$$

$\psi_3(x_{3,1}) = k+3$, $\psi_3(x_{3,2}) = 2k+7$, $\psi_3(x_{3,3}) = 3k+9$ and $\psi_3(x_{4,1}) = 4k+9$.

(ii) Define a labelling $\psi_4 : V(C(1, 2k+1, 7, 1)) \to \mathbb{Z}_{2k+14}$ by $\psi_4(x_1) = 1$, $\psi_4(x_2) = k+7$, $\psi_4(x_3) = 0$, $\psi_4(x_4) = 2k+13$, $\psi_4(x_{1,1}) = 2k+11$,

$$\psi_4(x_{2,i}) = \begin{cases} 2 & \text{if } i = 1, \\ i+3 & \text{if } 2 \le i \le k-1, \\ k+5 & \text{if } i = k, \\ i+7 & \text{if } i = k+1, k+2, \\ i+9 & \text{if } k+3 \le i \le 2k, \\ 2k+12 & \text{if } i = 2k+1, \end{cases}$$

$\psi_4(x_{3,1}) = 3$, $\psi_4(x_{3,2}) = 4$, $\psi_4(x_{3,3}) = k+3$, $\psi_4(x_{3,4}) = k+4$, $\psi_4(x_{3,5}) = k+6$, $\psi_4(x_{3,6}) = k+10$, $\psi_4(x_{3,7}) = k+11$ and $\psi_4(x_{4,1}) = 2k+10$. ■

Theorem 5. For all $k \ge 1$, the caterpillars $C(1, 2k+1, 5, 1)$, $C(1, 13, 4k+1, 1)$, $C(1, 0, k, k)$, $C(k, 1, 2k+3, k)$ and $C(k, 5, 2k+1, k)$ have orthogonal labellings.

Proof. (i) Define a labelling $\psi_5 : V(C(1, 2k+1, 5, 1)) \to \mathbb{Z}_{2k+12}$ by $\psi_5(x_1) = 2k+11$, $\psi_5(x_2) = 0$, $\psi_5(x_3) = k+6$, $\psi_5(x_4) = 1$, $\psi_5(x_{1,1}) = 2k+8$,

$$\psi_5(x_{2,i}) = \begin{cases} i+2 & \text{if } 1 \le i \le k+1, \\ k+5 & \text{if } i = k+2, \\ i+6 & \text{if } k+3 \le i \le 2k+1, \end{cases}$$

$\psi_5(x_{3,1}) = 2$, $\psi_5(x_{3,2}) = k+4$, $\psi_5(x_{3,3}) = k+7$, $\psi_5(x_{3,4}) = k+8$, $\psi_5(x_{3,5}) = 2k+10$ and $\psi_5(x_{4,1}) = 2k+9$.

(ii) Define a labelling $\psi_6 : V(C(1, 13, 4k+1, 1)) \to \mathbb{Z}_{4k+20}$ by $\psi_6(x_1) = 4k+19$, $\psi_6(x_2) = 0$, $\psi_6(x_3) = 2k+10$, $\psi_6(x_4) = 1$, $\psi_6(x_{1,1}) = 4k+16$,

$$\psi_6(x_{2,i}) = \begin{cases} i+1 & \text{if } i = 1, 2, 3, \\ k+5 & \text{if } i = 4, \\ 2k+1+i & \text{if } 5 \le i \le 8, \\ 2k+3+i & \text{if } 9 \le i \le 11, \\ 3k+15 & \text{if } i = 12, \\ 4k+18 & \text{if } i = 13, \end{cases}$$

$$\psi_6(x_{3,i}) = \begin{cases} i+4 & \text{if } 1 \le i \le k, \\ i+5 & \text{if } k+1 \le i \le 2k, \\ 2k+11 & \text{if } i = 2k+1, \\ i+13 & \text{if } 2k+2 \le i \le 3k+1, \\ i+14 & \text{if } 3k+2 \le i \le 4k+1 \end{cases}$$

and $\psi_6(x_{4,1}) = 4k+17$.

(iii) For $k = 1$, label the vertices $x_1, x_2, x_3, x_4, x_{1,1}, x_{3,1}$ and $x_{4,1}$ by 5, 3, 0, 1, 6, 2 and 4, respectively. So assume that $k \ge 2$. Define a labelling $\psi_7 : V(C(1,0,k,k)) \to \mathbb{Z}_{2k+5}$ by $\psi_7(x_1) = k+3$, $\psi_7(x_2) = k+1$, $\psi_7(x_3) = 0$, $\psi_7(x_4) = 2$, $\psi_7(x_{1,1}) = k+4$,

$$\psi_7(x_{3,i}) = \begin{cases} 1 & \text{if } i = 1, \\ i+1 & \text{if } 2 \le i \le k-1, \\ k+2 & \text{if } i = k \end{cases}$$

and $\psi_7(x_{4,i}) = k+4+i$ if $1 \le i \le k$.

(iv) Define a labelling $\psi_8 : V(C(k,1,2k+3,k)) \to \mathbb{Z}_{4k+8}$ by $\psi_8(x_1) = 1$, $\psi_8(x_2) = 2k+4$, $\psi_8(x_3) = 0$, $\psi_8(x_4) = 4k+7$, $\psi_8(x_{1,i}) = 2k+5+2i$ if $1 \le i \le k$, $\psi_8(x_{2,1}) = 2k+5$,

$$\psi_8(x_{3,i}) = \begin{cases} i+1 & \text{if } 1 \le i \le 2k+2, \\ 4k+6 & \text{if } i = 2k+3 \end{cases}$$

and $\psi_8(x_{4,i}) = 2k+4+2i$ if $1 \le i \le k$.

(v) Define a labelling $\psi_9 : V(C(k,5,2k+1,k)) \to \mathbb{Z}_{4k+10}$ by $\psi_9(x_1) = 1$, $\psi_9(x_2) = 2k+5$, $\psi_9(x_3) = 0$, $\psi_9(x_4) = 4k+9$, $\psi_9(x_{1,i}) = 2k+7+2i$ if $1 \le i \le k$, $\psi_9(x_{2,1}) = 2$, $\psi_9(x_{2,2}) = 2k+3$, $\psi_9(x_{2,3}) = 2k+6$, $\psi_9(x_{2,4}) = 2k+7$, $\psi_9(x_{2,5}) = 4k+8$,

$$\psi_9(x_{3,i}) = \begin{cases} i+2 & \text{if } 1 \le i \le 2k, \\ 2k+4 & \text{if } i = 2k+1 \end{cases}$$

and $\psi_9(x_{4,i}) = 2k+6+2i$ if $1 \le i \le k$. ■

In the caterpillar $C(1,0,k,k)$, $k \ge 2$, delete the edge x_1x_2 and add the new edge $x_{1,1}x_{4,2}$. The resulting graph is isomorphic to $C(1,0,k-1,k+1)$. The labelling ψ_7, in the proof of Theorem 5, is also an orthogonal labelling of $C(1,0,k-1,k+1)$. Thus, we have

Theorem 6. Let $k \ge 2$ be an integer. The caterpillar $C(1,0,k-1,k+1)$ has an orthogonal labelling.

4 Caterpillars of diameter 6

Gronau, Mullin and Rosa [3], observed that among the caterpillars of diameter 6 with $n \le 10$ vertices, the caterpillars not admitting an orthogonal labelling is $C(1,0,0,0,1)$. Also, this result for $n \leq 10$ has been extended to $n \leq 14$ in [4]. In this section, we find an orthogonal labelling for the caterpillar $C(1,4r+1,1,4t+3,1)$, for $r \geq 1$ and $t \geq 1$.

Theorem 7. For all $r \geq 1$ and $t \geq 1$, the caterpillar $C(1,4r+1,1,4t+3,1)$ has an orthogonal labelling.

Proof. Define a labelling $\rho : V(C(1,4r+1,1,4t+3,1)) \to \mathbb{Z}_{4r+4t+12}$ by $\rho(x_1) = 1$, $\rho(x_2) = 2r+2t+6$, $\rho(x_3) = r+t+3$, $\rho(x_4) = 0$, $\rho(x_5) = 4r+4t+11$, $\rho(x_{1,1}) = 4r+4t+9$,

$$\rho(x_{2,i}) = \begin{cases} 2 & \text{if } i = 1, \\ t+2+i & \text{if } 2 \leq i \leq r, \\ t+3+i & \text{if } r+1 \leq i \leq 2r-1, \\ 2r+2t+4 & \text{if } i = 2r, \\ 2t+6+i & \text{if } i = 2r+1,\, 2r+2, \\ 3t+7+i & \text{if } 2r+3 \leq i \leq 3r+1, \\ 3t+8+i & \text{if } 3r+2 \leq i \leq 4r, \\ 4r+4t+10 & \text{if } i = 4r+1, \end{cases}$$

$\rho(x_{3,1}) = 3r+3t+9$,

$$\rho(x_{4,i}) = \begin{cases} i+2 & \text{if } 1 \leq i \leq t+1, \\ 2r+1+i & \text{if } t+2 \leq i \leq 2t+2, \\ 2r+2t+5 & \text{if } i = 2t+3, \\ 2r+5+i & \text{if } 2t+4 \leq i \leq 3t+4, \\ 4r+4+i & \text{if } 3t+5 \leq i \leq 4t+3 \end{cases}$$

and $\rho(x_{5,1}) = 4r+4t+8$. ■

In conclusion, we propose the following problems:

Problem A. Which caterpillar admits an orthogonal labelling?

In general,

Problem B. Which tree admits an orthogonal labelling?

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