Singularities of the Newton mapping and the Van der Monde determinant

PO de Wet
Department of Quantitative Management,
Unisa, 0003 Pretoria, South Africa

1 Introduction

The fact that complex polynomials can be written in two forms,

$$\prod_{i=1}^{k}(\xi + z_i) = \xi^k + \sigma_1\xi^{k-1} + \ldots + \sigma_k,$$

can be used to define a mapping $N(z) = \sigma$ from $\mathbf{C}^k$ to $\mathbf{C}^k$, known as the Newton mapping. It is clearly surjective and turns out to have particularly useful properties regarding its symmetry, for example the analytic theorem of Newton: Let $f(z_1, \ldots, z_k)$ be an analytic function which is symmetric in $z_1, \ldots, z_k$, then there exists a unique analytic function $g(\sigma_1, \ldots, \sigma_k)$ such that $f = g \circ N$. It is used by Lojasiewicz [1, 2] to prove the division theorem for smooth functions and in general to study differentiable functions [3].

Finding a formula for the singularities of N requires finding $|DN|$, the determinant of the Jacobian matrix of N. (By the inverse function theorem.) We discuss a combinatorial proof of this known result [4] to demonstrate the close relationship between this determinant and another well known determinant, the Van der Monde determinant. The method demonstrated might also be useful.

2 Notation

We define the Newton mapping by

$$N : \mathbf{C}^k \to \mathbf{C}^k$$
$$N(z) = \sigma$$

with

$$\begin{array}{rcccl} \sigma_1 & = & z_1 + z_2 + \cdots + z_k & = & \sum z_* \\ \sigma_2 & = & \displaystyle\sum_{1 \le i < j \le k} z_i z_j & = & \sum z_*^2 \\ \vdots & & \vdots & & \vdots \\ \sigma_k & = & z_1 z_2 \cdots z_k & = & \sum z_*^k \end{array}$$

where we use the notation $\sum z_*^p$ to indicate the sum of all the products consisting of p different variables z_i. We also use $\sum_{z_m=0} z_*^p$ to indicate that all terms containing z_m are omitted; for example if $k = 4$ we have $\sum z_*^3 = z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4$ and $\sum_{z_2=0} z_*^3 = z_1 z_3 z_4$.

With this notation, the Jacobian matrix of the Newton mapping can be written as

$$DN = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \displaystyle\sum_{z_1=0} z_* & \displaystyle\sum_{z_2=0} z_* & \cdots & \displaystyle\sum_{z_k=0} z_* \\ \vdots & \vdots & & \vdots \\ \displaystyle\sum_{z_1=0} z_*^{k-1} & \displaystyle\sum_{z_2=0} z_*^{k-1} & \cdots & \displaystyle\sum_{z_k=0} z_*^{k-1} \end{pmatrix}.$$

3 The absolute value of $|DN|$ is equal to that of the Van der Monde determinant

In calculating the determinant of a square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

one is ultimately left with adding 2×2 matrixes. Consequently the effect of substituting a row in the matrix by another row can be reduced to the effect this has on 2×2 matrixes. This observation leads to the following lemma.

Lemma 1 *Replacing a row $a_{n1}, \ldots, a_{nk}$ of A by a new row $b_1, \ldots, b_k$ changes only the sign of the determinant if there exists another row $a_{m1}, \ldots, a_{mk}$*

of A such that for all $i \neq j$ we have

$$-\begin{vmatrix} a_{mi} & a_{mj} \\ a_{ni} & a_{nj} \end{vmatrix} = \begin{vmatrix} a_{mi} & a_{mj} \\ b_i & b_j \end{vmatrix}.$$

We say in this case that we use the row $a_{m1}, \ldots, a_{mk}$ as a hinge for the replacement.

We are now ready to show that the absolute value of $|DN|$ is equal to that of the Van der Monde determinant; that is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum_{z_1=0} z_* & \sum_{z_2=0} z_* & \cdots & \sum_{z_k=0} z_* \\ \vdots & \vdots & & \vdots \\ \sum_{z_1=0} z_*^{k-1} & \sum_{z_2=0} z_*^{k-1} & \cdots & \sum_{z_k=0} z_*^{k-1} \end{vmatrix} = \pm \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_k \\ \vdots & \vdots & & \vdots \\ z_1^{k-1} & z_2^{k-1} & \cdots & z_k^{k-1} \end{vmatrix}.$$

Our aim is to alter the left-hand side until it corresponds to the right-hand side, by using induction on the indexes of the rows. We note that the first rows are already similar. Thus let us assume that the first p rows at the left can be replaced by rows that are similar to those at the right without changing the absolute value of the determinant. This means that we have

$$|DN| = \pm \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ z_1^{p-1} & z_2^{p-1} & \cdots & z_k^{p-1} \\ \sum_{z_1=0} z_*^p & \sum_{z_2=0} z_*^p & \cdots & \sum_{z_k=0} z_*^p \\ \vdots & \vdots & & \vdots \\ \sum_{z_1=0} z_*^{k-1} & \sum_{z_2=0} z_*^{k-1} & \cdots & \sum_{z_k=0} z_*^{k-1} \end{vmatrix}.$$

We want to show that in row $p+1$

$$\sum_{z_1=0} z_*^p, \ldots, \sum_{z_k=0} z_*^p$$

can be replaced by

$$z_1 \sum_{z_1=0} z_*^{p-1}, \ldots, z_k \sum_{z_k=0} z_*^{p-1},$$

then by

$$z_1^2 \sum_{z_1=0} z_*^{p-2}, \ldots, z_k^2 \sum_{z_k=0} z_*^{p-2},$$

and so on, until we have reached row $p+1$ as

$$z_1^p, \ldots, z_k^p.$$

Thus it would suffice to show that we can replace

$$z_1^n \sum_{z_1=0} z_*^{p-n}, \ldots, z_k^n \sum_{z_k=0} z_*^{p-n},$$

for any $n < p$, by

$$z_1^{n+1} \sum_{z_1=0} z_*^{p-n-1}, \ldots, z_k^{n+1} \sum_{z_k=0} z_*^{p-n-1}.$$

We use Lemma 1 with row $n+1$ as a hinge. This row is $z_1^n, \ldots, z_k^n$ by our inductive hypothesis. For $i \neq j$ we obtain

$$\begin{vmatrix} z_i^n & z_j^n \\ z_i^n \sum_{z_i=0} z_*^p & z_j^n \sum_{z_j=0} z_*^p \end{vmatrix} = z_i^n z_j^n \left(\sum_{z_j=0} z_*^p - \sum_{z_i=0} z_*^p \right).$$

Omitting terms which negate each other gives

$$z_i^n z_j^n \left(z_i \sum_{\substack{z_j = 0 \\ z_i = 0}} z_*^{p-1} - z_j \sum_{\substack{z_i = 0 \\ z_j = 0}} z_*^{p-1} \right)$$

and then, allowing some terms which negate each other, gives

$$z_i^n z_j^n \left(z_i \sum_{z_i=0} z_*^{p-1} - z_j \sum_{z_j=0} z_*^{p-1} \right) = - \begin{vmatrix} z_i^n & z_j^n \\ z_i^{n+1} \sum_{z_i=0} z_*^{p-1} & z_j^{n+1} \sum_{z_j=0} z_*^{p-1} \end{vmatrix}.$$

This completes the proof. We can summarise it in algorithmic form as:

```
for j = 2 to k do
  begin
    for i = 1 to j-1 do
      begin
      replace row j by using row i as a hinge
      end
  end
```

4 A formula for $|DN|$

The Van der Monde determinant is given by the well known formula

$$\prod_{1 \leq i < j \leq k} (z_j - z_i).$$

Since we used Lemma 1 exactly $1+2+\ldots+(k-1)$ times (see the algorithm) in the proof of the previous section, we have

$$|DN| = (-1)^{1+2+\ldots+(k-1)} \prod_{1 \leq i < j \leq k} (z_j - z_i) = \prod_{1 \leq i < j \leq k} (z_i - z_j).$$

References

[1] Martinet, J., *Singularities of smooth functions and maps*, London Mathematical Society Lecture Note Series 58, Cambridge University Press, Cambridge, 1982.

[2] Lojasiewicz, S., Whitney fields and the Malgrange-Mather preparation theorem, *Proceedings of Liverpool Symposium, I.*, (ed. C.T.C. Wall), 106-115, Springer, New York, 1970.

[3] Barbanson, G.P., Monodromy of the Newton mapping, *Analysis (2)*, 141-162(1982).

[4] Muir, T., *A Treatise on the Theory of Determinants*, Constable and Company Limited, London, 1960.

γ-Labelings of Graphs

Gary Chartrand, Western Michigan University
David Erwin, University of Kwazulu-Natal
Donald W. VanderJagt, Grand Valley State University
Ping Zhang, Western Michigan University

ABSTRACT

Let G be a graph of order n and size m. A γ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The value of a γ-labeling f is $\text{val}(f) = \sum_{e \in E(G)} f'(e)$. The maximum value of a γ-labeling of G is defined as

$$\text{val}_{\max}(G) = \max\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\},$$

while the minimum value of a γ-labeling of G is

$$\text{val}_{\min}(G) = \min\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

The values $\text{val}_{\max}(G)$ and $\text{val}_{\min}(G)$ are determined for some well-known classes of graphs G. A sharp lower bound for the minimum value of a γ-labeling of a connected graph is established in terms of its order and size.

Key Words: γ-labeling, value of a γ-labeling.

AMS Subject Classification: 05C78.

1 Introduction

For a graph G of order n and size m, a *γ-labeling* of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by

$$f'(e) = |f(u) - f(v)| \text{ for each edge } e = uv \text{ of } G.$$

Therefore, a graph G of order n and size m has a γ-labeling if and only if $m \geq n - 1$. In particular, every connected graph has a γ-labeling.

If the induced edge-labeling f' of a γ-labeling f of a graph is also one-to-one, then f is a *graceful labeling.* Among all labelings of graphs, graceful labelings are probably the best known and most studied. Graceful labelings originated with a 1967 paper of Rosa [3], who used the term β-valuations. Five years later, Golomb [2] called these labelings "graceful" and this is the terminology that has been used since then. A graph that has a graceful labeling is called a *graceful graph.* One of the major conjectures in graph theory concerns graceful graphs and is due to Kotzig (see Rosa [3]).

The Graceful Tree Conjecture *Every tree is graceful.*

Gallian [1] has written a survey on labelings of graphs that includes an extensive discussion of graceful labelings.

Each γ-labeling f of a graph G of order n and size m is assigned a *value* denoted by $\text{val}(f)$ and defined by

$$\text{val}(f) = \sum_{e \in E(G)} f'(e).$$

Since f is a one-to-one function from $V(G)$ to $\{0, 1, 2, \ldots, m\}$, it follows that $f'(e) \geq 1$ for each edge e in G and so

$$\text{val}(f) \geq m. \tag{1}$$

Figure 1 shows nine γ-labelings $f_1, f_2, \ldots, f_9$ of the path P_5 of order 5 (where the vertex labels are shown above each vertex and the induced edge labels are shown below each edge). The value of each γ-labeling is shown in Figure 1 as well.

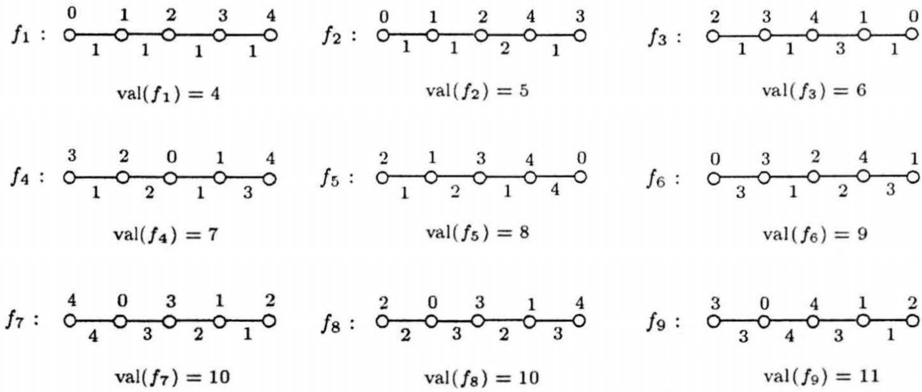

Figure 1: Some γ-labelings of P_5

The value of a graceful labeling of a graph G of order n and size m is necessarily $\binom{m+1}{2}$. For example, the γ-labeling f_7 of P_5 shown in Figure 1 is graceful and consequently $\mathrm{val}(f_7) = \binom{5}{2} = 10$. However, the labeling f_8 shows that it is not necessary for a γ-labeling to be graceful in order to have a value of $\binom{m+1}{2}$.

For a graph G of order n and size m, the *maximum value* of a γ-labeling of a graph G is defined as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\},$$

while the *minimum value* of a γ-labeling of G is

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

A γ-labeling g of G is a γ-max *labeling* if

$$\mathrm{val}(g) = \mathrm{val}_{\max}(G)$$

and a γ-labeling h is a γ-min *labeling* if

$$\mathrm{val}(h) = \mathrm{val}_{\min}(G).$$

Since $\mathrm{val}(f_1) = 4$ for the γ-labeling f_1 of P_5 shown in Figure 1 and the size of P_5 is 4, it follows that f_1 is a γ-min labeling of P_5. Although less clear, the γ-labeling f_9 shown in Figure 1 is a γ-max labeling.

For a γ-labeling f of a graph G of size m, the *complementary labeling* $\overline{f} : V(G) \to \{0, 1, 2, \ldots, m\}$ of f is defined by

$$\overline{f}(v) = m - f(v) \text{ for } v \in V(G).$$

Not only is $\overline{f}$ a γ-labeling of G as well but $\mathrm{val}(\overline{f}) = \mathrm{val}(f)$. This gives us the following.

Observation 1.1 *Let f be a γ-labeling of a graph G. Then f is a* γ-max *labeling* (γ-min *labeling*) *of G if and only if $\overline{f}$ is a* γ-max *labeling* (γ-min *labeling*).

By the *spectrum* of a graph G, we mean the set

$$\mathrm{spec}(G) = \{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

Consequently, if $G \cong P_5$, then $\{4, 5, 6, 7, 8, 9, 10, 11\} \subseteq \mathrm{spec}(G)$. If G is a graceful graph of size m, then $\binom{m+1}{2} \in \mathrm{spec}(G)$. As an illustration, we determine the spectrum of stars.

Proposition 1.2 *For each integer $t \geq 2$,*

$$\mathrm{spec}(K_{1,t}) = \left\{ \binom{t+1-k}{2} + \binom{k+1}{2} : 0 \leq k \leq t \right\}.$$

Proof. Suppose that $V(K_{1,t}) = \{v, v_1, v_2, \ldots, v_t\}$, where $\deg v = t$. Let f be a γ-labeling of $K_{1,t}$ such that $f(v) = k$, where $0 \le k \le t$. If $k = 0$, then we may assume that $f(v_i) = i$ for $1 \le i \le t$. Then

$$\mathrm{val}(f) = \sum_{i=1}^{t} |f(v_i) - f(v)| = \sum_{i=1}^{t} i = \binom{t+1}{2}.$$

If $k = t$, then $\mathrm{val}(f) = \binom{t+1}{2}$ by Observation 1.1. If $0 < k < t$, then we may assume that

$$f(v_i) = \begin{cases} i-1 & \text{if } 1 \le i \le k \\ i & \text{if } k+1 \le i \le t \end{cases}$$

Therefore,

$$\begin{aligned} \mathrm{val}(f) &= [k + (k-1) + \ldots + 1] + [1 + 2 + \ldots + (t-k)] \\ &= \binom{k+1}{2} + \binom{t-k+1}{2}, \end{aligned}$$

as desired. ■

Corollary 1.3 *For each integer $n \ge 3$,*

$$\mathrm{val}_{\max}(K_{1,n-1}) = \binom{n}{2} \text{ and } \mathrm{val}_{\min}(K_{1,n-1}) = \binom{\lfloor \frac{n+1}{2} \rfloor}{2} + \binom{\lceil \frac{n+1}{2} \rceil}{2}.$$

2 γ-Labelings of Subgraphs

We now describe the connection between the minimum and maximum values of a connected graph and that of a proper connected subgraph. For a graph G, let $m(G)$ denote the size of G.

Proposition 2.1 *If H is a proper connected subgraph of a connected graph G, then*

$$\mathrm{val}_{\min}(H) < \mathrm{val}_{\min}(G) \text{ and } \mathrm{val}_{\max}(H) < \mathrm{val}_{\max}(G).$$

Proof. Suppose that G has order n and f is a γ-min labeling of G. Let $f(V(H)) = \{a_1, a_2, \ldots, a_k\}$, where $k \le n$ and $a_1 < a_2 < \ldots < a_k$. Consider the function $g : \{a_1, a_2, \ldots, a_k\} \to \{0, 1, \ldots, k-1\}$ defined by $g(a_i) = i - 1$. Consequently, $g \circ (f|_{V(H)}) : V(H) \to \{0, 1, \cdots, m(H)\}$ is a γ-labeling of H and $\mathrm{val}(g \circ (f|_{V(H)})) \le \mathrm{val}(f|_{V(H)})$. Since H is a proper subgraph of G, there exists $e \in E(G) - E(H)$. Thus

$$\text{val}_{\min}(H) \le \text{val}(f|_H) \le \text{val}(f) - f'(e) = \text{val}_{\min}(G) - f'(e)$$

and so $\text{val}_{\min}(H) < \text{val}_{\min}(G)$.

Next, let f be a γ-max labeling of H. If H is a spanning subgraph of G, then surely the value of f on H is less than $\text{val}_{\max}(G)$. Hence we can assume that H is not a spanning subgraph of G. We note that if H' is the subgraph of G induced by $V(H)$, then the value of f on H' is at least as large as the value of f on H (and if $H \neq H'$, then the value of f on H' exceeds the value of f on H). We thus assume, without loss of generality, that H is a proper induced subgraph of G. Since H and G are both connected, there is a sequence $H_0, H_1, \ldots, H_t$ of connected induced subgraphs of G with $H_0 = H$ and $H_t = G$ such that for each integer i with $1 \le i \le t$, $|V(H_i)| = |V(H)| + i$ and $H_{i-1} \subset H_i$. Let $f_0 = f$, and for each integer i with $1 \le i \le t$, define f_i to be f_{i-1} when restricted to $V(H_{i-1})$, and $f_i(x) = m(H_i)$ for that vertex $x \in V(H_i) - V(H_{i-1})$. Then, for each i with $1 \le i \le t$, the function f_i is a γ-labeling and $\text{val}(f_{i-1}) < \text{val}(f_i)$. ∎

The *span* of a γ-labeling f of a graph G is defined as

$$\text{span}(f) = \max\{f(v) : V \in V(G)\} - \min\{f(v) : v \in V(G)\}.$$

We now consider a lemma.

Lemma 2.2 *Let G be a connected graph of order n and $f : V(G) \to \mathbf{Z}$ a one-to-one function. Then there is a γ-labeling g on G with* $\text{val}(g) \le \text{val}(f)$. *Furthermore, if* $\text{span}(f) \ge n$, *then there is a γ-labeling g with* $\text{val}(g) < \text{val}(f)$.

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $f(v_i) = a_i$ for $1 \le i \le n$, where $a_1 < a_2 < \cdots < a_n$. Consider the function $h : \{a_1, a_2, \ldots, a_n\} \to \{0, 1, \ldots, n-1\}$ defined by $h(a_i) = i - 1$ for $1 \le i \le n$. Certainly, $g = h \circ f$ is a γ-labeling of G. Furthermore, for every edge e of G, we have $g'(e) \le f'(e)$ and so $\text{val}(g) \le \text{val}(f)$. Suppose now that $\text{span}(f) \ge n$. Since $\text{span}(f) = n - 1$ if and only if $a_{i+1} - a_i = 1$ for every integer i with $1 \le i \le n-1$, there is some integer j for which $a_{j+1} - a_j \ge 2$. Since G is connected, there is an edge e joining two vertices x and y, where $x \in \{v_1, v_2, \cdots, v_j\}$ and $y \in \{v_{j+1}, v_{j+2}, \cdots, v_n\}$. Thus $f(x) = a_{j-\delta_x}$ and $f(y) = a_{j+1+\delta_y}$, where $\delta_x, \delta_y \ge 0$ and so

$$\begin{aligned} f'(e) &= a_{j+1+\delta_y} - a_{j-\delta_x} \ge (j + 1 + \delta_y) - (j - \delta_x) + 1 \\ &> 1 + \delta_x + \delta_y = g'(e), \end{aligned}$$

as desired. ∎

We state two consequences of Lemma 2.2.

Corollary 2.3 *If G is a connected graph of order n, then G has a γ-min labeling whose vertices are labeled $0, 1, \ldots, n-1$.*

Proposition 2.4 *If H is a subdivision of a connected graph G, then*

$$\mathrm{val}_{\min}(G) < \mathrm{val}_{\min}(H) \quad \text{and} \quad \mathrm{val}_{\max}(G) < \mathrm{val}_{\max}(H).$$

Proof. It is sufficient to consider the case when H is obtained by subdividing a single edge of G. Let uv be that edge of G that is subdivided to produce H, resulting in the edges uw and vw of H. We begin by verifying the first inequality. Let f be a γ-min labeling of H. Then the restriction $f|_{V(G)}$ satisfies $f|'_{V(G)}(uv) \leq f'(uw) + f'(wv)$ on the graph G. The first inequality now follows from Lemma 2.2. We now verify the second inequality. Let f be a γ-max labeling of G. We can extend f to a γ-labeling g of H by defining

$$g(x) = \begin{cases} m(H) & \text{if } x = w \\ f(x) & \text{if } x \neq w. \end{cases}$$

The result now follows from the triangle inequality. ∎

3 γ-Labelings of Paths

The γ-labeling f of the path $P_n : v_1, v_2, \cdots, v_n$ defined by $f(v_i) = i - 1$ has $\mathrm{val}(f) = n - 1$. Thus, by (1), we have the following observation.

Observation 3.1 *For each integer $n \geq 2$, $\mathrm{val}_{\min}(P_n) = n - 1$.*

Next, we determine $\mathrm{val}_{\max}(P_n)$. We begin by considering certain γ-labelings of P_n. Suppose first that $n = 2k + 1 \geq 3$ is odd. Consider the γ-labeling f of P_n defined by

$$f(v_i) = \begin{cases} k + \dfrac{i+1}{2} & \text{if } i \text{ is odd and } i < 2k+1 \\ k & \text{if } i = 2k+1 \\ \dfrac{i-2}{2} & \text{if } i \text{ is even.} \end{cases}$$

Then k edges of P_n are labeled $k+1$, one edge is labeled 1, and the remaining $k-1$ edges are labeled $k+2$. Thus

$$\mathrm{val}(f) = k(k+1) + 1 + (k-1)(k+2) = \frac{n^2 - 3}{2}. \tag{2}$$

Next, suppose that $n = 2k \geq 2$ is even. Consider the γ-labeling g of P_n defined by

$$g(v_i) = \begin{cases} k + \dfrac{i-1}{2} & \text{if } i \text{ is odd} \\ \dfrac{i-2}{2} & \text{if } i \text{ is even.} \end{cases}$$

Here, k edges of P_n are labeled k and the remaining $k-1$ edges are labeled $k+1$. Thus

$$\text{val}(g) = k \cdot k + (k-1)(k+1) = \frac{n^2-2}{2}. \tag{3}$$

Combining (2) and (3), we have the following.

Proposition 3.2 *For every integer $n \geq 2$,*

$$\text{val}_{\max}(P_n) \geq \left\lfloor \frac{n^2-2}{2} \right\rfloor.$$

In order to show that the lower bound for $\text{val}_{\max}(P_n)$ given in Proposition 3.2 is, in fact, the exact value of $\text{val}_{\max}(P_n)$ for all $n \geq 2$, we first establish a lemma.

Lemma 3.3 *For every integer $n \geq 3$, there exists a γ-max labeling f of $P_n : v_1, v_2, \ldots, v_n$ having the property that for every integer i with $1 \leq i \leq n-2$, the 3-term sequence*

$$s_i(f) = (f(v_i), f(v_{i+1}), f(v_{i+2}))$$

is not monotone.

Proof. For each γ-max labeling f of P_n, let

$$S(f) = \{s_1(f), s_2(f), \ldots, s_{n-2}(f)\}.$$

Assume that the lemma is false. Consequently, for each γ-max labeling f of P_n, some element of $S(f)$ is monotone. Among all γ-max labelings of P_n, let g be one for which t is the largest integer with $1 \leq t \leq n-2$ such that $s_t(g)$ is monotone and $s_i(g)$ is not monotone for $1 \leq i < t$.

We define a new γ-max labeling g' of P_n from g as follows:

$$g'(v_i) = \begin{cases} g(v_i) & \text{if } i \neq t+1, t+2 \\ g(v_{t+2}) & \text{if } i = t+1 \\ g(v_{t+1}) & \text{if } i = t+2. \end{cases}$$

It is now straightforward to show that $s_i(g')$ is not monotone for every integer i with $1 \leq i \leq t$ and that $\text{val}(g') \geq \text{val}(g)$. Since g is a γ-max labeling of P_n, it follows that $\text{val}(g') = \text{val}(g)$ and g' is also a γ-max labeling of P_n. This, however, contradicts the defining property of g. ∎

Proposition 3.4 *For every integer $n \geq 2$ and every γ-max labeling f of P_n,*

$$\text{val}(f) \leq \left\lfloor \frac{n^2-2}{2} \right\rfloor = \begin{cases} \dfrac{n^2-2}{2} & \textit{if } n \textit{ even,} \\ \dfrac{n^2-3}{2} & \textit{if } n \textit{ odd.} \end{cases}$$

Proof. A γ-max labeling f having the property introduced in Lemma 3.3 induces a partition of $V(P_n)$ into two independent sets, $T(f)$ and $B(f)$ (the *top* and *bottom* of f, respectively), such that for every edge tb joining a vertex $t \in T(f)$ to a vertex $b \in B(f)$, we have $f(t) > f(b)$. It is immediate that

$$\text{val}(f) = \sum_{v \in T(f)} f(v) \deg\, v - \sum_{v \in B(f)} f(v) \deg\, v. \tag{4}$$

Since the right hand side of (4) can be no larger than the quantity obtained by assigning the largest possible values to the vertices of $T(f)$ and the smallest possible values to the vertices of $B(f)$, it follows that val(f) is bounded above by

$$\left[2 \sum_{i=1}^{n/2-1} (n-i) + \frac{n}{2} \right] - \left[2 \sum_{i=1}^{n/2-1} (i-1) + \left(\frac{n}{2} - 1 \right) \right] = \frac{n^2-2}{2} \tag{5}$$

if n is even, and by

$$\left[2 \sum_{i=1}^{(n-3)/2} (n-i) + \frac{n-1}{2} + \frac{n+1}{2} \right] - \left[2 \sum_{i=1}^{(n-1)/2} (i-1) \right] = \frac{n^2-3}{2} \tag{6}$$

if n is odd. Simplifying (5) and (6), we get the desired upper bound. ■

Combining Propositions 3.2 and 3.4, we have the following.

Theorem 3.5 *For every integer $n \geq 2$,*

$$\text{val}_{\max}(P_n) = \left\lfloor \frac{n^2-2}{2} \right\rfloor .$$

4 γ-Labelings of Cycles

Next, we establish a formula for $\text{val}_{\min}(C_n)$ for all $n \geq 3$.

Theorem 4.1 *For every integer $n \geq 3$,*

$$\text{val}_{\min}(C_n) = 2(n-1).$$

Proof. Let $C_n : v_1, v_2, \ldots, v_n, v_1$. Consider the γ-labeling h of C_n defined by $h(v_i) = i - 1$ for $1 \le i \le n$. Then

$$\begin{aligned}\text{val}(h) &= \sum_{i=1}^{n-1} |h(v_{i+1}) - h(v_i)| + |h(v_1) - h(v_n)| \\ &= (n-1)\cdot 1 + (n-1) = 2(n-1).\end{aligned}$$

Therefore, $\text{val}_{\min}(C_n) \le 2(n-1)$. Hence it remains to show that $\text{val}_{\min}(C_n) \ge 2(n-1)$.

Let f be a γ-min labeling of C_n. By Corollary 2.3, we may assume that the vertices of C_n are labeled with the elements of the set $\{0, 1, \ldots, n-1\}$. We may further assume that $f(v_1) = 0$. Suppose that $f(v_t) = n - 1$, where $2 \le t \le n$. The cycle C_n contains two edge-disjoint $v_1 - v_t$ paths, namely

$$P : v_1, v_2, \ldots, v_t \quad \text{and} \quad P' : v_1, v_n, v_{n-1}, \ldots, v_t.$$

Let f_P be the restriction of f to P and $f_{P'}$ be the restriction of f to P'. Then f_P and $f_{P'}$ are γ-labelings of P and P', respectively, and

$$\text{val}(f) = \text{val}(f_P) + \text{val}(f_{P'}). \tag{7}$$

We show that $\text{val}(f_P) \ge n - 1$ and $\text{val}(f_{P'}) \ge n - 1$.

Consider the path P. If $f(v_1), f(v_2), \ldots, f(v_t)$ is an increasing sequence, then

$$\text{val}(f_P) = \sum_{i=1}^{t-1} [f(v_{i+1}) - f(v_i)] = f(v_t) - f(v_1) = n - 1.$$

If $f(v_1), f(v_2), \ldots, f(v_t)$ is not increasing, then this sequence can be divided into an odd number of subsequences that are alternately increasing and decreasing. Therefore, there exists an odd integer $s \ge 3$ such that

$$1 = i_0 < i_1 < \ldots < i_s = t$$

and

$f(v_1), f(v_2), \ldots, f(v_{i_1})$ is increasing,

$f(v_{i_1}), f(v_{i_1+1}), \ldots, f(v_{i_2})$ is decreasing, and so on, up to

$f(v_{i_{s-1}}), f(v_{i_{s-1}+1}), \ldots, f(v_{i_s})$ is increasing.

Then

$$\begin{aligned}\text{val}(f_P) &= [f(v_{i_1}) - f(v_{i_0})] + [f(v_{i_1}) - f(v_{i_2})] + [f(v_{i_3}) - f(v_{i_2})] + \\ &\quad \ldots + [f(v_{i_s}) - f(v_{i_{s-1}})] \\ &= [f(v_t) - f(v_1)] + 2\{[f(v_{i_1}) - f(v_{i_2})] + [f(v_{i_3}) - f(v_{i_4})] + \\ &\quad \ldots + [f(v_{i_{s-2}}) - f(v_{i_{s-1}})]\} \\ &\ge (n-1) + (s-1) > n - 1.\end{aligned}$$

In general then, $\text{val}(f_P) \geq n-1$. Similarly, $\text{val}(f_{P'}) \geq n-1$. It follows by (7) that $\text{val}(f) \geq 2(n-1)$. Therefore, $\text{val}_{\min}(C_n) = 2(n-1)$. ■

Since every edge labeling induced by a γ-labeling of a graph containing a vertex v with $\deg v \geq 3$ assigns a label of 2 or more to at least one edge incident with v, the following is a consequence of Theorem 4.1.

Corollary 4.2 *Let G be a connected graph of order n and size m. Then*

$$\text{val}_{\min}(G) = m \text{ if and only if } G \cong P_n.$$

In order to discuss $\text{val}_{\max}(C_n)$, we first establish the following result.

Proposition 4.3 *If G is a connected r-regular bipartite graph of order n and size m, where $r \geq 2$, then*

$$\text{val}_{\max}(G) = \frac{rn(2m-n+2)}{4}.$$

Proof. Let $n = 2k$ and let $V_1 = \{u_1, u_2, \ldots, u_k\}$ and $V_2 = \{v_1, v_2, \ldots, v_k\}$ be partite sets of G. Define a γ-labeling g of G by

$$g(u_i) = i-1 \text{ and } g(v_i) = m-(i-1) \text{ for } 1 \leq i \leq k.$$

Since $m = rk \geq 2k$, it follows that $g(u) < g(v)$ if $u \in V_1$ and $v \in V_2$. Thus

$$\begin{aligned} \text{val}(g) &= r\left[\sum_{i=1}^{k} g(v_i) - \sum_{i=1}^{k} g(u_i)\right] \\ &= r\{[m+(m-1)+\ldots+(m-k+1)] \\ &\quad -[1+2+\ldots+(k-1)]\} \\ &= r\left\{\left[mk-\binom{k}{2}\right]-\binom{k}{2}\right\} = r[mk-k(k-1)] \\ &= \frac{rn(2m-n+2)}{4}. \end{aligned}$$

Therefore, $\text{val}_{\max}(G) \geq \text{val}(g) = \dfrac{rn(2m-n+2)}{4}$.

To show that $\text{val}_{\max}(G) \leq \dfrac{rn(2m-n+2)}{4}$, let f be a γ-max labeling of G. Suppose that $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $e_i = x_iy_i$ and $f(x_i) < f(y_i)$ for $1 \leq i \leq m$. Then

$$\text{val}(f) = \sum_{i=1}^{m}[f(y_i) - f(x_i)] = \sum_{i=1}^{m} f(y_i) - \sum_{i=1}^{m} f(x_i). \tag{8}$$

Let $X = \{x_i : 1 \leq i \leq m\}$ and $Y = \{y_i : 1 \leq i \leq m\}$. Then $|X| = |Y| = m = rk$. Since at most r vertices in X can be labeled by each of the labels $0, 1, \ldots, k-1$ and at most r vertices in Y can be labeled by each of the labels $m, m-1, \ldots, m-(k-1)$, it follows that

$$\begin{aligned}
\sum_{i=1}^{m} f(x_i) &\geq r[1+2+\ldots+(k-1)] = r\binom{k}{2} \\
\sum_{i=1}^{m} f(y_i) &\leq r[m+(m-1)+\ldots+(m-k+1)] \\
&= r\left[mk - \binom{k}{2}\right]
\end{aligned}$$

It then follows by (8) that

$$\mathrm{val}(f) \leq r\left[mk - \binom{k}{2}\right] - r\binom{k}{2} = \frac{rn(2m-n+2)}{4}.$$

Therefore, $\mathrm{val}_{\max}(G) = \mathrm{val}(f) \leq \dfrac{rn(2m-n+2)}{4}$. ■

There is now an immediate corollary.

Corollary 4.4 *For an even integer $n \geq 4$,*

$$\mathrm{val}_{\max}(C_n) = \frac{n(n+2)}{2}.$$

We now determine $\mathrm{val}_{\max}(C_n)$ where n is odd. A γ^+*-labeling* of a connected graph G of order n and size m is a one-to-one function $f : V(G) \to \{0, 1, 2, \cdots, m+1\}$, where γ^+-max labeling and $\mathrm{val}^+_{\max}(G)$ are defined as expected.

Lemma 4.5 *For every integer $k \geq 2$,*

$$\mathrm{val}_{\max}(C_{2k+1}) = \mathrm{val}^+_{\max}(C_{2k}).$$

Proof. Let f be a γ^+-max labeling of C_{2k}. Then there are two numbers $a, b \in \{0, 1, 2, \cdots, m+1\}$ that are assigned to no vertex of C_{2k} by f. Consequently, f can be extended to a γ-labeling h of C_{2k+1} by viewing C_{2k+1} as a subdivision of C_{2k} and then assigning either of the numbers a, b to the unique vertex in $V(C_{2k+1}) - V(C_{2k})$. From the triangle inequality, the value of h is at least as large as that of f. Thus $\mathrm{val}_{\max}(C_{2k+1}) \geq \mathrm{val}^+_{\max}(C_{2k})$.

We now establish the reverse inequality. Let g be a γ-max labeling of C_{2k+1}. We construct an oriented graph D from C_{2k+1} by assigning to each edge uv the orientation (u, v) if $g(v) > g(u)$. Necessarily, D contains a directed path x, y, z of order 3. If we delete the vertex y from C_{2k+1} and join the vertices x and z, the resulting graph G is isomorphic to C_{2k} and the restriction g' of g to $V(C_{2k+1}) - \{y\}$ has the same value on G as g does on C_{2k+1}. The function g' is thus a γ^+-labeling of C_{2k}, and the result follows. ■

Theorem 4.6 *For every odd integer* $n \geq 3$,

$$\text{val}_{\max}(C_n) = \frac{(n-1)(n+3)}{2}.$$

Proof. The result is clear for $n = 3$, so we may assume that $n = 2k+1 \geq 5$. From Lemma 4.5, it is sufficient to show that $\text{val}^+_{\max}(C_{n-1}) = (n-1)(n+3)/2$. Let $C_{n-1} : x_1, y_1, x_2, y_2, \cdots, x_k, y_k, x_1$. Define a γ^+-labeling of C_{n-1} by

$$f(x_i) = i - 1 \text{ and } f(y_i) = 2k - i + 2 \text{ for } 1 \leq i \leq k.$$

Then $\text{val}(f) = (n-1)(n+3)/2$ and so $\text{val}^+_{\max}(C_{n-1}) \geq (n-1)(n+3)/2$.

It remains to verify that $\text{val}^+_{\max}(C_{n-1}) \leq (n-1)(n+3)/2$. Let g be a γ^+-max labeling of C_{n-1}, where $E(C_{n-1}) = \{e_1, e_2, \ldots, e_{n-1}\}$. For each integer i with $1 \leq i \leq n-1$, let $e_i = u_i v_i$, where $g(u_i) < g(v_i)$. Then $\text{val}(g) = \sum_{i=1}^{n-1} g(v_i) - \sum_{i=1}^{n-1} g(u_i)$. Since at most two vertices in $\{u_1, u_2, \ldots, u_{n-1}\}$ can be assigned each of the labels $0, 1, \ldots, k-1$ and at most two vertices in $\{v_1, v_2, \ldots, v_{n-1}\}$ can be assigned each of the labels $k+2, k+3, \ldots, 2k+1$, it follows that

$$\sum_{i=1}^{n-1} g(u_i) \geq k^2 - k \text{ and } \sum_{i=1}^{n-1} g(v_i) \leq 3k^2 + 3k$$

and so $\text{val}(g) \leq (3k^2 + 3k) - (k^2 - k) = (n-1)(n+3)/2$, producing the desired result. ■

5 γ-Labelings of Complete Graphs

First, we establish the minimum value of complete graphs. In order to do this, we recall a well-known combinatorial identity (which is sometimes called the *hockey stick property* of the Pascal triangle): For every two integers r and s with $0 \leq r \leq s$,

$$\binom{s+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \ldots + \binom{s}{r}. \quad (9)$$

Proposition 5.1 *For each integer $n \geq 3$,*

$$\text{val}_{\min}(K_n) = \binom{n+1}{3}.$$

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. By Corollary 2.3, the γ-labeling f of K_n defined by

$$f(v_i) = i - 1 \text{ for } 1 \leq i \leq n$$

is a γ-min labeling. Since

$$\begin{aligned} \text{val}(f) &= \sum_{i=1}^{n-1} \left(\sum_{j=i+1}^{n} [(j-1) - (i-1)] \right) \\ &= \sum_{i=1}^{n-1} \left(\sum_{k=1}^{n-i} k \right) = \binom{n+1}{3}, \end{aligned}$$

where the last equality follows from the combinatorial identity in (9), it follows that $\text{val}_{\min}(K_n) = \binom{n+1}{3}$. ■

Next, we determine the maximum value of complete graphs.

Proposition 5.2 *For every positive integer n,*

$$\text{val}_{\max}(K_n) = \begin{cases} \dfrac{n(3n^3 - 5n^2 + 6n - 4)}{24} & \text{if } n \text{ even} \\ \dfrac{(n^2-1)(3n^2 - 5n + 6)}{24} & \text{if } n \text{ odd.} \end{cases} \tag{10}$$

Proof. If f is a γ-labeling of K_n and $f(V(K_n)) = \{a_1, a_2, \ldots, a_n\}$, where $a_1 < a_2 < \ldots < a_n$, then

$$\text{val}(f) = \sum_{i=1}^{n} (2i - 1 - n)a_i.$$

For $N = \binom{n}{2}$, the number val(f) is maximized by assigning the labels

$$\{0, 1, \ldots, \lceil n/2 \rceil - 1, N - \lfloor n/2 \rfloor + 1, N - \lfloor n/2 \rfloor + 2, \ldots, N\}$$

to the vertices of K_n, thereby obtaining the result. ■

6 A Bound on the Minimum Value of a Graph in Terms of Its Order and Size

For integers n and m with $1 \leq n-1 \leq m \leq \binom{n}{2}$, let $S = \{1, 2, \cdots, n-1\}$ and

$$\alpha(n,m) = \max\{k \in S : \sum_{i=1}^{k}(n-i) \leq m\}.$$

There is an algebraic expression for $\alpha(n,m)$ in terms of n and m.

Lemma 6.1 *For integers n and m with $1 \leq n-1 \leq m \leq \binom{n}{2}$,*

$$\alpha(n,m) = \left\lfloor n - \frac{1}{2} - \sqrt{2\left[\binom{n}{2} - m\right] + \frac{1}{4}} \right\rfloor.$$

Proof. Since $\alpha(n,m) = n-1$ if $m = \binom{n}{2}$, we may assume that $m < \binom{n}{2}$. Let $\alpha(n,m) = k$. Then $k < n-1$ and

$$\sum_{i=1}^{k}(n-i) \leq m < \sum_{i=1}^{k+1}(n-i).$$

Therefore, $\binom{n}{2} \leq m + \binom{n-k}{2}$. Solving $\binom{n}{2} = m + \binom{n-x}{2}$ for x, we obtain two solutions:

$$\begin{aligned} x_1 &= n - \frac{1}{2} - \sqrt{2\left[\binom{n}{2} - m\right] + \frac{1}{4}}; \\ x_2 &= n - \frac{1}{2} + \sqrt{2\left[\binom{n}{2} - m\right] + \frac{1}{4}}. \end{aligned}$$

Since k is the largest integer less than $n-1$ for which $\binom{n}{2} \leq m + \binom{n-k}{2}$, it follows that

$$k = \lfloor x_1 \rfloor = \left\lfloor n - \frac{1}{2} - \sqrt{2\left[\binom{n}{2} - m\right] + \frac{1}{4}} \right\rfloor,$$

as desired. ■

We can now provide a lower bound for the minimum value of a connected graph in terms of its order and size.

Proposition 6.2 *If G is a connected graph of order n and size m with $\alpha(n,m) = k$, then*

$$\mathrm{val}_{\min}(G) \geq \binom{k+1}{2}\left(n + \frac{k+2}{3}\right) + (m - nk)(k+1). \tag{11}$$

Proof. First, consider the function $g : E(G) \to \{1, 2, \ldots, n-1\}$ defined as follows. Choose $n-1$ edges of G and assign 1 to each of these edges. From the remaining $m-(n-1)$ edges, choose another $\min\{n-2, m-(n-1)\}$ edges of G and assign 2 to each of these edges. At each step, if some edges of G have not been assigned a number, choose the smallest positive integer s not assigned to any edge of G, and assign s to $\min\{n-s, m-\sum_{i=1}^{s-1}(n-i)\}$ of the remaining edges of G.

Next, let f be a γ-min labeling of G. By Corollary 2.3, we may assume that $f(V(G)) = \{0, 1, \ldots, n-1\}$. Notice that, for each integer s with $1 \le s \le n-1$, there are exactly $n-s$ pairs i, j of integers with $0 \le i < j \le n-1$ and $j - i = s$. Consequently, for each such s, at most $n-s$ edges e of G have value $f'(e) = s$. From the way in which the function g is constructed, it follows that

$$\mathrm{val}(f) \ge \sum_{e \in E(G)} g(e).$$

We can now express $\sum_{e \in E(G)} g(e)$ in terms of n, m and k. In particular,

$$\begin{aligned}
\sum_{e \in E(G)} g(e) &= \sum_{i=1}^{k} i(n-i) + \left(m - \sum_{i=1}^{k}(n-i)\right)(k+1) \\
&= n\binom{k+1}{2} - \frac{k}{6}(k+1)(2k+1) \\
&\quad + \left(m - nk + \binom{k+1}{2}\right)(k+1) \\
&= \binom{k+1}{2}\left(n + \frac{k+2}{3}\right) + (m-nk)(k+1),
\end{aligned}$$

as desired. ∎

We now consider the sharpness and some consequences of Proposition 6.2. When $G = P_n$, the right hand side of (11) is $n-1$; while if $G = K_n$, the right hand side of (11) is $\binom{n+1}{3}$. Consequently, by Observation 3.1 and Proposition 5.1, the bound is sharp for paths and complete graphs.

For a connected graph G of order n and a positive integer k, the *kth power* G^k of G is that graph with $V(G^k) = V(G)$ such that $uv \in E(G^k)$ if $d_G(u, v) \le k$. Now let $P_n : v_1, v_2, \cdots, v_n$. Define the γ-labeling f of P_n^k by $f(v_i) = i - 1$ for $1 \le i \le n$. Then P_n^k has order n and size $\sum_{i=1}^{k}(n-i)$, and $\mathrm{val}(f) = \sum_{i=1}^{k} i(n-i)$. Since

$$\mathrm{val}_{\min}(P_n^k) \ge \sum_{i=1}^{k} i(n-i)$$

by Proposition 6.2, it follows that $\text{val}_{\min}(P_n^k) = \sum_{i=1}^{k} i(n-i)$. Thus the bound in Proposition 6.2 is sharp for P_n^k for all integers k and n with $1 \leq k \leq n-1$, including $P_n^1 = P_n$ and $P_n^{n-1} = K_n$.

Let's now consider connected graphs G of order n and size

$$\sum_{i=1}^{2}(n-i) = (n-1) + (n-2) = 2n-3.$$

By Proposition 6.2, any such graph G satisfies

$$\text{val}_{\min}(G) \geq \sum_{i=1}^{2} i(n-i) = 3n-5.$$

We have already noted that P_n^2 is a graph of order n and size $2n-3$ having minimum value $3n-5$. Actually P_n^2 is a maximal outerplanar graph for every positive integer n. (See Figure 2 for P_7^2.) Since every maximal outerplanar graph of order n has size $2n-3$, we have the following observation.

P_7^2 :

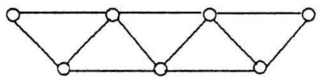

Figure 2: A maximal outerplanar graph

Proposition 6.3 *If G is a maximal outerplanar graph of order $n \geq 2$, then*

$$\text{val}_{\min}(G) \geq 3n-5.$$

Furthermore, this bound is obtained if and only if $G = P_n^2$.

Next we turn to connected graphs G of order n and size

$$\sum_{i=1}^{3}(n-i) = (n-1) + (n-2) + (n-3) = 3n-6.$$

By Proposition 6.2, any such graph G satisfies

$$\text{val}_{\min}(G) \geq \sum_{i=1}^{3} i(n-i) = 6n-14.$$

As mentioned before, P_n^3 is a graph of order n and size $3n-6$ having minimum value $6n-14$. Of course, every maximal planar graph of order $n \geq 3$ has size $3n-6$. Indeed, P_n^3 is maximal planar.

Proposition 6.4 *For every positive integer n, the graph P_n^3 is maximal planar.*

Proof. Let $P_n : v_1, v_2, \cdots, v_n$. The result is true for $n = 1$ and $n = 2$. For $n \geq 3$, we show by induction that there is a planar embedding of P_n^3 in which there is a (triangular) region whose boundary vertices are v_{n-2}, v_{n-1}, and v_n. Clearly, this holds for $n = 3$. Assume, for an integer $k \geq 3$, that there is a planar embedding of P_k^3 in which there is a region R whose boundary vertices are v_{k-2}, v_{k-1}, and v_k. We now place a new vertex v_{k+1} in R and joining v_{k+1} to v_{k-2}, v_{k-1}, and v_k. This produces a planar embedding of the graph P_{k+1}^3 containing a triangular region whose boundary vertices are v_{k-1}, v_k, and v_{k+1}. Since the size of P_{k+1}^3 is $3(k+1) - 6$, it follows that P_{k+1}^3 is maximal planar. ■

Corollary 6.5 *If G is a maximal planar graph of order $n \geq 3$, then* $\mathrm{val}_{\min}(G) \geq 6n - 14$. *Furthermore, this bound is attained if and only if $G = P_n^3$.*

Figure 3 shows the maximal planar graphs P_6^3, P_7^3, and P_8^3 and a γ-min labeling of each. Indeed, there are only two maximal planar graphs of order 6, where $\mathrm{val}_{\min}(P_6^3) = 22$ and $\mathrm{val}_{\min}(K_{2,2,2}) = 26$.

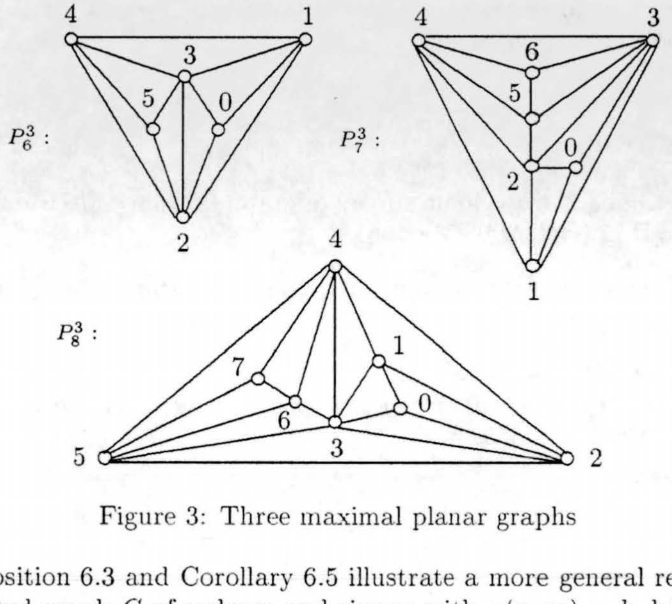

Figure 3: Three maximal planar graphs

Proposition 6.3 and Corollary 6.5 illustrate a more general result. For a connected graph G of order n and size m with $\alpha(n, m) = k$, let $L(n, m)$

denote the lower bound for $\text{val}_{\min}(G)$ given in Proposition 6.2, that is,

$$L(n,m) = \binom{k+1}{2}\left(n + \frac{k+2}{3}\right) + (m-nk)(k+1).$$

For $1 \le k < n-1$ and an integer t with $1 \le t < n-(k+1)$, let $\mathcal{P}_n^{(k,t)}$ denote the class of graphs obtained by adding t edges to P_n^k, where each such edge joins two vertices of P_n^k whose distance is $k+1$ in P_n. Thus $\mathcal{P}_n^{(k,0)} = \{P_n^k\}$. Consequently, if $F \in \mathcal{P}_n^{(k,t)}$, then $\text{val}_{\min}(F) = L(n,m)$. Moreover, if H is a graph of order n having the same size as F but $H \notin \mathcal{P}_n^{(k,t)}$, then $\text{val}_{\min}(H) > \text{val}_{\min}(F)$. These observations give us the following result.

Proposition 6.6 *Let G be a connected graph of order n and size m with $\alpha(n,m) = k$. Then*

$$\text{val}_{\min}(G) = L(n,m)$$

if and only if (i) $k = n-1$ and $G = K_n$ or (ii) $1 \le k \le n-2$ and $G \in \mathcal{P}_n^{(k,t)}$ for some integer t with $0 \le t < n-(k+1)$.

7 Acknowledgments

We are grateful to the referee whose valuable suggestions resulted in producing an improved paper.

References

[1] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* #DS6 (Oct. 2003 Version).

[2] S. W. Golomb, How to number a graph, in *Graph Theory and Computing*. Academic Press, New York (1972) 23-37.

[3] A. Rosa, On certain valuations of the vertices of a graph, in *Theory of Graphs, Pro. Internat. Sympos. Rome 1966*. Gordon and Breach, New York (1967) 349-355.