Knight's Tour on Boards with Holes is NP-Complete*

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Abstract

Can a knight using legal moves visit every square on a chessboard exactly once? This is a classical problem and it is known which rectangular chessboards admit a knight's tour. Discovering whether a chessboard has a knight's tour is more difficult if some squares are removed from the chessboard. We show that this problem is NP-complete by reduction from the Hamiltonian path problem for grid graphs.

1 Introduction

The knight's tour problem goes back at least to Euler[4]. It is a standard problem in the recreational mathematics books like Ball and Coxeter[2]. In computer science courses, it is often used as an example of a problem which can be solved by backtracking[1, 9]. We will not give a history here. It seems to be well known that certain rectangular boards have knight's tours. Theorem 1 gives this result. It also seems to be well known that knight's tours can be calculated quickly. Cull and De Curtins[3] give a method which runs in time linear in the number of squares on the board. Parberry[7] gives a neat recursive method for boards whose side lengths are a power of 2. We state these results in Theorem 2.

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Similar results hold for the knight's circuit problem where the last square of its path is required to be one knight's move from the first square. The status of other problems like the number of knight's tours on a board is still open.

Since knight's tour is a special case of the well known NP-complete Hamiltonian path problem, one could wonder what about the knight and the chessboard make this problem easy. An interesting observation is that the $(4n+1) \times (4n+1)$ board has a very special tour. In this tour, the knight starts in one of the corners, progresses around a width 2 border and then spirals inward towards the center square. This means that if we remove the central $[4(n-1)+1] \times [4(n-1)+1]$ board, the knight will still have a tour of the remaining border squares. Therefore, we surmised that allowing the chessboard to have holes (i.e. removed squares) might make an interesting and possibly NP-Complete problem. (Even if we didn't have the results in Theorems 1 and 2, it would be unlikely that the existence of a knight's tour on $n \times m$ rectangular boards could be NP-complete because there is only one instance for each problem size, and all known NP-complete sets have exponentially many elements. If the problem size were $\log n + \log m$, then this argument against NP-completeness would fail.)

We begin by recalling some known facts about knight's tour, and then we show that knight's tour on boards with holes in NP-complete. The proof is by reduction from the Hamiltonian path problem for grid graphs. (It may be worth noting that a grid graph can be viewed as a chessboard as seen by a rook. That is, the rook can move one square in any one of the four cardinal directions. For any rectangular board, there is a rook's tour. To make the rook's problem hard, barriers are erected to prevent the rook from moving in certain directions from certain squares.)

Here is a known theorem regarding the existence of a knight's tour on rectangular chessboards. (See Schwenk[8].)

Theorem 1 *There is a knight's tour of an $n \times m$ rectangular chessboard where $n \leq m$ unless:*

- $n = 1$ *and* $m > 1$
- $n = 2$
- $n = 3$ *and* $m = 3, 5, 6$
- $n = 4$ *and* $m = 4$

Theorem 1 says that it is easy to tell which rectangular boards have a knight's tour. The following theorem says that constructing such a knight's tour is also easy.

Theorem 2 *If a knight's tour of a rectangular chessboard exists, then we can construct one in polynomial time.*

Constructions of these rectangular boards can be seen in Cull and De Curtins[3]. Theorems 1 and 2 tell us that the decision problem and the construction problem for knight's tour on rectangular chessboards can both be solved in polynomial time. Now we wish to consider the knight's tour problem on chessboards where an arbitrary number of squares have been removed.

Definition 1 *A chessboard with holes, or simply, a board with holes, is a chessboard where any number of individual squares have been removed.*

We aim to prove that the decision problem for knight's tour on boards with holes is NP-complete. We will do this via a reduction from the Hamiltonian path problem for grid graphs. Thus we need the following definition.

Definition 2 *A grid graph is a graph in the plane where all vertices have integer coordinates. An edge can only connect two vertices if the Euclidean distance between them equals 1.*

The following theorem regarding grid graphs is due to Itai, Papadimitriou, and Swarcfiter.[6]

Theorem 3 *The Hamiltonian path problem on grid graphs is NP-complete.*

2 Reduction from Grid Graph

Given any grid graph, the following construction will show how we construct our chessboard. We will follow an example grid graph through this construction.

Construction 1 *First we take an arbitrary grid graph G and rotate it 45 degrees clockwise. (See Figure 1.)*

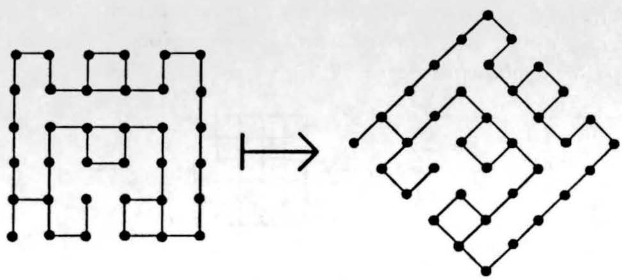

Figure 1: Rotate the grid graph 45 degrees clockwise

We now replace each grid node with a 5×5 chessboard, leaving one space vertically and zero spaces horizontally. This will form the board with holes as shown in Figure 2 below. The rotation performed in Figure 1 makes it easier to see this correspondence.

We have a set $\{v_0, \ldots, v_n\}$ of vertices on our grid graph in Figure 1 and a corresponding set $\{B_0, \ldots, B_n\}$ of sub-boards seen in Figure 2. Assume that these sets are ordered in such a way that v_i corresponds to B_i for each i.

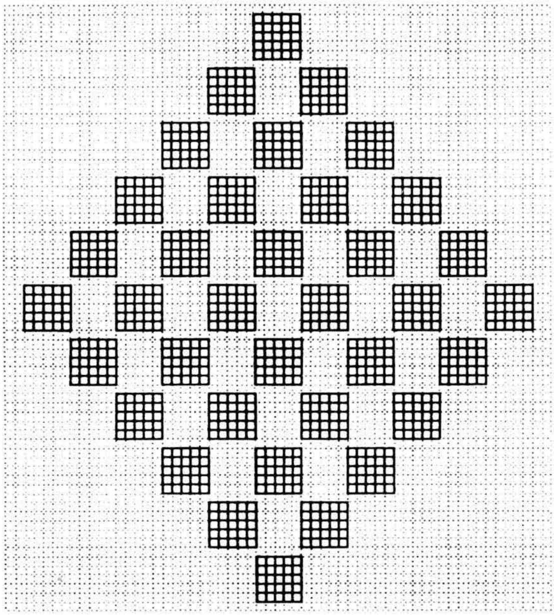

Figure 2: Sub-boards placed onto a larger grid

We "trim" these sub-boards so that the knight may travel from B_i to B_j if and only if v_i and v_j are adjacent vertices. Up to rotations there are 6 patterns of edges on any vertex v_i. We trim the corresponding B_i for the appropriate connection pattern following Figure 3.

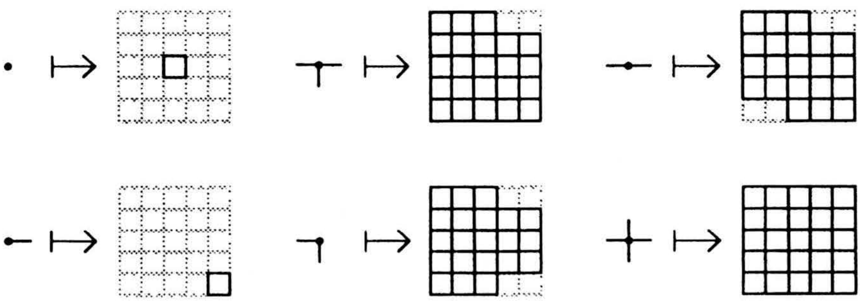

Figure 3: Transform each sub-board.

Applying the mapping defined in Figure 3 above, we obtain the chessboard with holes seen below in Figure 4.

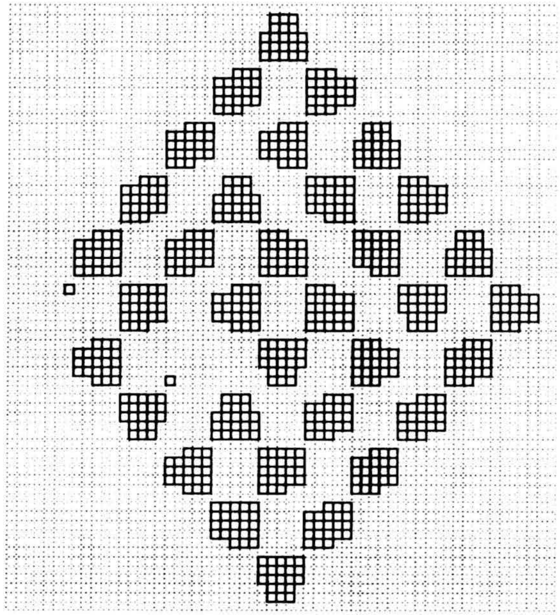

Figure 4: The resulting chessboard with holes.

Definition 3 *For any grid graph G, we will denote the chessboard constructed in Construction 1 above by $C(G)$. For example, if we denote the example grid graph in Figure 1 by $\widehat{G}$, then $C(\widehat{G})$ is the constructed chessboard seen in Figure 4.*

Proposition 1 *If there is a Hamiltonian path on the grid graph G, then there is a knight's tour on the chessboard $C(G)$.*

Proof. In Figure 5 below, we exhibit a tour of each of the possible sub-boards given in Figure 3. The in-arrow indicates where our tour will enter a sub-board, and the out-arrow indicates where the tour will exit. The numbers in the squares indicate the order in which the squares are visited.

The tours above account for every possible way to enter and exit a sub-board, up to symmetry (rotation, reflection, and path reversal).

Now we can easily find a tour of $C(G)$. We just travel from sub-board to sub-board following the Hamiltonian path on the grid graph G, and we tour each individual sub-board as shown in Figure 5. □

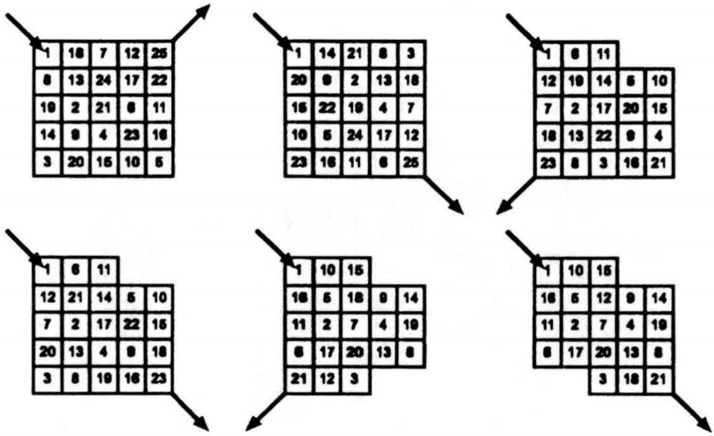

Figure 5: Touring sub-boards.

The tour of $\mathcal{C}(\widehat{G})$ in this manner is shown below in Figure 6.

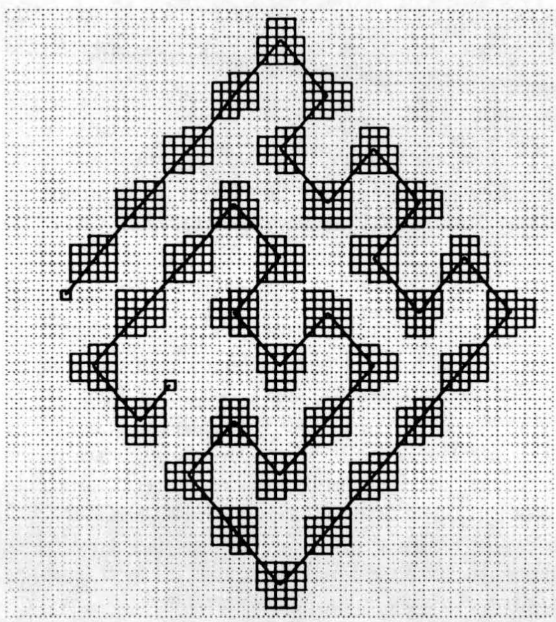

Figure 6: Touring the constructed chessboard.

The next three lemmas give limitations on knight's tours for $\mathcal{C}(G)$. We will use these lemmas to prove the converse of Proposition 1. These limitations come from parity properties of chessboards.

Traditionally, a chessboard is colored with two colors, say black and white. It is easy to check that the knight may only travel from a black square to a white square and vice versa. (That is, the knight's moves must alternate black, white, black, etc.) Let's assume that the 5×5 sub-board is colored so that the corner squares are black. Notice that there will be 13 black squares and only 12 white squares.

Lemma 1 *No sub-board in $\mathcal{C}(G)$ may be entered twice and exited twice in a knight's tour.*

Proof. Suppose that a sub-board B is entered and exited twice. This sub-board must have four corners, and is therefore the 5×5 sub-board. Now by parity, two more black squares than white squares are used. Hence there must be an unvisited white square. □

Lemma 2 *If in a knight's tour of $\mathcal{C}(G)$, a sub-board B is exited and then entered again, B must be the starting sub-board or the ending sub-board.*

This is clear from the previous lemma.

Definition 4 *We call a tour of $\mathcal{C}(G)$ repeat-free if the knight never visits any sub-board twice.*

Lemma 3 *If there exists a knight's tour of $\mathcal{C}(G)$, then there exists a knight's tour of $\mathcal{C}(G)$ that is repeat free.*

Proof. Suppose there is a knight's tour on $\mathcal{C}(G)$. Let $B_{t_1}, \ldots, B_{t_\ell}$ be the sequence of sub-boards visited by the knight. Each sub-board in $\mathcal{C}(G)$ is listed at least once, and the knight can legally move from B_{t_i} to $B_{t_{i+1}}$ for each i. Lemma 2 tells us that the only repeats can be B_{t_1} or B_{t_ℓ}. If $B_{t_1} = B_{t_\ell}$ then remove B_{t_ℓ}. Otherwise, if either of B_{t_1} or B_{t_ℓ} is a repeat, remove it. We may now use this repeat-free sequence and the sub-board tours shown in Figure 5 to construct a knight's tour of $\mathcal{C}(G)$. □

Proposition 2 *Let G be any grid graph. If there is a knight's tour on $\mathcal{C}(G)$, then there is a Hamiltonian path on G.*

Proof. Suppose there is a knight's tour of $\mathcal{C}(G)$. Then, by Lemma 3, there exists a knight's tour of $\mathcal{C}(G)$ that is repeat-free. Consider the sequence of sub-boards $B_{t_1}, \ldots, B_{t_\ell}$ visited by the knight on this repeat-free tour. This corresponds to a sequence of vertices $v_{t_1}, \ldots, v_{t_\ell}$ on the grid graph G. Since the knight was able to travel from sub-board B_{t_i} to $B_{t_{i+1}}$ for each i, the mapping in Figure 3 ensures the there is an edge between v_{t_i} and $v_{t_{i+1}}$ for each i. Furthermore, our sequence of sub-boards $B_{t_1}, \ldots, B_{t_\ell}$ lists each sub-board of $\mathcal{C}(G)$

exactly once and hence our sequence of vertices $v_{t_1}, \ldots, v_{t_\ell}$ lists each vertex in G exactly once. Therefore this sequence of vertices is a Hamiltonian path of G. □

So we have:

Theorem 4 *The knight's tour problem with holes is NP-Complete.*

3 Conclusion

Although both the knight's tour decision problem and construction problem on rectangular boards are computationally easy, the knight's tour problems on boards with holes are NP-complete.

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Iterations of eccentric digraphs

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Abstract

The *eccentricity* $e(u)$ of vertex u is the maximum distance of u to any other vertex of G. A vertex v is an *eccentric vertex* of vertex u if the distance from u to v is equal to $e(u)$. The *eccentric digraph* $ED(G)$ of a digraph G is the digraph that has the same vertex set as G and the arc set defined by: there is an arc from u to v if and only if v is an eccentric vertex of u. In this paper we consider the behaviour of an iterated sequence of eccentric graphs or digraphs of a graph or a digraph. The paper concludes with several open problems.

Keywords: Eccentricity, eccentric vertex, distance, eccentric graph, eccentric digraph.

1 Introduction and definitions

The study of distance properties of graphs is a classic area of graph theory; see, for example, the books of Buckley and Harary [5] and Brouwer, Cohen,

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Neumaier [3]. We study here an iterated version of a distance dependent mapping introduced by Buckley [4] and refined by others, including Boland and Miller [1]. The mapping is very simple but leads naturally to rather subtle questions. The questions posed are of the type studied by extremal graph theorists, but even they may consider our problems rather extreme!

A directed graph $G = G(V, E)$ consists of a vertex set $V(G)$ and an arc set $E(G)$. For the purposes of this paper, a *graph* is a digraph for which $(u, v) \in E$ implies $(v, u) \in E$. The least number of arcs in a directed path from u to v is the *distance* from u to v, denoted $d(u, v)$. If there is no directed path from u to v in G then we define $d(u, v) = \infty$. The *eccentricity*, $e(u)$, of u is the maximum distance from u to any other vertex in G. The *radius* is the minimum eccentricity of the vertices in G; the *diameter* is the maximum eccentricity of the vertices in G. Vertex v is an *eccentric vertex* of u if $d(u, v) = e(u)$. Note that if a vertex has out-degree zero, that vertex has all the other vertices of the given digraph as its eccentric vertices.

The *eccentric digraph* of a digraph G, denoted $ED(G)$, is the digraph on the same vertex set as G, but with an arc from vertex u to vertex v in $ED(G)$ if and only if v is an eccentric vertex of u. The eccentric digraph of a graph was introduced by Buckley [4] and Boland and Miller [1] introduced the concept of the eccentric digraph of a digraph. An example of a graph and its eccentric digraph is given in Figure 1. Note that arcs of graphs are drawn not as a pair of directed edges with arrows, but in the usual form.

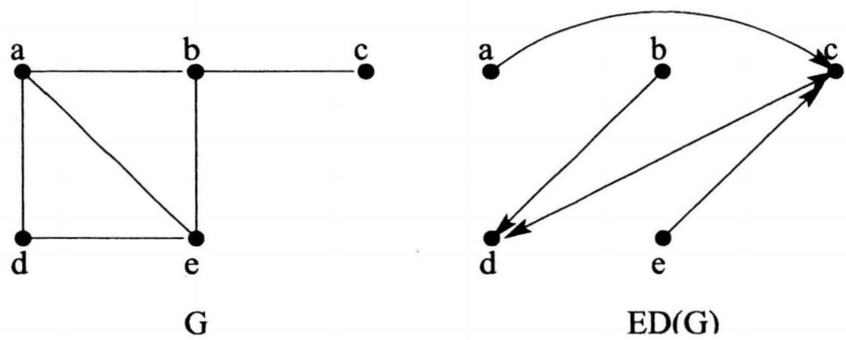

Figure 1: A graph and its eccentric digraph.

Given a positive integer $k \geq 2$, the k^{th} iterated eccentric digraph of G is written as $ED^k(G) = ED(ED^{k-1}(G))$ where $ED^0(G) = G$. Figure 2 illustrates these definitions showing digraph G and its iterated eccentric digraphs $ED(G)$, $ED^2(G)$, $ED^3(G)$, and $ED^4(G)$. Note that in this case $ED^5(G) = ED^3(G)$.

An interesting line of investigation concerns the iterated sequence of ec-

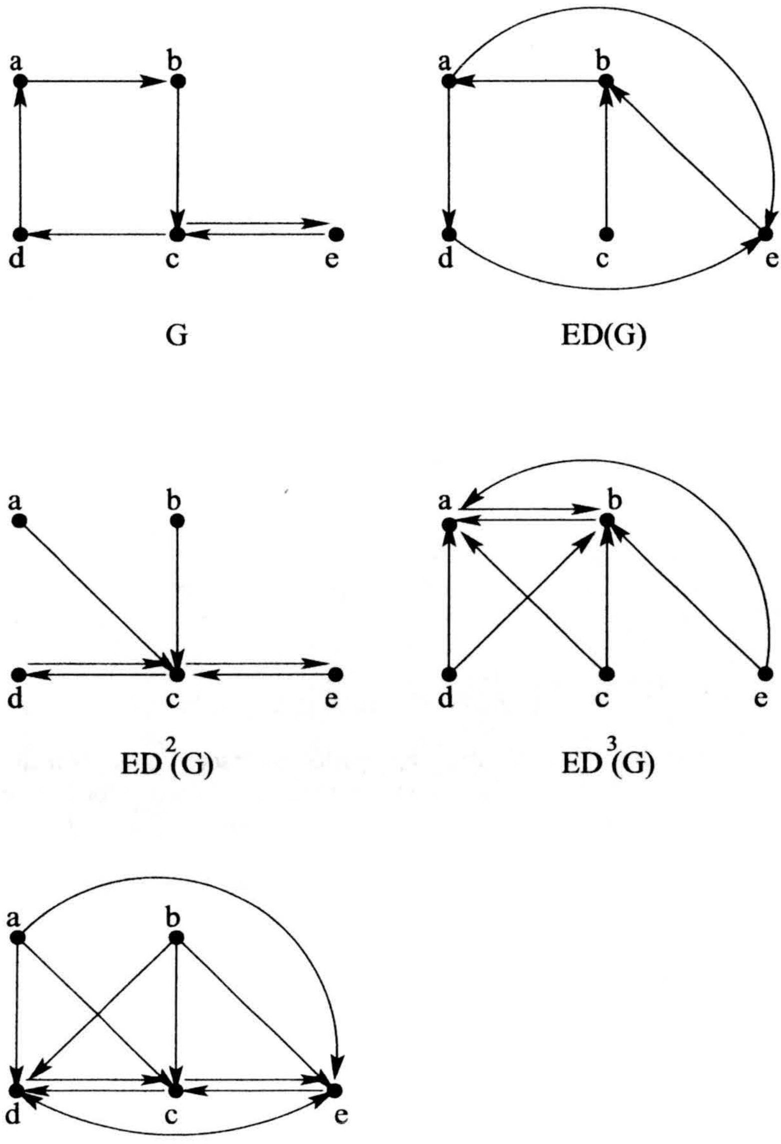

Figure 2: An eccentric digraph iteration sequence.

centric digraphs. For every digraph G there exist smallest integer numbers $p > 0$ and $t \geq 0$ such that $ED^t(G) = ED^{p+t}(G)$. For example, in Figure 2, $t(G) = 3$ and $p(G) = 2$. We call p the *period* of G and t the *tail* of G; these quantities are denoted $p(G)$ and $t(G)$ respectively. We say that a graph is *periodic* if it has no tail; i.e., if $t(G) = 0$. In the definitions just given, we assumed that the vertices of the graphs are labelled. It is also natural to consider the corresponding unlabelled version.

For every digraph G there exist smallest integer numbers $p > 0$ and $t \geq 0$ such that $ED^t(G) \cong ED^{p+t}(G)$, where $\cong$ denotes graph isomorphism. We call p the *iso-period* of G and t the *iso-tail* of G; these quantities are denoted $p'(G)$ and $t'(G)$ respectively. We say that a graph is *iso-periodic* if it has no iso-tail; i.e., if $t'(G) = 0$. Clearly $p'(G) \mid p(G)$.

2 Previous results

The following observations, theorems, and open problems first appeared in [1] or [2].

Observation 2.1 If a digraph G is the union of $k > 1$ vertex disjoint strongly connected digraphs of orders $n_1, n_2, \ldots, n_k$, for $m > 0$, where each $n_i \geq 2$, then

$$ED^m(G) = \begin{cases} K_{n_1,n_2,\ldots,n_k} & \text{if } m \text{ odd} \\ K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k} & \text{if } m \text{ even.} \end{cases}$$

Observation 2.2 The eccentric digraph of a directed cycle is a directed cycle, $ED(\overrightarrow{C}_n) \cong \overrightarrow{C}_n$. However, note that the direction of the arcs in $ED(\overrightarrow{C}_n)$ is opposite to the direction of the arcs in the given cycle $\overrightarrow{C}_n$.

Observation 2.3 A nontrivial eccentric digraph has no vertex of out-degree zero. However, the converse is not true: there exist digraphs with the out-degree of every vertex non-zero which are, nevertheless, not the eccentric digraphs of any graph or digraph. An example of such a digraph is the graph P_4, the path of four vertices.

It seems likely that a classification of all digraphs as to whether or not they are an eccentric digraph is not a trivial problem.

Question 2.1 Find necessary and sufficient conditions for a digraph to be an eccentric digraph.

The fact that there exist digraphs which are not eccentric digraphs of any graph or digraph leads to the question: "If a digraph G is not an eccentric digraph, can G be always embedded in an eccentric digraph?" This question was considered in [2]. The *eccentric digraph appendage number* of G is the minimum number of vertices that must be added to a digraph G so that there exists a digraph G' which is the eccentric digraph of some digraph and G is an induced subgraph of G'.

Theorem 2.1 *If G is not the eccentric digraph of some graph H, then the eccentric digraph appendage number of G equals one.*

Question 2.2 Find the period and the tail of various classes of graphs and digraphs.

Observation 2.4 The only digraph G with $p(G) = 1$ and $t(G) = 0$ is the complete digraph K_n.

Observation 2.5 For $p = 2$, $t = 0$ examples include the complete multipartite digraph $K_{n_1,n_2,\ldots,n_k}$, the disjoint union of complete digraphs $K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_k}$ and the directed cycle $\overrightarrow{C}_n$.

Question 2.3 Characterize periodic digraphs with period two.

Observation 2.6 For $p = 2$, $t = 1$ examples include the (disjoint) union of strongly connected digraphs $H_{n_1} \cup H_{n_2} \cup \ldots \cup H_{n_k}$, where at least one of them is not a complete digraph.

3 Examples, open problems, and conjectures

In this section we present some examples and new open problems and questions, all designed to stimulate further interest in the iterated eccentric mapping. Many examples of digraphs G with $p(G) = 2$ have been found. In fact, if you pick a digraph at random on a computer then it usually occurs that $p(G) = 2$ and you have to work quite hard to find one of larger period. This observation leads to our first conjecture.

Conjecture 3.1 In the standard model of n-vertex random digraphs where arcs are chosen at random with probability q, if $0 < q < 1$, then

$$\lim_{n\to\infty} Prob_q(p(G) = 2) = 1.$$

Here we present for the first time examples of eccentric digraph iteration cycles of length more than 2. The following three examples give eccentric digraph iteration cycles of lengths 4 and 8.

Example 3.1 Let R be the (undirected) cubic Cayley graph with the two generators (01)(23)(4567) and (56)(78). The directed version is shown in Figure 3. The graph R has 20 vertices and is periodic with $p(R) = 4$. However, the graphs $ED^k(R)$ are all isomorphic to R and so $p'(R) = 1$.

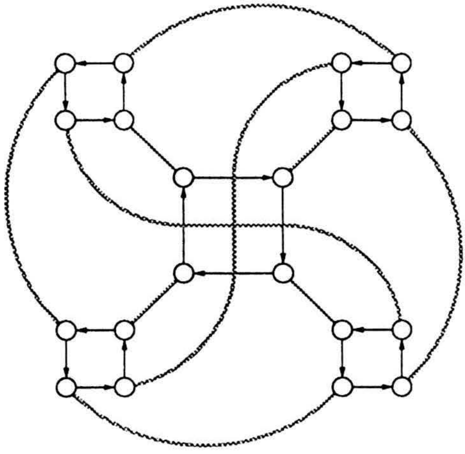

Figure 3: The Cayley graph with generators (01)(23)(4567) and (56)(78).

Example 3.2 Let R be as in Example 1. The *conjunction* (or *tensor product*) $G = G_1 \wedge G_2$ of two digraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ has $V = V_1 \times V_2$ as its vertex set, and $u = (u_1, u_2)$ is adjacent to $v = (v_1, v_2)$ in G iff $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$. For this graph, $p(R \wedge R) = 8$.

Example 3.3 The smallest digraph G found so far with $p'(G) > 2$ has 10 vertices and iso-period 4. Such a digraph G is shown in Figure 4.[1]

Example 3.4 Let C_n denote the cyclic graph of n vertices. Consider the odd cycles, C_{2m+1}. Figure 5 illustrates that $p(C_9) = 3$. Below we show a table of $p(C_{2m+1})$. This is sequence A003558 in Sloane's Encyclopedia of Integer Sequences.

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p(C_{2m+1})$	1	2	3	3	5	6	4	4	9	6	11	10	9	14

m	15	16	17	18	19	20	21	22	23	24	25	26
$p(C_{2m+1})$	5	5	12	18	12	10	7	12	23	21	8	26

It is not difficult to determine that

$$p(C_{2m+1}) = \min\{k \geq 1 \mid m(m+1)^{k-1} = \pm 1 \pmod{2m+1}\}.$$

[1]Note added in press: Brendan McKay reports that the 9 is the smallest order of a graph with $p'(G) > 2$. As an example he gives $E(G) = \{\{0,5\},\{0,6\},\{1,6\},\{1,8\},\{2,6\},\{2,8\},\{3,7\},\{3,8\},\{4,7\},\{4,8\},\{5,7\}\}$; here $p(G) = p'(G) = 3$ and $t(G) = t'(G) = 0$.

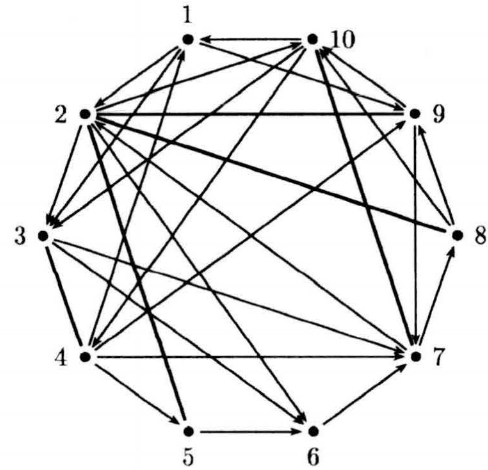

Figure 4: A digraph G with of order 10 such that $p(G) = p'(G) = 4$ and $t(G) = t'(G) = 1$.

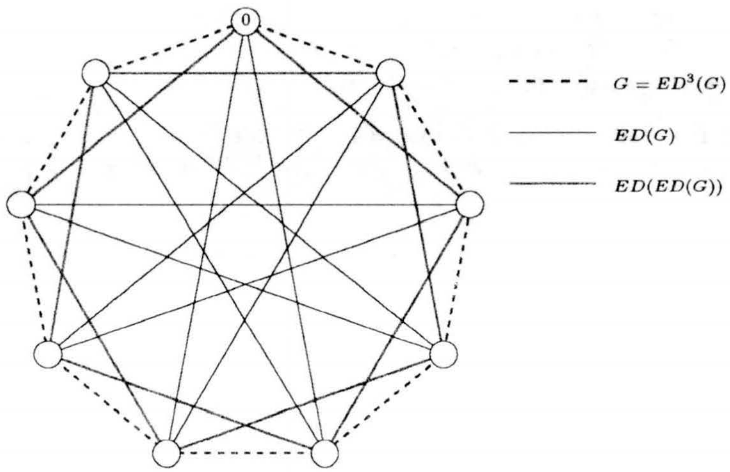

Figure 5: The graph C_9 and its iterated eccentric digraphs.

In particular, if $m = 2^k$, then $p(C_{2m+1}) = k+1$, showing that the period may take on any value. The numbers m for which $m = p(C_{2m+1})$ have been called the "Queneau numbers" (e.g. Sloane's A054639). Note that if m is a Queneau number, then the sequence of iterated eccentric digraphs gives a very pretty Hamilton decomposition of K_{2m+1}.

Example 3.5 *The circulant graph of order 23 with steps* $\{1, -1, 2, -2, 3, -3\}$ *has period 33.*

This example shows that the constant in the following conjecture is at least 33/23.

Conjecture 3.2 There is a constant c such that, for any digraph G of n vertices

$$p(G) \leq cn.$$

Conjecture 3.3 We have observed, but not proven, that

$$p(C_{2m+1} \times C_{2m+1}) = p(C_{2m+1}) + p(C_{2m+1}),$$

where $\times$ denotes the usual Cartesian product of graphs.

Example 3.6 The 336 vertex cubic Cayley graph with the two generators (23)(45)(67) and (025)(146) leads to a period 4 sequence G_0, G_1, G_2, G_3, where $G_0 \cong G_2$ is an 8-regular graph, and $G_1 \cong G_3$ is a 14-regular graph. Thus, for this example, $p(G) = 4$ and $p'(G) = 2$.

Observation 3.1 A digraph G of order n satisfies that $p(G) = t(G) = 1$ if and only if G has $k \geq 1$ vertices with out-degree 0 and $n - k$ vertices with out-degree $n - 1$.

Observation 3.2 Clearly, the eccentric digraph of a vertex transitive digraph is a vertex transitive digraph. A little thought reveals that the eccentric digraph of a vertex transitive graph is a vertex transitive graph. Similarly, the eccentric digraph of a Cayley (di)graph also a Cayley (di)graph. The generators of $ED(G)$ are the products of the generators along the longest paths in G.

Clearly ED induces a partition of the set of all graphs (and on the set of all unlabelled graphs). Let $\langle G \rangle$ denote the equivalence class of (labelled) graphs induced by ED; and let $[G]$ represent the corresponding unlabelled equivalence class.

What are the properties of that partition? In Figure 6 we show the partitions of labelled graphs (on the left) and unlabelled graphs (on the right) induced by ED for $n = 3$. Note that there are $2^{n(n-1)} = 64$ graphs represented on the left and 16 on the right.

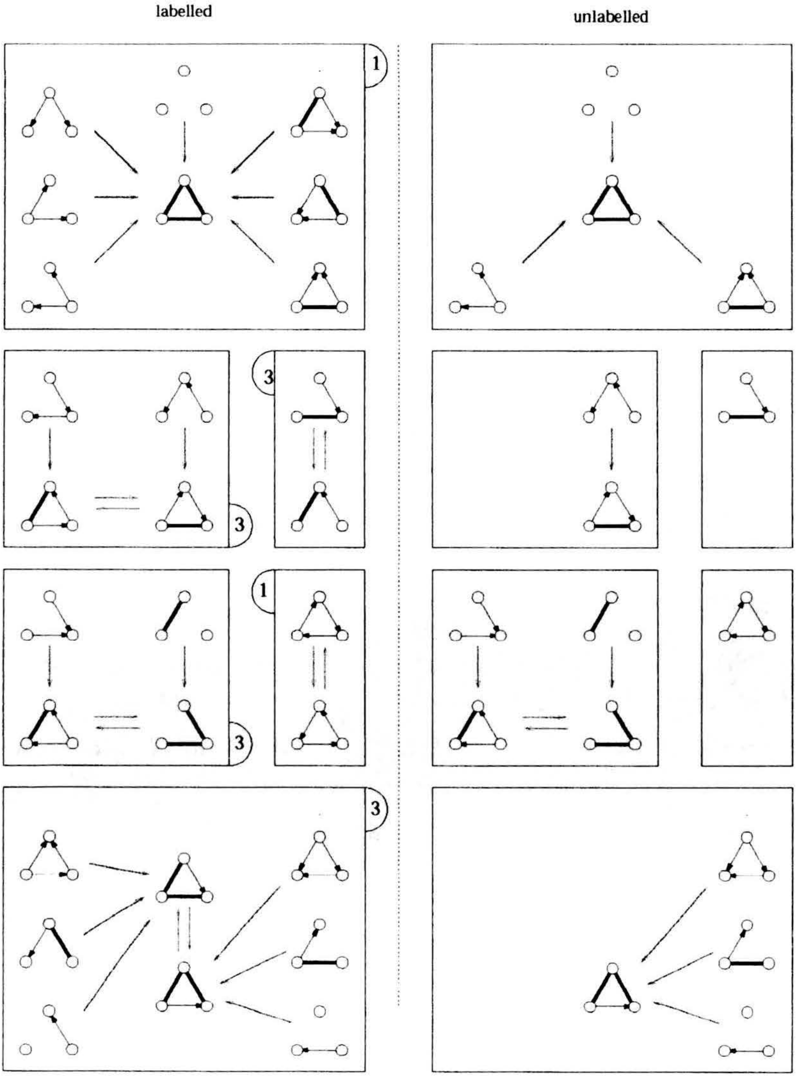

Figure 6: The equivalence classes induced by ED on the sets of unlabelled and labelled graphs for $n = 3$. The number enclosed in semi-circles are the number of classes of that form.

Question 3.1 Among all digraphs G on n vertices, what is the minimum size of $\langle G \rangle$? The maximum size? The average size? What about $[G]$?

Question 3.2 Let us say that a class is *periodic* if every graph in the class is periodic. For general n, identify some periodic classes. Can the periodic classes be characterized?

Question 3.3 Which unlabelled graphs are fixed points; i.e., such that $ED(G) = G$? For example, for $n = 3$ there are five such graphs. As observed earlier, for labelled graphs, only the complete graph is a fixed point.

Question 3.4 For every digraph G, is it true that $t(G) = t'(G)$?

4 Acknowledgements

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CONSTRUCTIONS OF CONNECTED GRAPHS WITH A GIVEN MATCHABLE RATIO

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Abstract

Given a positive rational number $\frac{a}{b}(0 \le a \le b)$, we identify families of connected graphs G, such that the ratio of the number of matchable edges to the total number of edges in G -denoted by $\pi(G)$, is $\frac{a}{b}$. We call $\pi(G)$, the matchable ratio of G. For certain kinds of rational numbers, we identify the smallest connected graphs with the property. This problem was initially discussed in [1].

1. Introduction

The graphs considered here are finite, and contain neither loops nor multiple edges. Let G be a graph. An edge of G is called **matchable**, if and only if it belongs to a perfect matching in G. This paper continues the investigation into matchable edges started in [1], where we focused on **totally matchable graphs**, that is, graphs in which every edge is matchable. We now turn our attention to graphs which are not totally matchable. These are graphs which contain non-matchable edges.

Definition

Let G be a graph with b edges; i.e. of size b, and with a matchable edges. Then the ratio $\pi(G) = \frac{a}{b}$ is called the **matchable ratio** of G.

The following are additional definitions, which will apply to the material that follows.

Definitions

(i) A **1-cycle** and a **2-cycle** is a vertex and an edge respectively. A cycle with more than two vertices is called a **proper cycle**.

In the material that follows, "cycle" will mean "proper cycle", unless otherwise specified.

(ii) A graph G is **non-matchable (matchable) saturated** if and only if no more non-matchable (matchable)edges can be added to it.

(iii) An edge joining two non-adjacent vertices of a cycle is a **chord** (The 3-cycle has no chords).

(iv) An even n-cycle is **canonically labeled,** if an only if its vertices are labeled in some agreed order, with the consecutive integers from 1 to n. For purposes of this paper, we will take the order to be a clockwise.

(v) In a canonically labeled cycle, a chord joining two odd (even) labeled vertices is called **odd (even)**; otherwise it is **mixed**.

(vi) A graph, consisting of a canonically labeled r-cycle C_r, with s odd chords and no even chords added, will be denoted by $G_{r,s}$. The subgraph C_r, is called its **boundary**.

(vii) A **chain** is a tree with nodes of valency 1 and 2 only. A **boundary chain** of $G_{r,s}$ is any connected subgraph of its boundary.

From the definition, the graph $G_{r,s}$ contains $r \geq 4$ vertices and $r+s$ edges. The graph $G_{r,0}$ is the cycle C_r.

In [1] we identified graphs with certain matchable ratios. We also established the existence of a graph for any given matchable ratio (Theorem 3.3). The proof of this result also provides an algorithm for the construction of such graphs. The graphs obtained from this theorem, may be disconnected. It is therefore interesting to be able to construct connected graphs with a given matching ratio $\frac{a}{b}$. Even more interesting, is the construction of a connected graph having precisely size b and with a matchable edges. In this paper, we give constructions for such graphs. Moreover, we identify the smallest order graphs with this property.

2. Graphs with matchable ratio a/b, where $a = 2n$ $(n > 1)$, $0 \leq a \leq b$ and $b - a \leq \binom{n}{2}$

Lemma 1

In the graph $G_{r,s}$, every chord is non-matchable.

Proof

Figure 1 shows a canonical drawing of $G_{r,s}$.

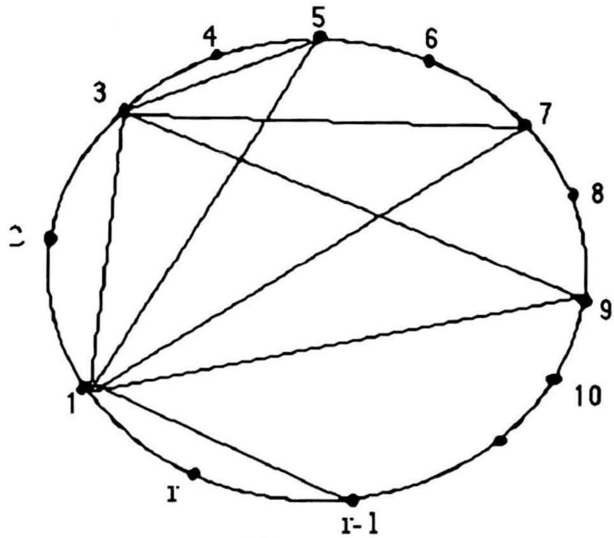

Figure 1

All chains referred to, will be boundary chains. It can be easily seen that a chain has odd order, if and only if the labels of its end-vertices have the same parity. Let us assume that there exists a perfect matching *M* containing the (odd) chord xy Now, remove vertices x and y from $G_{r,s}$. Since x and y are odd and there are no even or mixed chords in $G_{r,s}$, we have in $G_{r,s}$-x-y vertices x-1, x +1, y -1 and y+1 each having valency 1 (see Figure 2) and all other even vertices have valency 2. In particular, the edges (x-1, x-2), (x+1, x+2),(y -1, y -2) and (y+1, y+2) must all be in *M*.

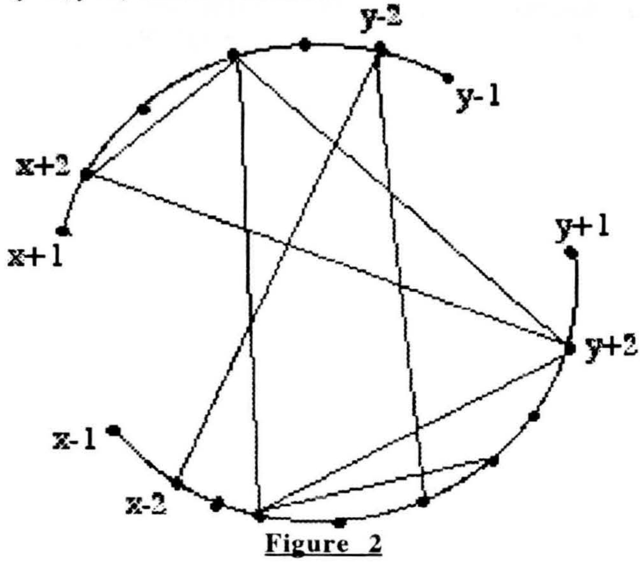

Figure 2

The edge joining x+1 to x+2 must belong to M. This forces the edge joining x+3 to x+4 to be in M. By continuing in this manner, we get that the edge joining y-3 and y-2 must belong to M. But the edge (y-1, y-2) is in M. This is a contradiction; since(by assumption) M is a perfect matching. Thus our assumption is false. The chord xy is non-matchable. Hence the result follows.

□

Theorem 1

Let $r = 2n$ $(n \geq 1)$. If $s = \binom{n}{2}$, then the graph $G_{r,s}$ is non-matchable saturated.

Proof

From the lemma, all the chords of $G_{r,s}$ are non-matchable. There are n odd labelled vertices in $G_{r,s}$. Any pair of these can be joined to form an odd chord. The number of such pairs is $\binom{n}{2}$. Therefore, when s takes this value, every pair of odd labelled vertices are joined by an edge, so that all odd chords are included.

We must now show that no more non-matchable chords can be added to $G_{r,s}$, that is, every new chord is matchable. Let us add a new chord xy. Then xy must either be (i) even or (ii) mixed. Call the resulting graph G.

Case (i) (xy is even)

Let us remove vertices x and y from G. In the resulting graph G', vertices x-1 and x+1, being odd vertices, will be joined by an edge (being odd labelled vertices); and so too will be vertices y-1 and y+1. Thus, the resulting graph will contain a new boundary cycle C_{r-2}. Since r is even, r-2 is even. Thus, G' has a perfect matching; and the chord xy is matchable in $G_{r,s}$.

Case (ii) (*ij* is mixed)

Without loss in generality, we will assume that x is odd and that y is even. Again, let us remove vertices x and y from G to obtain a graph G'. Then, G' will contain two boundary chains-one chain connecting vertex x+1 (even) to vertex y-1 (odd); the other, connecting vertex x-1 to vertex y+1. Since the chains have endnodes with different parities. They will be even chains; and consequently, have perfect matchings. Hence G' has a perfect matching. Adding the chord xy to this matching, yields a perfect matching in $G_{r,s}$.

We conclude therefore, that no more non-matchable edges can be added to $G_{r,s}$. Hence $G_{r,s}$ is saturated. □

Corollary 1.1

Let $a = 2n$ $(n \geq 2)$ and b be positive integers, with $a \leq b$ and $0 \leq b\text{-}a \leq \binom{n}{2}$.

Then the graph $G_{a,b-a}$ is a connected graph such that $\pi(G_{a,b-a}) = \frac{a}{b}$.

Proof

Let the number of added (odd) chords be ε. Then (from Theorem 1) we get

$$0 \le \varepsilon \le \binom{n}{2}.$$

Since C_a has a edges, total number of edges in G is $a+\varepsilon$.

$$\Rightarrow \pi(G) = \frac{a}{a+\varepsilon}.$$

But $\varepsilon = b-a$. Therefore, the result follows. □

3. Graphs with matchable ratio a/b, with $a = 2n+1$ ($n > 1$), $0 < a < b$ and $b-a \le \frac{4}{400}$.

Definition

The graph H_r is a canonically labeled r-cycle, with a chain of length 2, attached to one of its vertices; and with the vertex of valency 2 and 1 , labeled $r+2$ and $r+1$ respectively. The graph $H_{r,s}$ is the labeled graph H_r, with s odd chords added (See Figure 3).

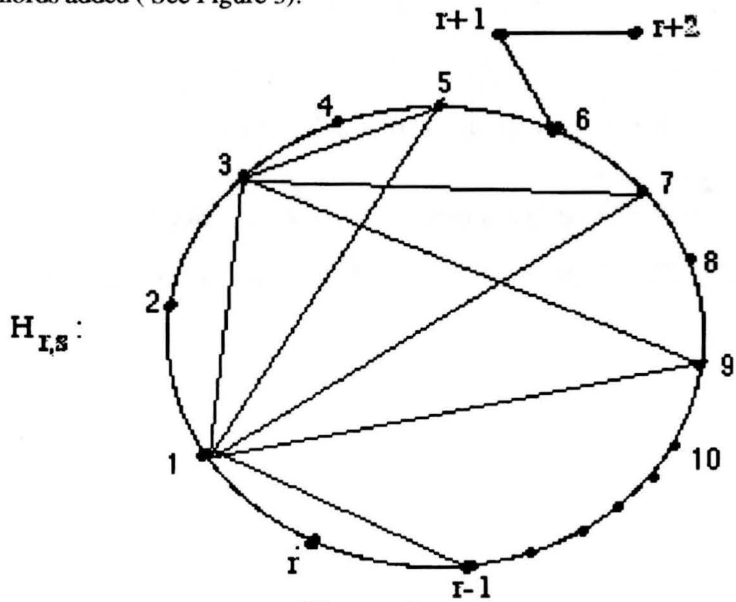

Figure 3

This graph is essentially the graph $G_{r,s}$, with a chain of length 2 attached to one of its boundary vertices., By definition, the graph $H_{r,s}$ contains r+2 vertices and r+2+s edges.

The following lemma is analogous to Lemma 1.

Lemma 2

In the graph $H_{r,s}$, every chord is non-matchable.

Proof

In Figure 3, we show a canonical drawing of a labeled graph $H_{r,s}$. The attached chain does not affect the arguments given in the proof of Lemma 1, since the edge joining r+1 to r+2 must be used in every perfect matching. Hence the result follows. □

This lemma implies that we can still add non-matchable odd (or even) chords to the subgraph $G_{r,s}$ of $H_{r,s}$. Also, it is clear that the edge joining the vertex r+1 to the boundary is non -matchable. Therefore, $H_{r,s}$ will have r+1 matchable edges and s+1 non-matchable edges. In $H_{r,s}$ new kinds of edges can join the" external" vertices r+1 and r+2 to vertices on the boundary of $G_{r,s}$. It is difficult to tell which ones are non-matchable, since the presence of these edges could even spoil the non-matchability of chords. We can however saturate the $G_{r,s}$ subgraph of $H_{r,s}$, so that the resulting graph is saturated with non-matchable chords. This yields the following analogy to Theorem 1.

Theorem 2

Let n be an integer greater than 2, and let $a = 2n+1$. Then the graph $H_{a-1,s}$ is saturated with non-matchable chords , when $s = \binom{n}{2}$.

The following result is immediate from Theorem 2; and is analogous to Corollary 1.1.

Corollary 2.1

Let $a = 2n + 1$ ($n \geq 2$) and b be positive integers, with $a \leq b$ and $0 \leq b\text{-}a \leq \binom{n}{2}$. Then the graph $H_{a-1,b-a+1}$ is a connected graph such that

$\pi(H_{a-1,b-a+1}) = \frac{a}{b}$. □

4. Graphs with matchable ratio a/b in which b-a is not bounded above by $\binom{n}{2}$.

We now consider the case in which $0 \leq a < b$ and $b\text{-}a$ is not bounded above by $\binom{n}{2}$. In the results above, $b\text{-}a$ is bounded above by $\binom{n}{2}$-the number of chords that can be added to the boundary cycle. This excludes many classes of rational numbers. For example, the rational number $\frac{1}{100}$ is not covered by the results above, since they all are based on the condition that C_a is a proper cycle. In fact, the smallest even value of a is 4 (Corollary 1.1) and the smallest odd value is 5 (Corollary 2.1). Therefore the numerator of the fraction must be at least 4. In this case the fraction will be $\frac{4}{400}$.

When b-a is bounded, as defined above, we have identified a connected graph G of size b with a matchable edges, such that $\pi(G) = \frac{a}{b}$. However, for some rational numbers, it will be impossible to find a graph; connected or not , with this property. For example, for the rational number $\frac{1}{100}$, one would have to find a graph with 100 edges in which exactly one edge belongs to a perfect matching. No such graph exists. We will therefore consider the related problem of finding a connected graph G, such that $\pi(G) = \frac{a}{b}$. This means that there are no restrictions on the size of the graph. Our technique is based on the simple fact that the rational numbers $\frac{a}{b}$ and $\frac{ka}{kb}$ are equal, for all non-zero values of k. This will allow us to use the construction given in Sections 2 and 3, since we can always arrange for ka to be even. We will do better than this. We will identify a smallest order graph obtained by our construction, that is, a smallest order $G_{r,s}$.

The following result is crucial.

Lemma 3

For all positive integers a and b, with $b \geq a$, there exists a positive integer k such that

$kb - ka \leq \binom{ka/2}{2}$, when ka is even. Furthermore, the smallest value of k for which the condition holds is $\left\lceil \frac{2}{a}\left(\frac{4}{a}(b-a)+1\right)\right\rceil$.

Proof

Since $b \geq a$, then $b-a \geq 0$ and $\left\lceil \frac{2}{a}\left(\frac{4}{a}(b-a)+1\right)\right\rceil$ is a positive integer.

Choose $k \geq \left\lceil \frac{2}{a}\left(\frac{4}{a}(b-a)+1\right)\right\rceil$. Then $\frac{ka}{2} \geq \left(\frac{4}{a}\right)(b-a)+1$. From this, we obtain $\frac{ka}{2} - 1 \geq \left(\frac{4}{a}\right)(b-a)$, which in turn, implies that $b-a \leq \frac{a}{4}(\frac{ka}{2}-1)$.

Thus $k(b-a) \le \frac{ka}{4}(\frac{ka}{2}-1) = \binom{ka/2}{2}$, when ka is even. Hence there exists such a positive integer k. It can be easily shown that if $k < \left\lceil \frac{2}{a}\left(\frac{4}{a}(b-a)+1\right)\right\rceil$, the inequality no longer holds. Hence the result follows. □

This lemma identifies the range of values of *k* which would make the *ka*-cycle (when *ka* is even) large enough so that there would be enough "room" to add the necessary number (*kb-ka*) of chords. We can now construct a graph *G*, for which $\pi(G) = \frac{ka}{kb}$. This is the gist of the following theorem.

Theorem 3

Let $\frac{a}{b}$ be a positive rational number. Then, for $n = ka$ and $s = k(b-a)$, where $k \ge \left\lceil \frac{2}{a}\left(\frac{4}{a}(b-a)+1\right)\right\rceil$, the graph $G_{n,s}$ has the property that $\pi(G_{n,s}) = \frac{a}{b}$, when *ka* is even. When *ka* is odd, the graph $H_{n-1,s}$ has the property that $\pi(H_{n-1,s}) = \frac{a}{b}$.

Proof

The result follows immediately from Corollary 1.1, Corollary 2.1 and Lemma 3. □

This theorem gives the range of values of *k* for which graphs of the types $G_{r,s}$ and $H_{r,s}$ can be constructed with a given matchable ratio.

Example 1

Let $\frac{a}{b}$ =1. Then $a = b$. In this case, b-a = 0, so that $k = \frac{2}{a} = 1$, $n = ka = 2$ and $s = k(b-a) = 0$. The resulting graph is $G_{n,s} = G_{2,0}$; which is a 2-cycle with no chords added. Therefore, the graph is an edge.

Example 2

Let $\frac{a}{b}=\frac{1}{100}$. Then $a=1$ and $b=100$. In this case, the smallest value of k is

$$\left\lceil \frac{2}{1}\left(\frac{4}{1}(100-1)+1\right)\right\rceil = 2(397)=794. \text{ Therefore } \frac{ka}{kb}=\frac{794}{79400}.$$

Therefore resulting graph is $G_{794,79400}$. This graph has 794 vertices, 79400 edges; and contains

79400 -794 =78606 chords.

It will be interesting to find out how good is this lowest value of k. We will therefore find the maximum number of (odd) chords that the 794-gon can contain. It is

$$\binom{794/2}{2}=\binom{397}{2}=78606.$$

This means that the graph $G_{794,79400}$ is saturated. Thus, we have indeed found the smallest order graph of the form $G_{r,s}$. In the above Example (ii), the smallest order graph belonging to the family of graphs of the form $G_{r,s}$ was found. However, it is a large graph. The natural question at this stage is the following. Can we find a smaller order graph; maybe from an entirely different family with $\frac{a}{b}=\frac{1}{100}$? This question motivates the material in the next section.

5. The smallest connected graphs with matchable ratio

If the matchable ratio $\frac{a}{b}$ is 0 or 1, then the smallest connected graphs are obvious, For $\frac{a}{b}=0$, the smallest connected graph is P_3. For $\frac{a}{b}=1$, every edge is matchable. In this case, the smallest graph is an edge. We will therefore consider only matchable ratios which are neither 0 nor 1.

We will denote vertex and edge sets of a graph G, by V(G) and E(G), respectively.

Definition

Let A and B be graphs. We sill say that A is **smaller than** B if and only if $|V(A)| \leq |V(B)|$ and $|E(A)| \leq |E(B)|$ and at least one of the inequalities is strict.

Lemma 4

Let r be a positive integer.such that $r = \binom{n}{2}$, for some even positive integer n. Then the smallest order connected graph with r matchable edges is K_n. Otherwise, the smallest order connected graph with r matchable edges is $G_{r,0}$, if r is even and $r \geq 4$; and is H_{r-1}, if r is odd and $r \geq 5$.

Proof

If a graph has r matchable edges, then it has at least r edges. Hence, any connected graph G with $r > 0$ matchable edges must have the following properties.

(i) G has at least r edges.

(ii) G has an even number of vertices (since it must have a perfect matching).

Case (1) **r-even**

For $r = 2$, the smallest graph is P_4 - the chain with 4 vertices. When r is even, and greater than 2, two connected graphs satisfy the minimum value of Condition(i), i.e. every edge is matchable; and Condition (ii). They are the r-cycle and the complete graph K_n, where

$r = \binom{n}{2}$, for some even positive integer n. It follows that when r is even, and greater than 2, the smallest connected unsaturated graph with r matchable edges is the r-cycle. Case (2) **r-odd**

When $r = 1$, the smallest connected graph, must have at least two vertices, by Condition (ii). Hence it must be an edge, When $r = 3$, two graphs satisfy the above conditions; the chain P_6 and the chain P_5, with a pendant edge attached to its centre vertex. Both graphs are trees with six vertices. For $r \geq 5$, we can start off the unique smallest connected matching unsaturated graph with r-1 ($\geq$4) matchable edges-which is C_{r-1}, and then add one more matchable edge, in the "cheapest" way. The desired graph G must have at least r+1 vertices. So ideally, we would like to add one matchable edge and exactly two vertices (if possible). This can be achieved in only one way; that is, by attaching a P_3 to a C_{r-1}. It follows that when r is odd, and greater than 3, the smallest connected graph with r matchable edges is an r-cycle, with a P_3 attached. Hence the result folllows. □

In order to construct a minimum graph with a prescribed number of matchable and non-matchable edges, we would start off with the smallest graph with the desired number of matchable edges and then add non-matchable edges, so as to minimize the number of additional vertices. Clearly, if we can add the non-matchable edges to the smallest graphs, without adding any new vertices, then the resulting graphs must be the smallest possible. This means that our initial smallest graph should be unsaturated. Thus, a complete graph cannot be used. Therefore, the smallest graphs with r matchable edges and s non-matchable edges and with appropriate restriction on s, are the graphs $G_{r,s}$ and $H_{r,s}$ defined in Section 2. Our discussion, together with Corollaries 1.1 and 2.1, lead to the following result.

Theorem 4

Let $a = 2n$ ($n > 1$) and b be positive integers, such that $a < b$ and $0 < b\text{-}a \leq \binom{n}{2}$. Then the smallest connected graph of size b with a matchable edges and with matching ratio $\frac{a}{b}$, is the graph $G_{a,b-a}$. If $a = 2n+1$ ($n > 1$), then the smallest connected graph of size b with a matchable edges and with matching ratio $\frac{a}{b}$ is the graph $H_{a-1,b-a-1}$.

6. Discussion

At this stage, there is still one unanswered question.

Problem Given $\frac{a}{b}$ ($0 \leq a \leq b$), find a smallest graph G, relative to either order of size, such that $\pi(G) = \frac{a}{b}$.

As discussed in Section 4, for some matchable ratios $\frac{a}{b}$, it might be impossible for any graph with a matchable edges to have size b. Therefore, a smallest graph does not exist. So the next best thing, is to look for the smallest graph with equal matchable ratio. This case is still unsolved. Is the graph with 794 vertices and 79400 edges, in Example (ii), the smallest connected graph with matchable ratio $\frac{1}{100}$?

Lemma 3 gives the smallest multiple of the ratio, which will yield a cycle large enough to accommodate the necessary unmatchable edges. We are assured that the smallest graph of the $G_{r,s}$ and $H_{r,s}$ forms are found. But for the cases where $b\text{-}a > \binom{n}{2}$, we are not sure that these types of graphs are the smallest connected graphs.

7. References

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