The prevalence of "paradoxical" dice

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Abstract

"Common sense" might seem to indicate that if two fair dice are rolled against each other repeatedly, then over the long term the result will be predicted by their average rolls. We observe that dice which satisfy this "common sense" prediction are actually rather rare.

In September of 2003 Peter Winkler (the Director of Fundamental Mathematics Research at Bell Labs) came to Lafayette College, where he inaugurated the lecture series sponsored by our new math club. His talk was about several games, and over dinner he talked about more games. One involved three dice he carried with him. Each of the dice was a cube with 21 "spots" or "pips," like an ordinary die, but the pips were distributed in different ways. One had four pips on each of five faces, and a single pip on the last face; we will denote this one $B = (1, 4, 4, 4, 4, 4)$. Another was $C = (1, 1, 1, 6, 6, 6)$, and a third was $D = (3, 3, 3, 3, 3, 6)$.

To simplify our discussion we will presume that the faces of dice are labeled with integers rather than collections of pips. The natural game to play with two dice is to roll them, and declare the winner to be the die which lands with a larger number on the top face. One of the two dice is *stronger* if it wins more of the possible rolls than the other; if neither is stronger then the two dice are *tied*. For instance, the die B is stronger than the die D because it wins 25 of the 36 possible rolls (each of the five rolls of 4 wins against any of the five rolls of 3). D in turn is stronger than C (with 18 wins and only 15 losses among the 36 possible rolls), and C is stronger than B (C wins 18 of the 36 possible rolls, and loses only 15).

"Common sense" might suggest that the relative strengths of two dice are typically determined by their average rolls, and that B, C, D are exceptional in that they do not tie each other despite having the same average roll. It turns out instead that among the 32 six-sided dice whose faces are

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labeled with elements of $\{1,2,3,4,5,6\}$ which have the average $21/6$, only the standard die $A = (1,2,3,4,5,6)$ always ties.

Theorem 1. *Let $X = (x_1, ..., x_6)$ with $x_1, ..., x_6 \in \{1,2,3,4,5,6\}$.*

(a) A and X are tied if and only if $\sum x_i = 21$.

(b) If $X \neq A$ and $\sum x_i = 21$, then there is another die $Y = (y_1, ..., y_6)$ with $y_1, ..., y_6 \in \{1,2,3,4,5,6\}$ and $\sum y_i = 21$ which is not tied with X.

Proof. (a) If X is rolled and x_i comes up, it will lose against any higher roll of A and win against any lower roll of A. That is, x_i will win $x_i - 1$ rolls and lose $6 - x_i$ rolls. Hence X wins $\sum(x_i - 1)$ out of the 36 rolls, and loses $\sum(6 - x_i)$. Consequently A and X are tied if and only if $-6 + \sum x_i = 36 - \sum x_i$.

(b) Suppose X has m 1s and n 6s. If $m > n$ then X loses to $C = (1,1,1,6,6,6)$, because the three 6s on C win $3(6-n)$ of 36 rolls, and the three 1s on C only lose $3(6-m)$. Similarly, if $m < n$ then C loses to X.

Suppose $m = n$ and X has p 2s and q 5s. $E = (2,2,2,5,5,5)$ wins $3m + 3(6-n-q)$ of the 36 rolls against X, because each 2 wins against a roll of 1 and each 5 wins against a roll not 6 or 5. Similarly, E loses $3n+3(6-m-p)$ of the 36 rolls against X. If $p \neq q$ then E and X are not tied. If $p = q$ then a similar calculation shows that X and $F = (3,3,3,4,4,4)$ are not tied unless X has the same number of 3s as it has 4s.

Suppose now that X has n 1s, n 6s, p 2s, p 5s, r 3s and r 4s. Then X wins $6n + 6p + r + r + p$ times against B, and B wins $5(n+p+r)$ times against X. Hence if B and X are tied then $n + 2p = 3r$. Against $G = (1,3,3,3,5,6)$, X wins $5n + 4p + 4r + r + p = 5n + 5p + 5r$ times and loses $3(n+p) + (n+p+2r) + (n+2p+2r) = 5n + 6p + 4r$ times. Hence if G and X are tied then $p = r$; $n + 2p = 3r$ implies that $n = p = r$. $\sum x_i = 7n + 7p + 7r = 21$, so it follows that $n = p = r = 1$. That is, $X = A$. ■

Peter Winkler has suggested generalizing Theorem 1 to other kinds of dice. Given integers $n > 0$ and a, b, s let $D(n,a,b,s)$ denote the set of all lists $X = (x_1, ..., x_n)$ of integers such that $a \leq x_1 \leq ... \leq x_n \leq b$ and $\sum x_i = s$. An element of $D(n,a,b,s)$ is an *n-sided generalized die with integer labels between a and b which sum to s.* Let $L(n,a,b,s) = \{x | \exists X \in D(n,a,b,s)$ with $x = x_i$ for some $i\}$. Note that $L(n,a,b,s)$ might not include all of the integers from a to b; indeed $L(n,a,b,s)$ might be empty. If $L(n,a,b,s) \neq \emptyset$

then let $p(n,a,b,s) = \min L(n,a,b,s)$ and $q(n,a,b,s) = \max L(n,a,b,s)$; we use p and q unless there is danger of confusion. Clearly $L(n,a,b,s) = \{x \in \mathbb{Z} | p \leq x \leq q\}$ and $D(n,a,b,s) = D(n,p,q,s)$.

We are interested in the elements of $D(n,a,b,s)$ which are *balanced* in the sense that they tie all other elements of $D(n,a,b,s)$.

For each $X = (x_1, ..., x_n) \in D(n,a,b,s)$ let $f_X : L(n,a,b,s) \to \mathbb{Z}$ give the win-loss difference of a roll of x against the die X; that is,

$$f_X(x) = |\{i|x_i < x\}| - |\{i|x_i > x\}| .$$

This function may be used to determine the relative strengths of X and other dice: if $Y = (y_1, ..., y_n) \in D(n,a,b,s)$ then $\sum f_X(y_i)$ is positive, negative or 0 according to whether X is weaker than Y, stronger than Y, or tied with Y.

Theorem 2. (*a*) *All the elements of* $D(2,a,b,s)$ *are balanced.*

(*b*) *If* $n \geq 3$ *then a given* $X \in D(n,a,b,s)$ *is balanced if and only if there is a* $c_X \in \mathbb{Z}$ *with*

$$f_X(x) - f_X(p) = c_X \cdot (x - p)$$

for every $x \in L(n,a,b,s)$.

Proof. (a) Two distinct 2-sided dice with the same sum must be (x_1, x_2) and (y_1, y_2) with $x_1 < y_1 < y_2 < x_2$, so each die wins two of the four possible rolls.

(b) If $q - p \leq 1$ then $|D(n,a,b,s)| = 1$ and the equivalent conditions stated in (b) are both trivially true.

Suppose $q - p \geq 2$. If there is a c_X as in the statement of (b) then for any $Y = (y_1, ..., y_n) \in D(n,a,b,s)$,

$$\begin{aligned}
\sum_{i=1}^{n} f_X(y_i) &= nf_X(p) - npc_X + c_X \sum_{i=1}^{n} y_i \\
&= nf_X(p) - npc_X + sc_X \\
&= nf_X(p) - npc_X + c_X \sum_{i=1}^{n} x_i \\
&= \sum_{i=1}^{n} f_X(x_i),
\end{aligned}$$

which is 0 because X must be tied with itself. Hence X and Y are tied.

Suppose conversely that X is tied with every $Y \in D(n,a,b,s)$. Observe that if the labels $v > p$ and $w < q$ both appear in Y then there is a $Z \in D(n,a,b,s)$ with the same labels as Y (including multiplicities) except for the replacement of one v by $v-1$ and the replacement of one w by $w+1$. Then $\sum f_X(y_i) = 0 = \sum f_X(z_i)$ implies that $f_X(v) - f_X(v-1) = f_X(w+1) - f_X(w)$.

There is a unique k with $1 \le k \le n-1$ and $(k+1)p+(n-k-1)q < s \le kp+(n-k)q$. Hence there is a $Y^0 = (y_1^0, ..., y_n^0) \in D(n,a,b,s)$ with $y_i^0 = p$ for $i \le k$ and $y_i^0 = q$ for $i \ge k+2$; moreover if $k = n-1$ then $y_n^0 = q$. In any case, p and q both appear as labels in Y^0. We obtain elements $Y^1, ..., Y^{q-p}$ of $D(n,a,b,s)$ by replacing single appearances of $p+i$ and $q-i$ in Y^i with appearances of $p+i+1$ and $q-i-1$ in Y^{i+1}; note that because $n \ge 3$ there is at least one label which appears in every one of $Y^0, ..., Y^{q-p}$. Applying the observation of the preceding paragraph to $v = q-i$ and $w = p+i$ in Y^i, we conclude that $f_X(q-i) - f_X(q-i-1) = f_X(p+i+1) - f_X(p+i)$ for each i. Considering that $Y^0, ..., Y^{q-p}$ all share a label, the observation of the preceding paragraph also implies that these differences are all equal, i.e., that $f_X(x) - f_X(x-1)$ is the same for all $x \in \{p+1, ..., q\}$. This common value is c_X. ■

If $n \ge 3$ and $|L(n,a,b,s)| \ge 3$ then the condition that there be a $c_X \in \mathbb{Z}$ with $f_X(x) - f_X(p) \equiv c_X \cdot (x-p)$ is easily described using the *characteristic vector* $(v_p, ..., v_q)$ of $X = (x_1, ..., x_n)$, where $v_j = |\{i | x_i = j\}|$. If X satisfies the condition then $c_X = f_X(j+1) - f_X(j) = v_j + v_{j+1}$ for every $j \in \{p, ..., q-1\}$, so the characteristic vector must be of the form $(v, w, v, w, ...)$. For instance, the balanced elements of $D(6,1,6,s)$ include $(1,1,3,3,5,5)$ with $v = 2$ and $w = 0$, $(2,2,4,4,6,6)$ with $v = 0$ and $w = 2$, and $(1,2,3,4,5,6)$ with $v = w = 1$.

A consequence is that balanced dice obey "common sense" not only by tying dice with the same average roll, but also by winning (resp. losing) against dice with lower (resp. higher) average rolls.

Corollary 3. *Suppose $n \ge 3$ and $X \in D(n,a,b,s)$ is balanced. Then X is weaker than every $Y \in D(n, p(n,a,b,s), q(n,a,b,s), s')$ with $s' > s$, and X is stronger than every $Y \in D(n, p(n,a,b,s), q(n,a,b,s), s')$ with $s' < s$.*

Proof. If $Y = (y_1, ..., y_n) \in D(n, p(n,a,b,s), q(n,a,b,s), s')$ then X is weaker or stronger than Y according to whether $\sum f_X(y_i)$ is positive or negative. X must tie itself, so $\sum f_X(x_i) = 0$. Theorem 2 implies that

$$\begin{aligned}
\sum_{i=1}^{n} f_X(y_i) &= \sum_{i=1}^{n} f_X(y_i) - \sum_{i=1}^{n} f_X(x_i) \\
&= \sum_{i=1}^{n} (f_X(p) + c_X \cdot (y_i - p)) - \sum_{i=1}^{n} (f_X(p) + c_X \cdot (x_i - p)) \\
&= \sum_{i=1}^{n} c_X \cdot (y_i - x_i) = c_X \cdot (s' - s),
\end{aligned}$$

so X is weaker or stronger than Y according to whether $s' - s$ is positive or negative. ■

The next corollary generalizes part (b) of Theorem 1.

Corollary 4. *If $q(n,a,b,s) - p(n,a,b,s) \geq 3$ is odd and $n \geq 3$ then there is at most one balanced $X \in D(n,a,b,s)$.*

Proof. Suppose $X, X' \in D(n,a,b,s)$ are both tied with all the elements of $D(n,a,b,s)$; let their characteristic vectors be $(v,w,v,w,...)$ and $(v',w',v',w',...)$ respectively. Let $n_1 = |\{i \geq 0 | p + 2i \leq q\}|$ and $n_2 = |\{i \geq 0 | p + 1 + 2i \leq q\}|$. Then $vn_1 + wn_2 = n = v'n_1 + w'n_2$, so $(v-v')n_1 = (w'-w)n_2$. If $v = v'$ then it follows that $w = w'$, and hence $X = X'$.

Suppose $v \neq v'$. The label-sums of X and X' are both s, so

$$\begin{aligned}
s &= v \sum_{i=0}^{n_1-1} (p+2i) + w \sum_{i=0}^{n_2-1} (p+1+2i) \\
&= v' \sum_{i=0}^{n_1-1} (p+2i) + w' \sum_{i=0}^{n_2-1} (p+1+2i).
\end{aligned}$$

These equalities imply

$$(v-v')(p+n_1-1)n_1 = (w'-w)(p+n_2)n_2.$$

Then $(v-v')n_1 = (w'-w)n_2$ implies

$$(v-v')(p+n_1-1)n_1 = (v-v')(p+n_2)n_1,$$

and hence $n_1 - 1 = n_2$ by cancellation. This contradicts the hypothesis that $q - p = n_1 + n_2 - 1$ is odd. ■

If n, a, b are as in Corollary 4 then there may be several different s-values for which $D(n, a, b, s)$ has a balanced element. For instance each of $D(6, 1, 6, 18)$, $D(6, 1, 6, 21)$, $D(6, 1, 6, 24)$ has a balanced element, as observed above.

If $q(n, a, b, s) - p(n, a, b, s) > 3$ is even and $n \geq 3$ then the situation is reversed: there is only one s-value for which $D(n, a, b, s)$ may have a balanced element, and such an element need not be unique. If $D(n, a, b, s)$ has a balanced element then this element's characteristic vector is $(v, w, v, w, ..., v, w, v)$ with n_1 entries equal to v and $n_2 = n_1 - 1$ entries equal to w. The labels other than the "middle" label $\frac{p+q}{2}$ may be paired so that the sum of each pair is $p + q$; then if n_1 is odd

$$\begin{aligned} s &= \left(\frac{n_1 - 1}{2}\right) v(p+q) + v\left(\frac{p+q}{2}\right) + \left(\frac{n_2}{2}\right) w(p+q) \\ &= \left(\frac{p+q}{2}\right)(n_1 v + n_2 w) = \left(\frac{p+q}{2}\right) n, \end{aligned}$$

and if n_1 is even

$$\begin{aligned} s &= \left(\frac{n_1}{2}\right) v(p+q) + w\left(\frac{p+q}{2}\right) + \left(\frac{n_2 - 1}{2}\right) w(p+q) \\ &= \left(\frac{p+q}{2}\right)(n_1 v + n_2 w) = \left(\frac{p+q}{2}\right) n. \end{aligned}$$

$D(n, a, b, (\frac{p+q}{2})n)$ may have several different balanced elements; for instance each of $(2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4)$, $(1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5)$ and $(1, 1, 1, 1, 3, 3, 3, 3, 5, 5, 5, 5)$ is tied with all the elements of $D(12, 1, 5, 36)$.

The special case most directly suggested by familiar 6-sided dice occurs when $a = 1$ and $b = n$.

Corollary 5. *If $n \geq 2$ is even then $D(n, 1, n, s)$ has a balanced element if and only if $s \in \{n, n+1, \frac{n^2}{2}, \frac{n(n+1)}{2}, \frac{n(n+2)}{2}, n^2 - 1, n^2\}$, and such an element is unique. If $n \geq 3$ is odd then $D(n, 1, n, s)$ has a balanced element if and only if $s \in \{n, n+1, \frac{n(n+1)}{2}, n^2 - 1, n^2\}$, and such an element is unique unless $n = 3$.*

Proof. If $1 \leq |L(n, 1, n, s)| \leq 2$ then $s \in \{n, n+1, n^2 - 1, n^2\}$ and $|D(n, 1, n, s)| = 1$. The single element of $D(n, 1, n, s)$ is balanced, of course.

Suppose $|L(n, 1, n, s)| > 2$; then $n > 2$. If $n + 1 < s < 2n$ then $(1, 1, ..., 1, s - n + 1)$ is weaker than every other element of $D(n, 1, n, s)$, so there is no balanced element. If $n^2 - n < s < n^2 - 1$ then $(s - n^2 + n, n, n, ..., n)$ is stronger than every other element of $D(n, 1, n, s)$, so there is no balanced element.

Suppose $2n \le s \le n^2 - n$; then $p = 1$ and $q = n$. If n is even then a balanced element of $D(n, 1, n, s)$ has characteristic vector $(v, w, ..., v, w)$, with $v\left(\frac{n}{2}\right) + w\left(\frac{n}{2}\right) = n$. The only possibilities are $v = w = 1$ and $v \ne w \in \{0, 2\}$, corresponding to $s \in \{\frac{n^2}{2}, \frac{n(n+1)}{2}, \frac{n(n+2)}{2}\}$. If n is odd, on the other hand, the discussion preceding this corollary implies that $2s = (p+q)n = n(n+1)$. The characteristic vector of a balanced element is $(v, w, v, ..., w, v)$ with $n = v\left(\frac{n+1}{2}\right) + w\left(\frac{n-1}{2}\right)$. This is impossible if $v > 1$. If $v = 1$ then $w(n-1) = 2n - (n+1) = n - 1$ and hence $v = w = 1$. If $v = 0$ then $2n = w(n-1)$ and hence $w = \frac{2n}{n-1} = 2 + \frac{2}{n-1}$, so $\frac{2}{n-1} \in \mathbb{Z}$; necessarily then $n = 3$ and $w = 2 + \frac{2}{2} = 3$. ■

Dice may be thought of as candidates in a multi-candidate election. Each voter gives each candidate a number of points from a to b; for instance a voter who likes all the candidates may give every candidate b points, and a voter who likes none of them may give every candidate a points. The die (2,2,3,3,4,5) represents a candidate who has attracted the following support: one of the six voters gives the candidate 5 points, one of the six voters gives the candidate 4 points, two of the six voters give the candidate 3 points apiece, and two of the six voters give the candidate 2 points apiece. With this interpretation, the results above indicate that almost all possible candidates in this kind of election are involved in "paradoxical" election results, winning and losing one-on-one contests against candidates with the same average level of voter support. Such electoral "paradoxes" have been carefully studied in recent years; see [1] for an engaging and accessible introduction.

Acknowledgment

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References

[1] D. G. Saari, *Chaotic elections!*, Amer. Math. Soc., Providence, 2001.