Disjoint union-free designs with block size three

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Abstract

Let $n \geq k$ be positive integers. A famous question of Erdös asks for the largest size of a family $\mathcal{F}$ of k-subsets of an n-set such that there are no distinct $A, B, C, D \in \mathcal{F}$ with $A \cap B = C \cap D = \emptyset$ and $A \cup B = C \cup D$. In the case $k = 3$, Füredi has conjectured that for sufficiently large n, $|\mathcal{F}| \leq \binom{n}{2}$ and has constructed a family of examples achieving equality in which $\mathcal{F}$ is the block set of a design. Here, we characterize the designs meeting this conjectured bound.

1 Introduction

Let $n \geq k$ be positive integers, and let X be an n-set. Let $\binom{X}{k}$ denote the set of all $\binom{n}{k}$ k-subsets of X. A set $\mathcal{F} \subset \binom{X}{k}$ is *disjoint union-free* (DUF) if all disjoint pairs of elements of $\mathcal{F}$ have distinct unions; that is, if for every $A, B, C, D \in \mathcal{F}$, $A \cap B = C \cap D = \emptyset$ and $A \cup B = C \cup D$ implies $\{A, B\} = \{C, D\}$. For a given n, k, the maximum size of such a family $\mathcal{F}$ is denoted $f_k(n)$. Erdös asked in [2] to determine the values of $f_k(n)$. Füredi [3] determined the exact value of $f_2(n)$ for infinitely many values of n. Füredi also conjectured in [4] that $f_3(n) \leq \binom{n}{2}$ for sufficiently large n.

An (n,k,λ) *design* is a pair $(X,\mathcal{B})$ with $|X| = n$ and $\mathcal{B}$ a collection of k-subsets, or *blocks*, of X such that every pair of distinct points in X is contained in exactly λ blocks. We call λ the *index* of the design. An (m,k,λ) design $(Y,\mathcal{B}')$ is a *sub-design* of $(X,\mathcal{B})$ if $Y \subset X$ and $\mathcal{B}' \subset \mathcal{B}$ (counting multiplicities).

We will abuse terminology and say an $(n,3,\lambda)$ design $(X,\mathcal{B})$ is DUF if $\mathcal{B}$ is DUF. It is not hard to see that $\lambda \leq 3$ for a DUF $(n,3,\lambda)$ design to exist (see the proof of Lemma 2.2). Any $(n,3,1)$ design is trivially DUF, and certain $(n,3,2)$ designs with the DUF property are discussed in [1]. When $\lambda = 3$, n must be odd, and there are exactly $\binom{n}{2}$ blocks in an $(n,3,3)$ design.

The following gives a family of examples of DUF $(n,3,3)$ designs. This was first shown in [4] and we recall the argument here for completeness.

Proposition 1.1. [4] *DUF* $(n,3,3)$ *designs exist for* $n \equiv 1,5 \pmod{20}$.

Proof: For $n \equiv 1,5 \pmod{20}$ there exists an $(n,5,1)$ design [5]. Replace each block E of this design with $\binom{E}{3}$. Since each $(E,\binom{E}{3})$ is a $(5,3,3)$ design, the result is an $(n,3,3)$ design, say with blocks $\mathcal{B}$. We show $\mathcal{B}$ is DUF. Suppose $A \cap B = C \cap D = \emptyset$ but $A \cup B = C \cup D$ for distinct $A,B,C,D \in \mathcal{B}$. Observe A and B must come from different blocks E,F of the underlying $(n,5,1)$ design, and $|E \cap F| \leq 1$. A similar statement is true for C and D. So we have, say, $A,C \subset E$ and $B,D \subset F$. But A and C are distinct, so $|A \cap D| = 1$. Similarly $|B \cap C| = 1$. So $|E \cap F| \geq 2$, a contradiction. □

Perhaps surprisingly, our main theorem in this note (Theorem 2.6) characterizes DUF $(n,3,3)$ designs as only those coming from the construction in Proposition 1.1.

2 The main result

Suppose $(X,\mathcal{B})$ is a DUF $(n,3,3)$ design. Define the mapping $\tau : \binom{X}{2} \to \binom{X}{3}$ by $\tau(\{a,b\}) = \{x,y,z\}$ if and only if $\{a,b,x\}$, $\{a,b,y\}$, $\{a,b,z\} \in \mathcal{B}$. Let $\mathcal{A} = \tau(\binom{X}{2})$ be the multiset whose $\binom{n}{2}$ elements are the images all pairs in X under τ. (It will turn out in fact that $\mathcal{A}$ has no repeated elements, but *a priori* this may not be the case.) We now give a series of lemmas, the first of which is a restatement of the DUF property in terms of τ.

Lemma 2.1. $|\tau(\{u,v\}) \cap \tau(\{x,y\})| \geq 2$ *implies* $\{u,v\} \cap \{x,y\} \neq \emptyset$.

Lemma 2.2. $(X, \mathcal{A})$ *is an* $(n,3,3)$ *design.*

Proof: Since $\mathcal{A}$ consists of $\binom{n}{2}$ 3-subsets of X, it suffices to show that every pair $\{x,y\}$ of distinct points appears in *at most* three elements of $\mathcal{A}$ (counting multiplicities). Suppose for contradiction that $\{x,y\} \subset \tau(P_i)$, $i = 1,2,3,4$, where $P_i \in \binom{X}{2}$ are distinct. By Lemma 2.1, we must have any two of the P_i intersecting pairwise. But four distinct pairs which intersect pairwise must all intersect in a point. So for some $a \in X$, $P_i = \{a, b_i\}$, $i = 1,2,3,4$, with the b_i distinct. But then $\{a, x, b_i\} \in \mathcal{B}$ for $i = 1,2,3,4$, a contradiction to $(X, \mathcal{B})$ having index 3. □

In light of Lemma 2.2, we may call the elements of $\mathcal{A}$ blocks. We now claim that there are no blocks appearing exactly twice in $\mathcal{A}$.

Lemma 2.3. *If* $\{x,y,z\}$ *appears twice in* $\mathcal{A}$ *then it appears thrice in* $\mathcal{A}$.

Proof: Suppose $\tau(\{a,b\}) = \tau(\{a,c\}) = \{x,y,z\}$. Now by Lemma 2.2, each of $\{x,y\}$, $\{x,z\}$, and $\{y,z\}$ occur in a third block of $\mathcal{A}$, say $\{x,y,z'\}$, $\{x,y',z\}$, and $\{x',y,z\}$, respectively. Either these three blocks all equal $\{x,y,z\}$ or they are all distinct. In the latter case, we suppose these blocks are $\tau(P_1)$, $\tau(P_2)$, and $\tau(P_3)$, respectively, for $P_i \in \binom{X}{2}$. By Lemma 2.1, each of P_1, P_2, P_3 must intersect both $\{a,b\}$ and $\{a,c\}$. But at most one P_i can equal $\{b,c\}$. So, say, $P_1 = \{a,d\}$ and $P_2 = \{a,e\}$ with b,c,d,e distinct. Observe now that $\{a,x\}$ occurs in $\mathcal{B}$ with each of b,c,d,e, again contradicting index 3. □

Lemma 2.4. *Every pair of points in* X *is either contained in a thrice repeated block of* $\mathcal{A}$, *or in a* $(5,3,3)$ *sub-design of* $\mathcal{B}$ *(and hence of* $\mathcal{A}$*).*

Proof: Let $\{x,y\} \in \binom{X}{2}$ and suppose $\tau(\{a,b\}) = \{x,y,z\}$. Assume $\{x,y,z\}$ is not a thrice repeated block of $\mathcal{A}$. Then by Lemmas 2.2 and 2.3, we have blocks

$$\{x,y,z'\}, \{x,y,z''\}, \{x,y',z\}, \{x,y'',z\}, \{x',y,z\}, \{x'',y,z\} \in \mathcal{A}$$

with x, x', x'' distinct, y, y', y'' distinct, and z, z', z'' distinct. Suppose these blocks are $\tau(P_1), \ldots, \tau(P_6)$, respectively, for $P_1, \ldots, P_6 \in \binom{X}{2}$. Now each of $P_1, \ldots, P_6$ must intersect $\{a,b\}$ by Lemma 2.1, and moreover $P_1 \cap P_2$, $P_3 \cap P_4$, $P_5 \cap P_6 \neq \emptyset$.

Case 1. $P_i \cap P_{i+1} \cap \{a,b\} \neq \emptyset$ for all $i = 1,3,5$.

Assume (by relabeling if necessary) that $a \in P_1, P_2, P_3, P_4$. We have $\tau(\{a,c\}) = \{x,y,z'\}$, $\tau(\{a,d\}) = \{x,y,z''\}$, $\tau(\{a,c'\}) = \{x,y',z\}$, and $\tau(\{a,d'\}) = \{x,y'',z\}$ for some distinct c,d,c',d'. But then $\tau(\{a,x\}) = \{b,c,d\} = \{b,c',d'\}$, a contradiction.

CASE 2. $P_i \cap P_{i+1} \cap \{a,b\} = \emptyset$ for some $i = 1, 3$, or 5.

Assume $P_1 = \{a,c\}$ and $P_2 = \{b,c\}$ for some $c \in X$. Then we have $\{b,c\} \subset \tau(\{a,x\}), \tau(\{a,y\})$. Suppose first that $\{b,c\} \subset \tau(\{a,w\})$ with w,x,y distinct. Then $\{a,b\}$ occurs with w,x,y,z. So we conclude that $w = z$. Also, $\{a,c\}$ occurs with w,x,y,z'. So $z = w = z'$, a contradiction to z,z' being distinct. Otherwise, it must be that $\{b,c\} \subset \tau(\{x,y\})$. We apply a similar reasoning with $\{a,c\} \subset \tau(\{b,x\}), \tau(\{b,y\})$ to get either a contradiction to z,z'' being distinct, or $\{a,c\} \subset \tau(\{x,y\})$. Therefore, $\tau(\{x,y\}) = \{a,b,c\}$. Then $\tau(\{b,y\}) = \{a,x,c\}$, and $\tau(\{c,y\}) = \{a,x,b\}$. Suppose $\{a,x,d\}$ is the third block in $\mathcal{A}$ containing $\{a,x\}$. By Lemma 2.1, either $\tau(\{b,c\}) = \{a,x,d\}$ or $\tau(\{t,y\}) = \{a,x,d\}$ for some $t \neq b,c$. In the latter case, $\{a,y\}$ occurs with b,c,x,t in $\mathcal{B}$. So $x = t$, but this is impossible because $\{t,y,x\} \in \mathcal{B}$. Therefore, $\tau(\{b,c\}) = \{a,x,d\}$, and in particular, $\{a,b,c\} \in \mathcal{B}$. Taking an inventory of triples in $\mathcal{B}$, we see that every 3-subset of $\{a,b,c,x,y\}$ is a block in $\mathcal{B}$, and hence in $\mathcal{A}$. This proves $\{x,y\}$ is in a $(5,3,3)$ sub-design of both $\mathcal{A}$ and $\mathcal{B}$. □

Observe now that the characterization is "close" to complete. It remains only to rule out the existence of thrice repeated blocks in $\mathcal{A}$. To this end, we present the following result.

Lemma 2.5. *If $\{x,y,z\}$ is thrice repeated in $\mathcal{A}$, then for some distinct a,b,c,d we have $\tau(\{a,b\}) = \tau(\{a,c\}) = \tau(\{a,d\}) = \{x,y,z\}$.*

Proof: Suppose $\{x,y,z\}$ is thrice repeated in $\mathcal{A}$. By Lemma 2.1, the pre-images of $\{x,y,z\}$ under τ must consist of three intersecting pairs. If the three pairs all intersect in a point, we are done. So suppose $\tau(\{a,b\}) = \tau(\{a,c\}) = \tau(\{b,c\}) = \{x,y,z\}$ for some distinct a,b,c.

Consider the pair $\{a,b\}$. By Lemma 2.4, it is either in a thrice repeated block of $\mathcal{A}$, or belongs to a $(5,3,3)$ sub-design in $\mathcal{B}$. But this latter case is impossible, since then $\{x,y,z\}$ also belongs to this sub-design and could not be thrice repeated. So for some t, $\{a,b,t\}$ is thrice repeated in $\mathcal{A}$. Since $\{a,b\} \subset \tau(\{c,x\}), \tau(\{c,y\}), \tau(\{c,z\})$, it follows from Lemma 2.2 that $\{a,b,t\} = \tau(\{c,x\}), \tau(\{c,y\}), \tau(\{c,z\})$. But then $\tau(\{t,c\}) = \{x,y,z\}$. Using Lemma 2.2 again gives $t \in \{a,b\}$, and this is absurd. □

We are now in a position to characterize the DUF $(n, 3, 3)$ designs.

Theorem 2.6. *In a DUF $(n, 3, 3)$ design, every pair of points is contained in a unique $(5, 3, 3)$ sub-design.*

Proof: By Lemma 2.4, it suffices to show there are no thrice repeated blocks in $\mathcal{A}$. Suppose $\{x, y, z\}$ is thrice repeated in $\mathcal{A}$. By Lemma 2.5, we have $\{x, y, z\} = \tau(\{a, b\})$, $\tau(\{a, c\})$, $\tau(\{a, d\})$ for some distinct a, b, c, d. Then also $\tau(\{a, x\}) = \tau(\{a, y\}) = \tau(\{a, z\}) = \{b, c, d\}$. We must have $\tau(\{b, x\}) = \{a, r, s\}$ for some r, s. Consider the two cases from Lemma 2.4.

Case 1. $\{a, r, s\}$ is thrice repeated in $\mathcal{A}$.

Using Lemma 2.5, the three pre-images of $\{a, r, s\}$ under τ must all meet in either b or x. Suppose the former; that is, $\{a, r, s\} = \tau(\{b, x\})$, $\tau(\{b, t\})$, $\tau(\{b, u\})$, where t, u, x are distinct. Then dually we have $\tau(\{b, a\}) = \{t, u, x\}$. But from before, $\tau(\{b, a\}) = \{x, y, z\}$. So $\{t, u\} = \{y, z\}$. Now $\tau(\{b, r\}) = \{x, y, z\}$, so by Lemma 2.2 it must be that $\{b, r\}$ equals either $\{a, c\}$ or $\{a, d\}$. Either case is impossible as $a \neq b$. The case when the pre-images of $\{a, r, s\}$ all meet in x is similar.

Case 2. $\{a, r, s\}$ belongs to a $(5, 3, 3)$ sub-design of $\mathcal{B}$.

In particular, $\tau(\{a, x\}) = \{b, r, s\}$. But we already have $\tau(\{a, x\}) = \{b, c, d\}$. So $\{r, s\} = \{c, d\}$ and $\{a, b, c, d, x\}$ are points of a $(5, 3, 3)$ sub-design in $\mathcal{A}$ and in $\mathcal{B}$. This contradicts $\{b, c, d\}$ being thrice repeated in $\mathcal{A}$.
□

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A few more $(K_v, K_5 - e)$-designs

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Abstract

Let G be a simple graph without isolated vertices. A (K_v, G)-design is a partition of the edges of K_v into subgraphs each of which is isomorphic to G. In this note, we remove all the left values v summarized in [5] for the existence of a $(K_v, K_5 - e)$-design when $v \equiv 1 \pmod{18}$, and establish that a $(K_v, K_5 - e)$-design exists for any integer $v \equiv 1 \pmod{18}$ and $v \geq 19$. We also construct a $(K_v, K_5 - e)$-design for $v = 28, 46, 82$.

1 Introduction

Let K_v be a complete graph on v vertices. Let $G = (V(G), E(G))$ be a simple graph without isolated vertices. A (K_v, G)-*design* is a partition of edges of K_v into subgraphs (G-*blocks*) each of which is isomorphic to G. When the graph G is itself a complete graph K_k, the (K_v, K_k)-design is known as a $(v, k, 1)$-BIBD. If there exists a (K_v, G)-design, then

(1) $v(v-1) \equiv 0 \pmod{2|E(G)|}$, and

(2) $(v-1) \equiv 0 \pmod{d}$, where d is the greatest common divisor of the degrees of the vertices of G.

It was proved in [11] that the necessary conditions (1) and (2) for the existence of a (K_v, G)-design are asymptotically sufficient, that is, there exists an integer $N(G)$ such that there is a (K_v, G)-design for any integer $v \geq N(G)$ satisfying the necessary conditions (1) and (2).

The existence of a (K_v, G)-design for various graphs G has been studied in literatures (see, [3, 5, 6, 7, 12]). The case where G is a graph with at most four vertices has been solved completely in [2]. If G has no isolated vertices and $|V(G)| = 5$, the existence problem of a (K_v, G)-design has been very nearly solved in [1, 4, 8, 9, 10].

In what follows, we denote $K_5 - e$ by $[a, b, c, d, e]$ with vertex set $V = \{a, b, c, d, e\}$ and edge set $E = \{ab, ac, ad, bc, bd, be, cd, ce, de\}$.

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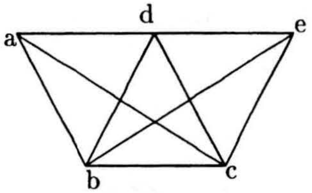

The necessary condition of the existence of a $(K_v, K_5 - e)$-design is $v \equiv 0, 1 \pmod 9$ and $v \neq 9, 10, 18$. For the sufficiency we have the following result (see, for example, [5]).

Lemma 1.1 *If* $v \equiv 1 \pmod{18}$ *and* $v \neq 37$, 55, 73, 109, 397, 415, 469, 487, 505, 541, 613, 685, *then there exists a* $(K_v, K_5 - e)$*-design.*

In this note, we remove all the left cases in Lemma 1.1. We also construct $(K_v, K_5 - e)$-design for $v = 28, 46, 82$.

2 Working lemmas

Denote the (K_v, G)-design by $K_v \to G$ sometimes for convenience. Let $K_{m_1, m_2, \cdots, m_n}$ be the complete multipartite graph with vertex set $V = \bigcup_{i=1}^{n} V_i$, where V_i $(1 \leq i \leq n)$ are disjoint sets with $|V_i| = m_i$ $(i = 1, 2, \cdots, n)$. We also denote the $(K_{m_1, m_2, \cdots, m_n}, G)$-design by $K_{m_1, m_2, \cdots, m_n} \to G$.

The following lemmas are well illustrated in [1].

Lemma 2.1 ([1]) *If* $K_{n_1, n_2, \cdots, n_h} \to G$ *and* $K_{n_i} \to G$ *for* $1 \leq i \leq h$, *then* $K_n \to G$ *where* $n = \sum_{i=1}^{h} n_i$.

Lemma 2.2 ([1]) *If* $K_{n_1, n_2, \cdots, n_h} \to G$ *and* $K_{n_i + 1} \to G$ *for* $1 \leq i \leq h$, *then* $K_n \to G$ *where* $n = \sum_{i=1}^{h} n_i + 1$.

Lemma 2.3 *If a* (K_{r,r,r,r_1}, G)*-design and a* (K_{r,r,r,r_2}, G)*-design both exist, then so does a* $(K_{pr,pr,pr,(p-q)r_1 + qr_2}, G)$*-design for* $p \neq 2, 6$, $0 \leq q \leq p$.

Proof The existence of a decomposition of $K_{p,p,p,p}$ into K_4 is equivalent to that of a pair of orthogonal Latin square of order p; the latter one is known to exist for $p \neq 2$ or 6. Take such a $K_{p,p,p,p}$ on sets $X_1 \bigcup X_2 \bigcup X_3 \bigcup X_4$ where $|X_i| = p$, and let X_i^* denote the sets obtained from X_i by replacing each $x_i \in X_i$ by its r copies, $x_i^1, x_i^2, ..., x_i^r$, for $i = 1, 2, 3$. To obtain the X_4^*, we replace each of q elements $x_4 \in X_4$ by its r_2 copies, $x_4^1, x_4^2, ..., x_4^{r_2}$, and the rest $p - q$ elements are respectively repeated r_1 times. Let $V(K_{pr,pr,pr,(p-q)r_1+qr_2}) = X_1^* \bigcup X_2^* \bigcup X_3^* \bigcup X_4^*$, and then we can obtain $K_{pr,pr,pr,(p-q)r_1+qr_2} \to \{K_{r,r,r,r_1}, K_{r,r,r,r_2}\}$ by the existence of a decomposition of $K_{p,p,p,p}$ into K_4. By the assumption, we obtain that a $(K_{pr,pr,pr,(p-q)r_1+qr_2}, G)$-design exists for $p \neq 2, 6$, $0 \leq q \leq p$. □

3 The case $v \equiv 1 \pmod{18}$

In this section we first construct a $(K_v, K_5 - e)$-design where $v =$ 37, 55, 73, 109, 397, 415, 469, 487, 505, 541, 613, 685.

Lemma 3.1 *There exists a $(K_v, K_5 - e)$-design for $v =$ 37, 55, 73, 109.*

Proof For $v =$ 37, 55, 73, 109, a $(K_v, K_5 - e)$-design is constructed by listing its base $(K_5 - e)$-blocks as follows (where $V(K_v) = Z_v$), respectively.

$K_{37} \to K_5 - e$: $[0, 1, 3, 8, 21]$, $[0, 4, 14, 26, 35]$.

$K_{55} \to K_5 - e$: $[0, 13, 21, 39, 1]$, $[0, 1, 11, 41, 34]$, $[0, 2, 6, 52, 33]$.

$K_{73} \to K_5 - e$: $[0, 4, 11, 5, 51]$, $[0, 12, 30, 22, 54]$, $[0, 2, 15, 50, 36]$, $[0, 3, 19, 56, 47]$.

$K_{109} \to K_5 - e$: $[0, 8, 18, 30, 43]$, $[0, 14, 29, 52, 96]$, $[0, 16, 33, 70, 101]$, $[0, 1, 6, 51, 3]$, $[0, 4, 11, 77, 57]$, $[0, 9, 69, 90, 43]$. □

Lemma 3.2 *There exists a $(K_v, K_5 - e)$-design for $v =$ 397, 415, 505, 541, 613, 685.*

Proof Note that $K_5 - e$ is isomorphic to $K_{1,1,1,2}$. By Lemma 2.3 with $p = 18$, $r = 1$, $r_1 = r_2 = 2$ and $q = 18$ there exists a $(K_{18,18,18,36}, K_5 - e)$-design. There exists a $(K_{6,6,6,6}, K_5 - e)$-design from [1]. Applying Lemma 2.3 with $p = 3$ and $r = r_1 = r_2 = 6$ we get a $(K_{18,18,18,18}, K_5 - e)$-design.

For $v =$ 397, 415, 505, 541, 613, 685, we construct a $(K_v, K_5 - e)$-design as follows.

$K_{397}, K_{415} \to K_5 - e$: Applying Lemma 2.3 with $p = 5$, $r = r_1 = 18$, $r_2 = 36$ and $q = 2, 3$, we get a $(K_{90,90,90,s}, K_5-e)$-design where $s = 126, 144$. By Lemma 1.1 there exists a $(K_{s+1}, K_5 - e)$-design for $s = 90, 126, 144$. By Lemma 2.2 there exists a $(K_v, K_5 - e)$-design for $v = 397, 415$.

$K_{505}, K_{541} \to K_5 - e$: Applying Lemma 2.3 with $p = 7$, $r = r_1 = 18$, $r_2 = 36$ and $q = 0, 2$, we get a $(K_{126,126,126,s}, K_5 - e)$-design where $s = 126, 162$. By Lemma 1.1 there exists a $(K_{s+1}, K_5 - e)$-design for $s = 126, 162$. By Lemma 2.2 there exists a $(K_v, K_5 - e)$-design for $v = 505, 541$.

$K_{613}, K_{685} \to K_5 - e$: Applying Lemma 2.3 with $p = 8$, $r = r_1 = 18$, $r_2 = 36$ and $q = 2, 6$, we get a $(K_{144,144,144,s}, K_5 - e)$-design where $s = 180, 252$. By Lemma 1.1 there exists a $(K_{s+1}, K_5 - e)$-design for $s = 144, 180, 252$. By Lemma 2.2 there exists a $(K_v, K_5 - e)$-design for $v = 613, 685$. □

Lemma 3.3 *There exists a* $(K_{469}, K_5 - e)$*-design.*

Proof Let $K_{m_1,m_2,\cdots,m_{25}}$ be the complete multipartite graph where $m_1 = \cdots = m_{24} = 18$ and $m_{25} = 36$. It is well known that there is a $(25, 4, 1)$-BIBD. From the existence of a $(25, 4, 1)$-BIBD, we know that $K_{m_1,m_2,\cdots,m_{25}} \to \{K_{18,18,18,18}, K_{18,18,18,36}\}$. From the proof of Lemma 3.2 we have a $(K_{18,18,18,18}, K_5 - e)$-design and a $(K_{18,18,18,36}, K_5 - e)$-design. Hence, we have $K_{m_1,m_2,\cdots,m_{25}} \to K_5 - e$. By Lemma 1.1 there exists a $(K_{s+1}, K_5 - e)$-design for $s = 18, 36$. Hence, we have a $(K_{469}, K_5 - e)$-design by Lemma 2.2. □

Lemma 3.4 *There exists a* $(K_{487}, K_5 - e)$*-design.*

Proof It is checked that the multiplicative order of 301 is 27 in Z_{487}. Let $G = \langle 301 \rangle$ denote the subgroup of order 27 generated by 301 in $Z_{487}^* = Z_{487} \setminus \{0\}$. It is readily checked that all the differences from the initial $(K_5 - e)$-block $[7, 0, 1, 3, 140]$ form a representative system of the coset classes of G in Z_{487}^*. A $(K_{487}, K_5 - e)$-design is thus constructed by developing the following base $(K_5 - e)$-blocks $[7x, 0, x, 3x, 140x]$ $(x \in G)$ in Z_{487}. □

Theorem 3.5 *If* $v \equiv 1 \pmod{18}$, *then there exists a* $(K_v, K_5 - e)$*-design.*

Proof It follows immediately by Lemma 1.1 and Lemmas 3.1-3.4. □

4 Remarks

We finally remove all the left values in Lemma 1.1 and establish that a $(K_v, K_5 - e)$-design exists for any integer $v \equiv 1 \pmod{18}$ and $v \geq 19$. But, for $v \equiv 0, 9, 10 \pmod{18}$, any example of a $(K_v, K_5 - e)$-design is unknown previous. We provide three examples as below.

$K_{28} \to K_5 - e$: Let $V(K_{28}) = Z_7 \times I_4$ where $I_4 = \{0, 1, 2, 3\}$. The base blocks are developed in $(Z_7, -)$:

$$
\begin{array}{ll}
[0_2, 1_2, 0_0, 0_3, 4_2], & [0_3, 0_1, 3_3, 2_1, 5_2], \\
[0_0, 0_1, 2_0, 6_0, 4_3], & [0_1, 4_0, 1_1, 2_2, 0_2], \\
[0_2, 1_0, 0_1, 3_1, 2_3], & [0_3, 1_0, 3_2, 5_3, 4_3].
\end{array}
$$

$K_{46} \to K_5 - e$: Let $V(K_{46}) = Z_{23} \times I_2$ where $I_2 = \{0, 1\}$. The base blocks are developed in $(Z_{23}, -)$:

$$
\begin{array}{ll}
[0_0, 1_0, 3_0, 7_0, 12_0], & [0_0, 0_1, 8_0, 1_1, 3_1], \\
[0_0, 2_1, 7_1, 11_1, 20_0], & [0_0, 3_1, 13_1, 20_1, 14_0], \\
[0_1, 2_0, 15_0, 11_1, 19_1]. &
\end{array}
$$

$K_{82} \to K_5 - e$: Let $V(K_{82}) = Z_{41} \times I_2$ where $I_2 = \{0, 1\}$. The base blocks are developed in $(Z_{41}, -)$:

$$
\begin{array}{ll}
[0_0, 2_1, 12_0, 22_0, 37_0], & [0_0, 4_1, 8_1, 13_1, 19_0], \\
[0_0, 7_1, 24_1, 14_1, 38_0], & [0_0, 18_1, 38_1, 5_1, 26_0], \\
[0_0, 16_1, 0_1, 22_1, 38_0], & [0_1, 4_0, 15_1, 27_1, 38_1], \\
[0_0, 9_1, 14_0, 18_0, 35_0], & [0_0, 28_1, 30_0, 29_1, 31_1], \\
[0_0, 1_0, 3_0, 9_0, 37_0]. &
\end{array}
$$

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