2-Colouring Doubly-Periodic Graphs

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Abstract

We give a polynomial-time algorithm for determining whether a doubly-periodic infinite graph is 2-colourable. This is a consequence of an upper bound established on the length of a shortest odd cycle.

1 Introduction

A *doubly-periodic graph*, or DP-graph, G is an infinite graph whose vertex-set can be paritioned into sets $V_{ij} = \{v_{ij}^1, \dots v_{ij}^n\}$, $i, j \in Z$, in such a way that

1. a vertex in V_{ij} is adjacent to vertices in V_{pq} only if $|i-p| \leq 1$ and $|j-q| \leq 1$ and,
2. if v_{ij}^k is adjacent to v_{pq}^r, then $v_{i+a,j+b}^k$ is adjacent to $v_{p+a,q+b}^r$ for all integers a, b.

In particular, the subgraphs induced by the sets $V_{i,j}$ are pairwise isomorphic. Given such a partition, these subgraphs are called the *cells* of the DP-graph, and denoted G_{ij}. Furthermore, for all i, j, p, and q; $f(v_{ij}^k) = v_{pq}^k$ is an isomorphism from G_{ij} to G_{pq}. We will say that vertices with the same superscript are of the same *type*.

Intuitively, we think of the cells as being the cells of an infinite square grid, each containing a copy of some finite graph, with edges only within cells or between neighbouring cells, and with the same pattern of adjacencies between any pair cells which are adjacent in the same way (horizontally, vertically, or either of the two diagonal orientations).

Thus, we may define a DP-graph by describing a single cell G_{00}, and giving the neighbours in G_{01}, G_{11}, G_{10}, and $G_{1,-1}$ of each $v_{00}^k \in G_{00}$. Henceforth we will assume our DP-graphs are presented in this way, and the partition into cells is the natural one implied by the this presentation.

The term *doubly-periodic* arises from the existence of a collection of isomorphisms T_{ab} $(a, b \in \mathbf{Z})$ of G, defined by $T_{ab}(v_{ij}^k) = v_{i+a,j+b}^k$ for all integers i and j. Each T_{ab} is simply a translation of the cells of G by a cells horizontally and b cells vertically. If v and w are two vertices with the same type, then there is a unique T_{ab} which maps v to w. For convenience, we will denote this by t_{vw}.

We use $x(v)$ (resp. $y(v)$) to indicate the unique i (resp. j) such that $v \in G_{ij}$, that is, the x or y coordinate of the cell containing v.

If H is a subgraph of a DP-graph G, then the *height* of H is defined to be $max_{v,w \in H}\{|y(v) - y(w)|\}$, that is, the maximum vertical distance between cells that intersect with H. We define the *width* of H similarly.

S. Burr proved that the problem of whether a pre-colouring of a finite number of vertices of a DP-graph F extends to a k-colouring of G is undecidable for every fixed integer $k \geq 3$ [1]. He notes that this remains true even if G is planar and has maximum degree four. The proof is to transform the Halting problem by constructing a DP-graph whose k-colourings model the computation of a universal Turing machine. The proof depends strongly on the pre-colouring, as it is how the initial settings of the machine are obtained. Burr's result was generalized to graph homomorphisms in [2], but the class of graphs considered needed to be enlarged slightly in order for the transformation to work. The transformation is from Burr's theorem, and is accomplished by showing that standard tools from the study of the complexity of graph homomorphisms also work for the graphs being considered.

In this paper we study the question of whether a DP-graph is 2-colourable. No vertices are pre-coloured. We show that this question is decidable in time polynomial in n, the number of vertices in each cell. In particular, we prove that if a DP-graph G has an odd cycle (i.e., is not 2-colourable), then there is an odd cycle contained in any $16n^2 \times 4n$ block of cells. Thus, it can be decided whether G is 2-colourable by examining a small, finite, subgraph. In the conclusion we briefly discuss the possibilty of proving such a result for k-colouring, $k \geq 3$, and examine colouring and homomorphism problems for singly-periodic graphs, which are defined similarly to DP-graphs.

2 2-colourability

We prove the following result, which immediately yields a polynomial-time algorithm for determining 2-colourability of a DP-graph.

Theorem 1 *If a DP-graph G contains an odd cycle, then the subgraph of G induced by the cells G_{ij}, $0 \leq i < 16n^2$, $0 \leq j < 4n$ contains an odd cycle.*

Before proceeding with the proof, we note some simple properties of a DP-graph G with n vertices in each cell.

Lemma 2 *If there is a path from a vertex v to a vertex w in G, then there is a path of length less than n from v to a vertex with the same type as w.*

Proof: If $P = (v = v_0, v_1, \ldots, v_r = w)$ is a shortest path from v to w and $r \geq n$, then P must contain two vertices of the same type, say v_i and v_j with $i < j$. Then $v_0, v_1, \ldots, v_i, t_{v_j v_i}(v_{j+1}), t_{v_j v_i}(v_{j+2}), \ldots, t_{v_j v_i}(v_r)$ is a walk from v to a vertex with the same type as w that is shorter than P, and thus contains a such a path that is shorter than P, a contradiction. ■

This lemma allows us to find short paths from a given vertex to a given type of vertex in the same connected component.

Lemma 3 *If a connected component H of a DP-graph G has height at least n, then H contains vertices v and w and a path $P = (v = v_0, v_1, \ldots, v_r = w)$ such that*

1. *$r \leq n$,*
2. *v and w are the same type,*
3. *$y(v) \neq y(w)$.*

Proof: Choose vertices $v, w \in H$ so that $y(v) - y(w) \geq n$, and let Q be a path from v to w. Since Q has length at least n, as in the proof of Lemma 2 it must contain two vertices v' and w' of the same type such that $y(v') \neq y(w')$. Choose such v' and w' so that the length of a shortest path R from v' to w' is minimised.

We claim that R, v', and w' have the properties required for P, v, and w. Properties (2) and (3) are satisfied by the choice of v' and w', so we need only verify that R has length less than or equal to n.

Suppose this is not the case. Write R as $v' = v_0, v_1, \ldots, v_r = w'$. The subpath $v_0, v_1, \ldots, v_{r-1}$ must contain two vertices v_i and v_j (with $i < j$) of the same type. If $y(v_i) \neq y(v_j)$, then v_i and v_j satisfy the same properties as v' and w', but have a shorter path between them, a contradiction. Otherwise $y(v_i) = y(v_j)$, but then $v_0, v_1, \ldots, v_i, t_{v_j v_i}(v_{j+1}), \ldots, t_{v_j v_i}(w')$ is a path from v to a vertex of the same type as v which is shorter than R. Furthermore, since $y(v_i) = y(v_j)$, we know that $y(t_{v_j v_i}(w')) = y(w') \neq y(v')$, and we again obtain a contradiction. ■

This implies that if a connected component H of a DP-graph has height at least n, then there is a type of vertex in H and a constant $0 < c <= n$ so that from any vertex v of that type there is a short path from v to a vertex w with the same type and with $y(v) - y(w) = c$ or $-c$, as desired. This is accomplished by simply applying an appropriate translation mapping to the path found in this lemma.

We may now proceed to place an upper bound on the minimum height of an odd cycle in a DP-graph. The proof of this result is constructive, so the following notation will be useful. If $P = v_0, v_1, \ldots, v_r$ and $Q = w_0, w_1, \ldots, w_s$ are paths with $v_r = w_0$, then $P \circ Q$ is the walk $P \circ Q = v_0, v_1, \ldots, v_r, w_1, \ldots, w_s$, *i.e.* the concatenation of P and Q. We denote by P^r the reverse of P, *i.e.* $P^r = v_r, v_{r-1}, \ldots, v_0$.

Lemma 4 *If C is an odd cycle in a DP-graph G then G contains an odd cycle C' with height less than $4n$.*

Proof: Let C be an odd cycle of minimum height in a DP-graph G. Suppose C has height h that is greater than $4n$, and suppose without loss of generality that C contains a vertex a with $y(a) = 0$ but no vertex a' with $y(a') < 0$. Then C contains at least one vertex b with $y(b) = h$ but no vertex b' with $y(b') > h$. Let C also be chosen so that $|\{b : y(b) = h\}|$ is minimised. We will obtain a contradiction by constructing an odd cycle with either smaller height or fewer such vertices.

Let v be a vertex in C with $y(v) = h$. Write C as $v_0, v_1, \ldots, v_r, v_0$, where $y(v_0) = 0$. Now choose i and j so that $P_0 = v_i, v_{i+1}, \ldots, v_j$ is the shortest path in C such that P_0 contains v, and $y(v_i) = y(v_j) = 2n$. This implies that $y(v_k) > 2n$ for all $i < k < j$. Let $P_1 = v_j, v_{j+1}, \ldots, v_r, v_0, v_1, \ldots, v_{i-1}, v_i$, *i.e.* the part of C induced by the edges not in P.

Since C has height greater than n, the connected component containing C contains a path $Q = q_0, q_1, \ldots, q_r$ as described in Lemma 3. Assume without loss of generality that $y(q_0) > y(q_r)$.

Now, let P_2 be a path of length less than n from v_i to a vertex q_0' with the same type as q_0, and let $P_3 = t_{q_0 q_0'}(Q)$. Label the end of P_3 with q_r', and let $P_4 = (t_{q_0' q_r'}(P_2))^r$. Observe that $P_2 \circ P_3 \circ P_4$ is a walk from v_i to $v_i' = t_{q_0' q_r'}(v_i)$, and that $y(v_i') < y(v_i)$.

Similary, we construct paths P_5, P_6, P_7 so that $P_5 \circ P_6 \circ P_7$ is a walk from v_j to $v_j' = t_{q_0' q_r'}(v_j)$.

Finally, let $P_8 = t_{q_0' q_r'}(P_0)$.

Now observe that $W = P_1 \circ P_2 \circ P_3 \circ P_4 \circ P_8 \circ P_7^r \circ P_6^r \circ P_5^r$ is a closed walk in G. Furthermore, since $P_1 \cup P_8$ contains an odd number of edges, and the paths in each pair P_2, P_4; P_5, P_7; and P_3, P_6 are translations of each other and so have the same number of edges, W has odd length. Hence W contains an odd cycle. To complete the proof, we need only show that there is no vertex w in P_i, $2 \leq i \leq 8$, with $y(w) < 0$ or $y(w) \geq h$.

If $w \in P_2$, then w has distance less than n from v_i, and $y(v_i) = 2n$, so $n < y(w) < 3n$.

For the same reason, $n < y(q_0') < 3n$. If $w \in P_3$, then (because P_3 has length no greater than n), we have $0 < y(w) < 4n$.

If $w \in P_4$, then $w = t_{q_0' q_r'}(w')$ for some $w' \in P_2$. Because of the inequality $0 < y(q_0') - y(q_r') \leq n$, we have $0 < y(w) < 3n$.

Similarly, for any $w \in P_5 \cup P_6 \cup P_7$ we have $0 < y(w) < 4n$.

Finally, if $w \in P_8$, then $w = t_{q_0' q_r'}(w')$ for some $w' \in P_1$. This implies $2n < y(w') \leq 4n$. Again, because of the inequality $0 < y(q_0') - y(q_r') \leq n$, we have $n < y(w) < 4n$.

Thus, the walk W has height no greater than h. If it has height h, then the only vertices b with $y(b) = h$ in W must be in P_1. Since P_1 is a subgraph of $C - \{v\}$, where $y(v) = h$, the number of such vertices in W is strictly less than the number in C. The odd cycle contained in W clearly has these properties as well, so we are done. ■

The following result will be convenient as well. The proof is identical, aside from reversal of the co-ordinates.

Corollary 5 *If C is an odd cycle in a DP-graph G then G contains an odd cycle C' with width less than $4n$.*

This implies that to determine if a DP-graph is 2-colourable, we need only consider the subgraph induced by the cells G_{ij}, $0 \leq j < 4n$. Our next result allows us to place a bound on the range of the first co-ordinate as well.

Proof of Theorem 1: We construct a new DP-graph G' by setting $G'_{ij} = \bigcup_{k=4nj}^{4n(j+1)-1} G_{ik}$. If v, w are in adjacent cells then $uv \in E(G')$ if and only if $uv \in E(G)$ and it is not the case that $y(u) = 4nk$ and $y(v) = 4nk - 1$ (or vice versa) for any k. In other words, we combine columns of G in chunks of height $4n$ into single cells of G', with no edges between vertically adjacent cells of G'. Observe that if G contains an odd cycle, then that cycle will be preserved in G' and will lie in a single row of G'. Also observe that each cell in G' contains $4n^2$ vertices.

Now, applying Corollary 5 we see that G' has a cycle of width less than $16n^2$. Since there are no edges between vertically adjacent cells of G', this cycle must have height one. Thus, the cycle is contained in the cells $G'_{i,0}$, $0 \leq i < 16n^2$, and so is contained in G as well in the cells G_{ij}, $0 \leq i < 16n^2$, $0 < j < 4n$. ■

This yields the promised algorithm.

Corollary 6 *If G is a DP-graph with n vertices in each cell, then 2-colourability of G can be determined in $O(n^8)$.*

Proof: By Theorem 1 we can restrict out attention to a finite subgraph F of G with $O(n^3)$ cells, or $O(n^4)$ vertices. This can be checked for 2-colourability in $O(|E(F)|)$, or $O(n^8)$ time. ■

3 Concluding remarks

The question of deciding whether a DP-graph is k-colourable seems to be related to tilings of the plane. One can think of the collection of k-colourings of the cells $G_{i,j}$ as being a collection of tiles, and regard the adjacencies between neighbouring cells are compatibility relations between tiles. The question is whether tiles can be used to tile the plane so that each tile is adjacent only to tiles with which it is compatible. Problems of this sort, involving as few as nine tiles, are known to be undecidable [3]. The main issue in using a similar technique seems to be whether there exists a finite graph G_{ij} whose collection of k-colourings is rich enough.

By contrast, the question of whether a *singly-periodic graph*, the subgraph induced by a single row of a DP-graph, is k-colourable is decidable for every fixed positive integer k. (One can also define singly periodic graphs formally in a manner similar to the definition of DP-graphs.) To see this, let G be a singly periodic graph and note that each cell has the same finite number of k-colourings, say N. Construct a finite digraph C whose vertices are the k-colourings of G, with an arc from colouring x to colouring y in C if y is permissible on cell $i+1$ whenever x is used on cell i. A k-colouring of G corresponds to an infinite, two way, directed walk in C. If such a walk exists, then C contains a directed cycle, and conversely. Thus, there is a k-colouring of G if and only if C contains a directed cycle. If $c_0, c_1, c_2, \ldots, c_{t-1}, c_0$ is a directed cycle in C, then the singly-periodic graph G can be coloured in blocks of t consecutive cells. In particular, if a singly-periodic graph has a k-colouring, then it has a *periodic* k-colouring (*i.e.*, there is a constant t such that the colouring of cell G_i is identical to the colouring of cell G_{i+t}, for all $i \in \mathbf{Z}$. Similar statements hold for homomorphisms of a singly-periodic graph to a fixed finite graph H.

The conjecture of Wang that any set of tiles which permits tiling the plane also permits a *doubly-periodic* tiling (defined in the obvious way), is known to be false (see [3]). This follows from the undecidability of the tiling problems, for (as is pointed out in [3]) if a periodic tiling always existed then there would be a decision procedure as to whether any given set of tiles suffices. Thus, another question that arises for k-colourings of DP-graphs is whether there always exists a doubly-periodic k-colouring.

References

[1] S. Burr, Some undecidable problems involving the edge-colouring and vertex colouring of graphs. *Discrete Mathematics* **50** (1984), 171-177.

[2] P. Dukes, H. Emerson and G. MacGillivray, Undecidable generalized colouring problems. *JCMCC* **26** (1998), 97-112.

[3] R.M. Robinson, Undecidability and nonperiodicity for tilings of the plane. *inventiones Math.* **12** (1971), 177-209.