More on Spanning 2-Connected Subgraphs of Alphabet Graphs, Special Classes of Grid Graphs

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Abstract

A grid graph G is a finite induced subgraph of the infinite 2-dimensional grid defined by $Z \times Z$ and all edges between pairs of vertices from $Z \times Z$ at Euclidean distance precisely 1. A natural drawing of G is obtained by drawing its vertices in $\Re^2$ according to their coordinates. Apart from the outer face, all (inner) faces with area exceeding one (not bounded by a 4-cycle) in a natural drawing of G are called the holes of G. We define 26 classes of grid graphs called alphabet graphs, with no or a few holes. We determine which of the alphabet graphs contain a Hamilton cycle, i.e.

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a cycle containing all vertices, and solve the problem of determining a spanning 2-connected subgraph with as few edges as possible for all alphabet graphs.

Keywords: alphabet graphs, grid graph, Hamilton cycle, spanning 2-connected subgraph

AMS Subject Classifications: 05C40, 05C85

1 Introduction

The *infinite grid graph* G^∞ is defined by the set of vertices $V = \{(x, y) \mid x \in Z,\ y \in Z\}$ and the set of edges E between all pairs of vertices from V at Euclidean distance precisely 1. For any integers $s \geq 1$ and $t \geq 1$, the *rectangular grid graph* $R(s, t)$ is the (finite) subgraph of G^∞ induced by $V(s,t) = \{(x, y) \mid 1 \leq x \leq s, 1 \leq y \leq t,\ x \in Z,\ y \in Z\}$ (and just containing all edges from G^∞ between pairs of vertices from $V(s, t)$). This graph $R(s, t)$ is also known as the *product graph* $P_s \times P_t$ of two disjoint paths P_s and P_t. A *grid graph* is a graph that is isomorphic to a subgraph of $R(s, t)$ induced by a subset of $V(s, t)$ for some integers $s \geq 1$ and $t \geq 1$. It is clear that a grid graph $G = (V, E)$ is *a planar graph*, i.e. it can be drawn in the plane $\Re^2$ in such a way that the edges only intersect at the vertices of the graph. In such a drawing, the regions of $\Re^2 \backslash (V \cup E)$ are called the *faces* of G. Exactly one of the faces is unbounded; this is called the *outer face*; the others are its *inner faces*. The *natural drawing* of a grid graph is just described by drawing its vertices in $\Re^2$ according to their coordinates. A *solid grid graph* is a grid graph all of whose inner faces have area one (are bounded by a cycle on four vertices) in a natural drawing. A grid graph that is not solid contains inner faces (in a natural drawing) that have area larger than one; these faces are called *holes*. A subgraph H of a graph $G = (V, E)$ is called a *spanning subgraph* if $V(H) = V$. A connected graph is called 2-*connected* if it remains connected if at most one vertex is removed. A *Hamilton cycle* in a graph $G = (V, E)$ is a cycle containing every vertex of V, i.e. a spanning 2-connected subgraph in which every vertex has degree 2 (the number of edges is $|V|$).

Itai, Papadimitriou and Szwarcfiter [2] proved that deciding whether a given grid graph has a Hamilton cycle is an NP-complete problem. This implies that the problem of finding a spanning 2-connected subgraph with as few edges as possible is also NP-hard for grid graphs. It has been conjectured that the first problem remains NP-complete when it is restricted to solid grid graphs. However, Umans and Lenhart [9] recently proved that this problem is polynomially solvable, by presenting a complicated algorithm with time complexity $O(|V|^4)$. In a recent paper of Sheffield [8] the work of [2] has been extended to grid graphs with a small number of holes.

For the second problem the complexity is not known when it is restricted to solid grid graphs. It remains an open problem -what the complexity of both problems is -when we restrict ourselves to grid graphs with a fixed number of holes.

Motivated by the above problems, we study the problem of the existence of a Hamilton cycle and the problem of determining a spanning 2-connected subgraph with as few edges as possible for a large number of classes of finite grid graphs with no or a few holes called *alphabet graphs.* This is a continuation of the work in [3], [4], [5], [6] and [7]. For all graphs of the defined classes we solve the second problem. All solutions are of the same type : first, we use the well-known *Grinberg-condition* and the properties of bipartite graphs to derive a lower bound for the number of edges in a spanning 2-connected subgraph. Secondly, we show by construction that this lower bound is in fact the optimum value.

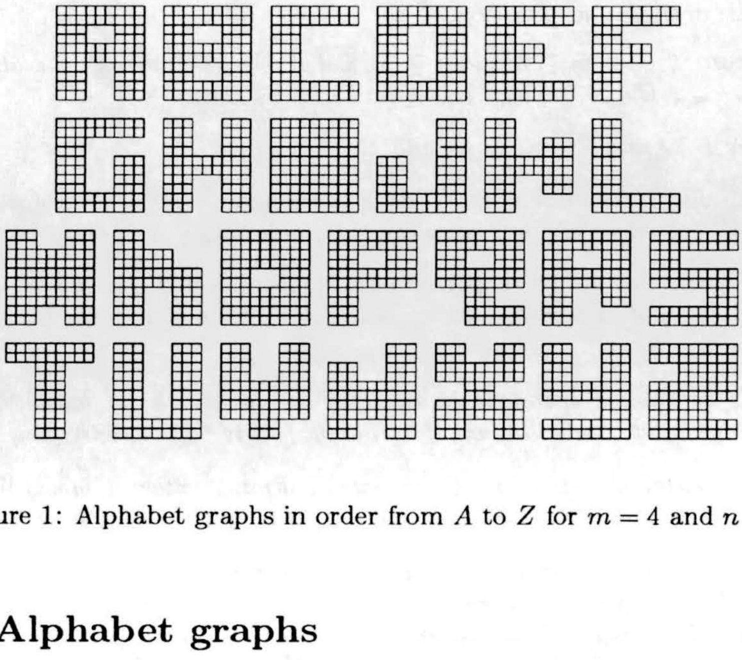

Figure 1: Alphabet graphs in order from A to Z for $m = 4$ and $n = 3$

2 Alphabet graphs

We now introduce the 26 classes of grid graphs which we call *alphabet graphs.*

For every letter λ of the alphabet $\{a, b, \ldots, z\}$ we define a corresponding subgraph $\Lambda_{m,n}$ of $R(3m-2, 5n-4)$ for all $m \geq 3$, $n \geq 3$. These alphabet graphs $\{A_{m,n}, B_{m,n}, \ldots, Z_{m,n}\}$ are shown in Figure 1 for $m = 4$ and $n = 3$. It is clear from these figures how these graphs should be extended for other values of m and n. We will avoid the tedious details of defining all these 26 graph classes formally. Note that the extension of these classes to $m = 2$ or $n = 2$ causes problems with the definition of grid graphs: for instance, the natural definition of $E_{2,2}$ would not result in an *induced* subgraph of G^{∞}.

Notice that from these 26 classes, there is one class of alphabet graphs with two holes, namely the graph $B_{m,n}$; six classes with one hole, namely the graphs $A_{m,n}$, $D_{m,n}$, $O_{m,n}$, $P_{m,n}$, $Q_{m,n}$ and $R_{m,n}$; the remaining 19 classes contain no holes, i.e. are solid grid graphs.

We refer to these classes in the next result just by the capital letters, omitting the indices. Spanning 2-connected subgraphs with a minimum number of edges of alphabet graphs with $m = n \geq 3$ were presented in [5]. In this paper we generalize that result. Our main result characterizes which of the alphabet graphs for any $m \geq 3$ and any $n \geq 3$ are hamiltonian and shows a spanning 2-connected subgraph with at most two edges more than their number of vertices. We postpone the proofs and constructions (figures) until the next section.

Theorem 1 *Let $m \geq 3$ and $n \geq 3$. Let $A, B, \ldots, Z$ denote the alphabet graphs $A_{m,n}, B_{m,n}, \ldots, Z_{m,n}$ as defined above. Then:*

(i) A, D, O and P are hamiltonian.

(ii) E and F contain a spanning 2-connected subgraph with (at most) $|V| + 1$ edges and are hamiltonian if and only if n is even.

(iii) N contains a spanning 2-connected subgraph with (at most) $|V| + 1$ edges and is hamiltonian if and only if m and n have a different parity.

(iv) Q contains a spanning 2-connected subgraph with (at most) $|V| + 1$ edges and is hamiltonian if and only if m is odd or n is even.

(v) R contains a spanning 2-connected subgraph with (at most) $|V| + 1$ edges and is hamiltonian if and only if m is even or n is odd.

(vi) W contains a spanning 2-connected subgraph with
- *$|V|$ edges if m is even;*
- *$|V| + 1$ edges if both m and n are odd;*
- *$|V| + 2$ edges if m is odd and n is even.*

These numbers of edges are all best possible.

(vii) X contains a spanning 2-connected subgraph with
- *$|V|$ edges if either (m is even) or (m is odd, $m \geq 7$ and n is even);*
- *$|V| + 1$ edges if either (m and n are odd) or (m=5 and n is even);*
- *$|V| + 2$ edges if m=3 and n is even.*

(*viii*) *The remaining alphabet graphs contain a spanning 2-connected subgraph with (at most) $|V|+1$ edges and are hamiltonian if and only if $m \cdot n$ is even.*

3 Proofs and constructions

A useful necessary condition for hamiltonicity is the following result due to Grinberg [1].

Lemma 2 *Suppose a planar graph G has a Hamilton cycle H. Let G be drawn in the plane, and let r_i denote the number of faces inside H bounded by i edges in this planar embedding. Let r_i' be the number of faces outside H bounded by i edges. Then the numbers r_i and r_i' satisfy the following equation.*

$$\sum_i (i-2)(r_i - r_i') = 0.$$

We use this lemma to prove the following corollaries.

Corollary 3 *E and F contain no Hamilton cycle if n is odd.*

Proof. We will divide the proof into two cases.

Case 1 We consider the alphabet graph E. There is exactly one face with $12(m-1)+10(n-1)$ edges and there are exactly $10(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of this graph. Let E be hamiltonian. Then, by Lemma 2, we have

$$(12(m-1)+10(n-1)-2)(-1)+(4-2)(r_4-r_4')=0.$$

Hence

$$r_4 - r_4' = 6m+5n-12. \tag{1}$$

It is known that the number of faces with four edges is

$$r_4 + r_4' = 10m\cdot n - 10m - 10n + 10. \tag{2}$$

From equations (1) and (2) we obtain

$$2r_4 = 10m\cdot n - 4m - 5n - 2. \tag{3}$$

So, n is even.

Case 2 We consider the alphabet graph F. There is exactly one face with $8(m-1)+10(n-1)$ edges and there are exactly $8(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of this graph. Let F

be hamiltonian. Then, by Lemma 2 and using a similar method as in the proof of Case 1, we obtain

$$2r_4 = 8m \cdot n - 4m - 3n - 2. \tag{4}$$

So, n is even. $\square$

Corollary 4 *N contains no Hamilton cycle if m and n have the same parity.*

Proof. There is exactly one face with $6(m-1)+14(n-1)$ edges and there are exactly $10(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of the graph N. Let N be hamiltonian. Then, by Lemma 2 and using a similar method as in the proof of Corollary 3, we obtain

$$2r_4 = 10m \cdot n - 7m - 3n - 1. \tag{5}$$

So, m and n have a different parity. $\square$

Corollary 5 *Q contains no Hamilton cycle if m is even and n is odd.*

Proof. It is easy to check that there is exactly one face with $8(m-1)+10(n-1)$ edges, one face with $2(m+n-2)$ edges and there are exactly $11(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of the graph Q. Let Q be hamiltonian. Then, by Lemma 2 and using a similar method as in the proof of Corollary 3, we obtain

$$2r_4 = 11m \cdot n - 7m - 6n + 1 - (m+n-3)(r_{2(m+n-2)} - r'_{2(m+n-2)}). \tag{6}$$

So, m is odd or n is even since $(r_{2(m+n-2)} - r'_{2(m+n-2)})$ is -1 or 1. $\square$

Corollary 6 *R contains no Hamilton cycle if m is odd and n is even.*

Proof. We can check that there is exactly one face with $6(m-1)+12(n-1)$ edges, one face with $2(m+n-2)$ edges and there are exactly $11(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of the graph R. Let R be hamiltonian. Then, by Lemma 2 and using a similar method as in the proof of Corollary 3, we obtain

$$2r_4 = 11m \cdot n - 8m - 5n + 1 - (m+n-3)(r_{2(m+n-2)} - r'_{2(m+n-2)}). \tag{7}$$

So, m is even or n is odd since $(r_{2(m+n-2)} - r'_{2(m+n-2)})$ is -1 or 1. $\square$

Corollary 7 *W, X and the remaining alphabet graphs in (viii) contain no Hamilton cycle if $m \cdot n$ is odd.*

Proof. We will divide the proof into two cases.

Case 1 We consider the alphabet graph B. There is exactly one face with $6(m-1)+10(n-1)$ edges, there are two faces with $2(m+n-2)$ edges and there are exactly $13(m-1)(n-1)$ faces with four edges in the planar (natural) drawing of B. Let B be hamiltonian. Then, by Lemma 2 and using a similar method as in the proof of Corollary 3, we obtain

$$2r_4 = 13m \cdot n - 10m - 8n + 4 - (m+n-3)(r_{2(m+n-2)} - r'_{2(m+n-2)}). \quad (8)$$

So, $m \cdot n$ is even since $(r_{2(m+n-2)} - r'_{2(m+n-2)})$ is $-2, 0$ or 2.

Case 2 We consider the alphabet graphs C, G, H, I, J, K, L, M, S, T, U, V, W, X, Y and Z. They are solid grid graphs. So the only faces to be considered in the planar (natural) drawing of every one of these graphs are the single outer face and the faces with four edges. Let these graphs be hamiltonian. The number of edges in the outer face is always even since they form a cycle and the graphs are bipartite. This number is then always $2x(m-1)+2y(n-1)$ for some positive integers x and y. The number of faces with four edges is always of the form $z(m-1)(n-1)$ for some positive integer z. Then, by Lemma 2 and using a similar method as in the proof of Corollary 3, we obtain

$$2r_4 = z \cdot m \cdot n + (x-z)m + (y-z)n + (z-x-y-1). \quad (9)$$

Since

$$(x,y,z) = \begin{cases} (5,5,9) & \text{for } C \\ (5,7,11) & \text{for } G, X, Y \\ (3,9,11) & \text{for } H, U \\ (3,5,15) & \text{for } I \\ (3,7,9) & \text{for } J, K \\ (3,5,7) & \text{for } L, T \\ (3,7,13) & \text{for } M \\ (7,5,11) & \text{for } S \\ (3,7,11) & \text{for } V, W \\ (5,5,13) & \text{for } Z, \end{cases}$$

z is odd, and $x-z$, $y-z$ and $z-x-y-1$ are even. So, $m \cdot n$ is even. □

Lemma 8 *W contains no spanning 2-connected subgraph with at most $|V|+1$ edges if m is odd and n is even.*

Proof. Consider the alphabet graph W for odd m and even n. We can partition this graph into three rectangles; name them A (on the left), B (in the middle) and C (on the right). A is $R(m, 4n-3)$, B is $R(m-2, 2n-1)$

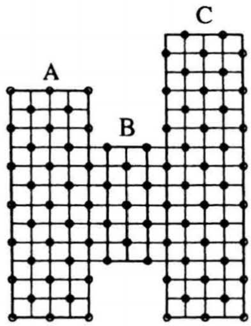

Figure 2: Partition of the alphabet graph $W_{5,4}$

and C is $R(m, 5n-4)$. For illustration, look at the partition of the alphabet graph $W_{5,4}$ in Figure 2. All of the rectangles are bipartite graphs with a bipartition of the vertices, say in S and T, where we start with S in a corner vertex of A. It is easy to check that A and B have one more vertex from S than from T, whereas C has the same number of vertices from S and from T. So, $|V(W) \cap S| = |V(W) \cap T| + 2$. In any spanning 2-connected subgraph G of W all vertices in S have degree at least 2, hence $|E(G)| \geq 2|S| = |S| + |T| + 2 = |V(G)| + 2$. This completes the proof of Lemma 8. □

We complete the proof of Theorem 1 by showing, through construction, the existence of a Hamilton cycle or a spanning 2-connected subgraph with at most $|V| + 2$ edges, in all cases where $m = 3, 5, 6$ or 7 and $n = 4$ or 5. Meanwhile, for other values of m and n, it is not difficult to see, from the patterns in the figures that now follow, how to extend the solutions.

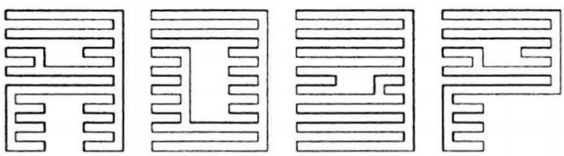

Figure 3: Hamilton cycles for the alphabet graphs A, D, O, and P for $m = 5$ and $n = 4$

Hamilton cycles for the alphabet graphs A, D, O, and P are shown in Figure 3 for $m = 5$ and $n = 4$, in Figure 4 for $m = 6$ and $n = 4$, and in Figure 5 for $m = 7$ and $n = 5$. The patterns in Figure 3 can be used for finding Hamilton cycles for these graphs for any odd number m and any

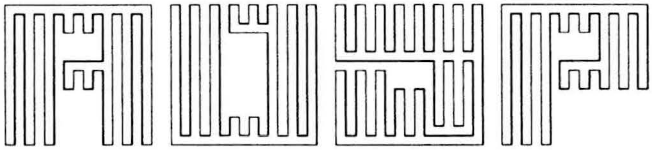

Figure 4: Hamilton cycles for the alphabet graphs A, D, O, and P for $m = 6$ and $n = 4$

even number n; the patterns in Figure 4 can be used for finding Hamilton cycles for these graphs for any even number m and any number n; and the patterns in Figure 5 can be used for finding Hamilton cycles for these graphs for any odd numbers m and n.

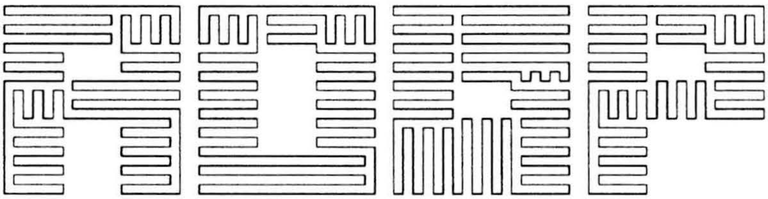

Figure 5: Hamilton cycles for the alphabet graphs A, D, O, and P for $m = 7$ and $n = 5$

In Figure 6, we show Hamilton cycles for the alphabet graphs $E_{5,4}$, $F_{5,4}$, $N_{5,4}$, $N_{6,5}$, $Q_{6,4}$, $Q_{7,5}$, $R_{6,4}$ and $R_{7,5}$. The patterns in Figure 6(a) and Figure 6(b) can be used for finding Hamilton cycles for the alphabet graphs E and F, respectively, for any number m and any even number n. The patterns in Figure 6(c) and Figure 6(d) can be used for finding Hamilton cycles for the alphabet graph N (the pattern in Figure 6(c) for any odd number m and any even number n, the pattern in Figure 6(d) for any even number m and any odd number n). The patterns in Figure 6(e) and Figure 6(f) can be used for finding Hamilton cycles for the alphabet graph Q (the pattern in Figure 6(e) for any number m and any even number n, the pattern in Figure 6(f) for any odd numbers m and n). The patterns in Figure 6(g) and Figure 6(h) can be used for finding Hamilton cycles for the alphabet graph R (the pattern in Figure 6(g) for any even number m and any number n, the pattern in Figure 6(h) for any odd numbers m and n).

In Figure 7, we show spanning 2-connected subgraphs with $|V|+1$ edges for the alphabet graphs $E_{6,5}$, $F_{6,5}$, $N_{6,4}$, $N_{7,5}$, $Q_{6,5}$ and $R_{5,4}$. The pattern

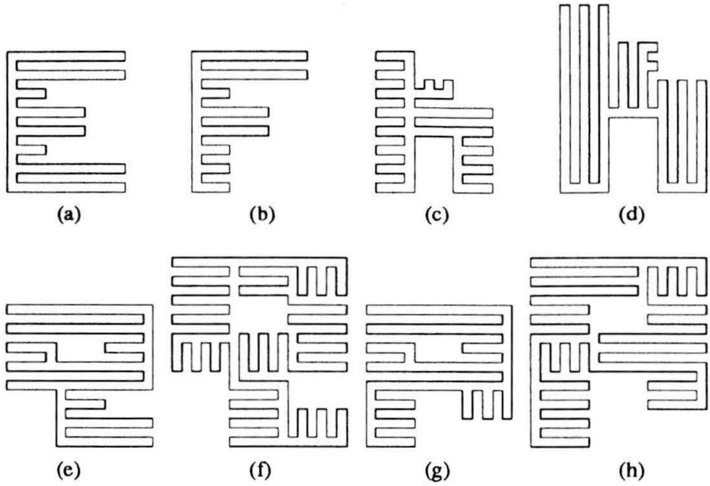

Figure 6: Hamilton cycles for the alphabet graphs (a) $E_{5,4}$ (b) $F_{5,4}$ (c) $N_{5,4}$ (d) $N_{6,5}$ (e) $Q_{6,4}$ (f) $Q_{7,5}$ (g) $R_{6,4}$ (h) $R_{7,5}$

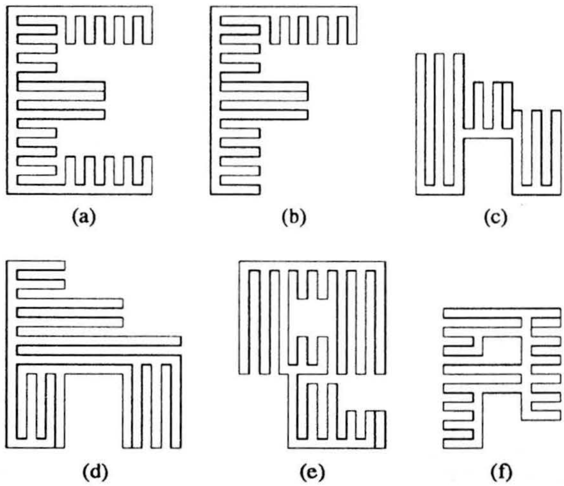

Figure 7: Spanning 2-connected subgraphs with $|V| + 1$ edges for the alphabet graphs (a) $E_{6,5}$ (b) $F_{6,5}$ (c) $N_{6,4}$ (d) $N_{7,5}$ (e) $Q_{6,5}$ (f) $R_{5,4}$

in Figure 7(a) can be used for determining such a spanning subgraph for the alphabet graph E for any number m and any odd number n. The pattern in Figure 7(b) can be used for determining such a spanning subgraph for the alphabet graph F for any number m and any odd number n. The patterns in Figure 7(c) and Figure 7(d) can be used for finding such a spanning subgraph for the alphabet graph N (the pattern in Figure 7(c) for any even numbers m and n, the pattern in Figure 7(d) for any odd numbers m and n). The pattern in Figure 7(e) can be used for determining such a spanning subgraph for the alphabet graph Q for any even number m and any odd number n. The pattern in Figure 7(f) can be used for determining such a spanning subgraph for the alphabet graph R for any odd number m and any even number n.

We show a Hamilton cycle for the alphabet graph $W_{6,4}$ in Figure 8(a). The pattern in Figure 8(a) can be used for finding a Hamilton cycle for the alphabet graph W for any even number m and any number n. In Figure 8(b) is shown a spanning 2-connected subgraph with $|V|+1$ edges for the alphabet graph $W_{7,5}$. The pattern in Figure 8(b) can be used for determining such a spanning subgraph for the alphabet graph W for any odd numbers m and n. In Figure 8(c) is shown a spanning 2-connected subgraph with $|V|+2$ edges for the alphabet graph $W_{5,4}$. The pattern in Figure 8(c) can be used for determining a spanning 2-connected subgraph with $|V|+2$ edges for the alphabet graph W for any odd number m and any even number n. This is the optimum value for the minimum number of edges in such a spanning 2-connected subgraph.

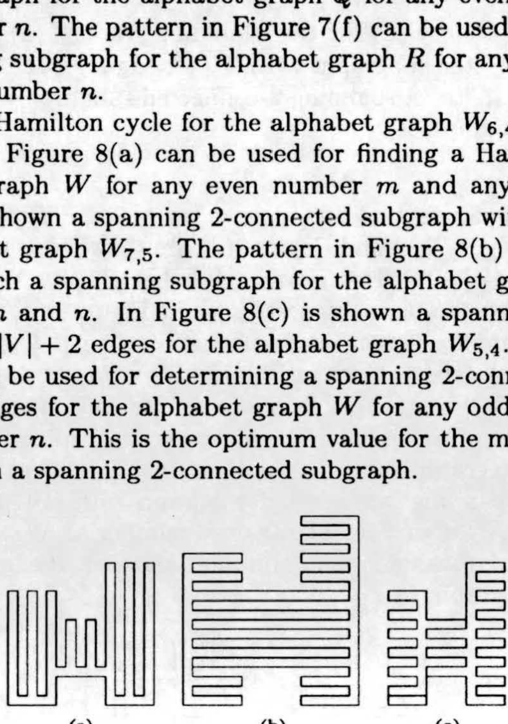

Figure 8: (a) A Hamilton cycle for the alphabet graph $W_{6,4}$ (b) A spanning 2-connected subgraph with $|V|+1$ edges for the alphabet graph $W_{7,5}$ (c) A spanning 2-connected subgraph with $|V|+2$ edges for the alphabet graph $W_{5,4}$

We show Hamilton cycles for the alphabet graphs $X_{6,4}$ in Figure 9(a) and $X_{7,4}$ in Figure 9(b). The pattern in Figure 9(a) can be used for finding a Hamilton cycle for the alphabet graph X for any even number m and any number n, whereas the pattern in Figure 9(b) can be used for finding

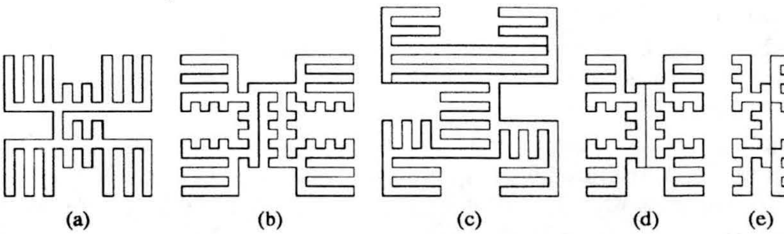

Figure 9: Hamilton cycles for the alphabet graphs (a) $X_{6,4}$ (b) $X_{7,4}$; Spanning 2-connected subgraphs with $|V|+1$ edges for the alphabet graphs (c) $X_{7,5}$ (d) $X_{5,4}$; (e) A spanning 2-connected subgraph with $|V|+2$ edges for the alphabet graph $X_{3,4}$

a Hamilton cycle for any odd number m, $m \geq 7$ and any even number n. Meanwhile, in Figure 9(c) and Figure 9(d) are shown spanning 2-connected subgraphs with $|V|+1$ edges for the alphabet graphs $X_{7,5}$ and $X_{5,4}$, respectively. The patterns in Figure 9(c) and Figure 9(d) can be used for determining spanning 2-connected subgraphs with $|V|+1$ edges for the alphabet graph X (the pattern in Figure 9(c) for any odd numbers m and n, the pattern in Figure 9(d) for $m = 5$ and any even number n). In Figure 9(e) is shown a spanning 2-connected subgraph with $|V|+2$ edges for the alphabet graph $X_{3,4}$. The pattern in Figure 9(e) can be used for determining a spanning 2-connected subgraph with $|V|+2$ edges for the alphabet graph X for $m = 3$ and any even number n. We are not sure that this is the optimum value for the minimum number of edges in a spanning 2-connected subgraph.

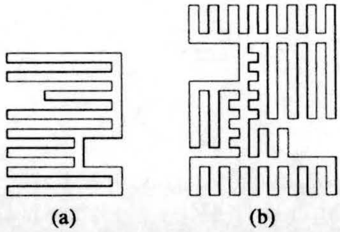

Figure 10: Hamilton cycles for the alphabet graphs (a) $Z_{5,4}$ (b) $Z_{6,5}$

We show Hamilton cycles for the alphabet graph $Z_{5,4}$ in Figure 10(a) and $Z_{6,5}$ in Figure 10(b). The pattern in Figure 10(a) can be used for finding a Hamilton cycle for the alphabet graph Z for any number m and

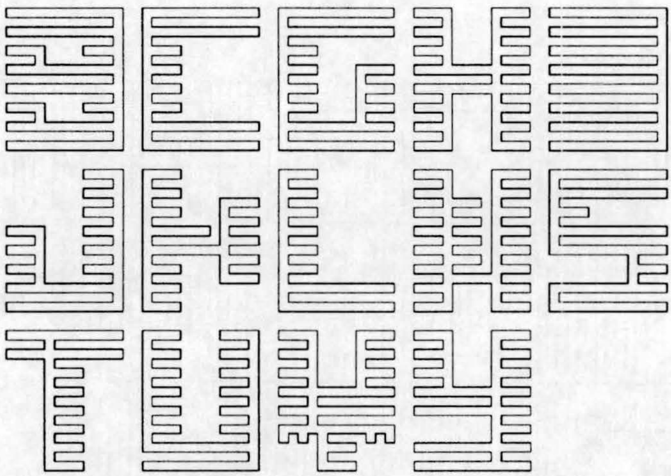

Figure 11: Hamilton cycles for the alphabet graphs B, C, G, H, I, J, K, L, M, S, T, U, V and Y for $m = 5$ and $n = 4$

any even number n, whereas the pattern in Figure 10(b) can be used for finding a Hamilton cycle for any even number m and any odd number n.

Hamilton cycles for the remaining alphabet graphs are shown in Figure 11 for $m = 5$ and $n = 4$ and in Figure 12 for $m = 6$ and $n = 4$. The patterns in Figure 11 can be used for finding Hamilton cycles for these graphs for any odd number m and any even number n. The patterns in Figure 12 can be used for determining Hamilton cycles for these graphs for any even number m and any number n. Finally, in Figure 13 we show spanning 2-connected subgraphs with $|V| + 1$ edges for the alphabet graphs in (viii) for $m = 7$ and $n = 5$. The patterns in this last figure can be used for determining such spanning subgraphs for these graphs for any odd numbers m and n.

To conclude this section, we present the remaining open problem.

Problem 9

(i) Is there a Hamilton cycle for the alphabet graph $X_{m,n}$ for $m = 5$ and any even n?

(ii) Is there a spanning 2-connected subgraph with (at most) $|V| + 1$ edges for the alphabet graph $X_{m,n}$ for $m = 3$ and any even n?

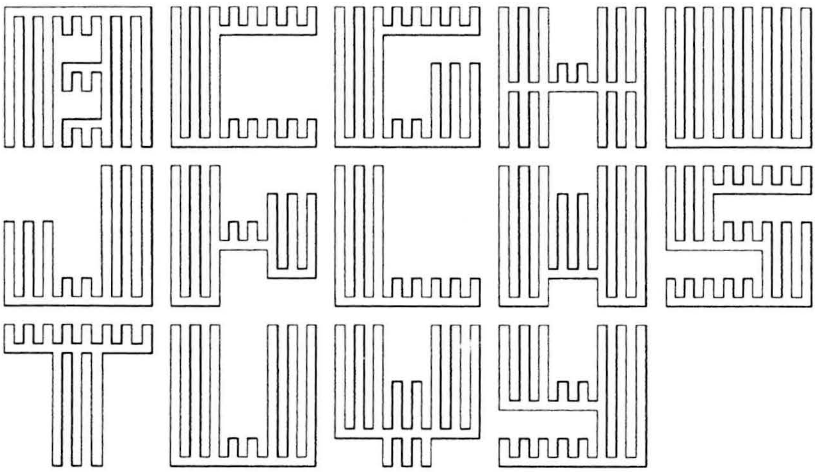

Figure 12: Hamilton cycles for the alphabet graphs B, C, G, H, I, J, K, L, M, S, T, U, V and Y for $m = 6$ and $n = 4$

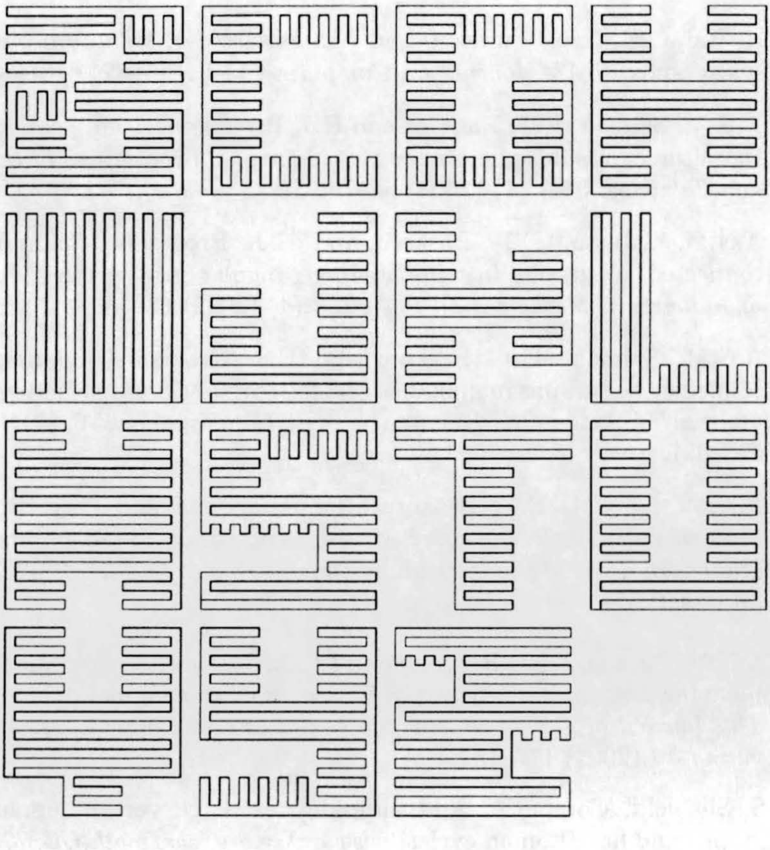

Figure 13: Spanning 2-connected subgraphs with $|V| + 1$ edges for the alphabet graphs in $(viii)$ for $m = 7$ and $n = 5$

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