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Harmonic Index of Cubic Polyhedral Graphs Using Bridge Graphs

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Abstract

Let G be a simple graph, the vertex-set and edge-set of which are represented by $V(G)$ and $E(G)$ respectively. In this paper, the Harmonic indices of cubic polyhedral graphs using bridge graphs and the relations among them have been found.

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1 Introduction

A graph is a mathematical concept to determine whether different points termed vertices are connected or not. This also includes the manner in which the points are connected with each other. However, it is not necessary that all points be connected with the rest. All graphs used in this paper are simple undirected which means no multiple edges, and no loops. A graph is k-regular if $d_v = k \ \forall \ v \in V(G)$ where d_v is the degree of the vertex v representing the edges incident to v and $V(G)$ is the vertex set of the graph G. Throughout this paper, n represents the nodes or vertices, b represents the number of bridges.

Cubic Polyhedral graphs are graphs in which all the vertices are of degree 3. Cubic polyhedral graphs are $3-$ regular graphs. Cubic polyhedral graphs start with vertices $n = 4, 6, 8, ...$ A cubic polyhedral graph with $n = 12$ nodes is known as Frucht graph. Frucht graph is the smallest cubic identity graph with 12 vertices and 18 edges. The bridge graph $B(G_i) = B(G_i; v_i)$ of $\{G_i\}_{i=1}^{d}$ and $\{v_i\}_{i=1}^{d}$ is the graph obtained from the graphs $G_1, G_2, ...G_d$ by connecting the vertices v_i and v_{i+1} by an edge for all $i = 1, 2, ...(d-1)$.

For works in operations on graphs see[2-4]. For more related aspects refer to [6-10]

The Harmonic index is given by

$$H(G) = \sum_{(u,v)\in E(G)} \frac{2}{d_u + d_v}. \tag{1.1}$$

If u and v are two vertices of G then let $d_G(u,v)$ denote the length of the shortest path connecting u and v. The work is motivated from the works of [5, 11]. Here we deal with Harmonic indices of Cubic polyhedral graphs using bridge graphs and interesting relationships are obtained.

Forthcoming section deals with elementary results useful to main results. The final section concerns with cubic graphs attached with bridges and corresponding relations.

2 Preliminaries Results

This section concentrates on results for the harmonic index of cubic graphs. As indicated earlier, n represents the nodes or vertices while b represents the number of bridges.

Lemma 2.1. *The Harmonic index $H(G)$ of a cubic graph $n \geq 4$ is*

$$H(G) = \frac{n}{2} \tag{2.1}$$

Proof. For $n \geq 4$ of a cubic graph, there are $\frac{3n}{2}$ edges. As each of the vertices is of degree 3, the Harmonic index is $\frac{n}{2}$. □

Theorem 2.1. *The Harmonic index of a bridge graph of a cubic graph with n even vertices (for $n \geq 4$) is*

$$H(G_n) = \frac{(3b+3)(n-2)}{2}\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \tag{2.2}$$

Proof. We prove in different cases:
Case(i): For $n = 4$.
For two 4 vertex cubic graphs, if a bridge is attached, there are $(3,3)$, $(3,4)$, $(3,5)$, $(4,4)$, $(4,5)$ and $(5,5)$ edges for $B(G_1, G_2) = B(G_1, G_2; v_1, v_2)$. The number of edges is $(7b+6)$. As the number of bridges increases and the graphs are increased, the Harmonic index is

$$H(G_4) = (3b+3)\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right].+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \tag{2.3}$$

Case(ii): For $n = 6$.
If a bridge is attached with two cubic graphs with 6 vertices, there are $(3,3)$, $(3,4)$, $(3,5)$, $(4,5)$ and $(5,5)$ edges. In particular, there are $(6b+6)$ number of $(3,3)$ edges, $(3b+3)$ number of $(3,4)$ edges, $(3b-3)$ number of $(3,5)$ edges, 2 number of $(4,5)$ edges, $(b-2)$ number of $(5,5)$ edges. Hence the total number of edges will be $(10b+9)$. Then the Harmonic index is

$$H(G_6) = (6b+6)\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \tag{2.4}$$

Case(iii): For $n = 8$.
If a bridge is attached with two cubic graphs of 8 vertices, there are $(3,3)$, $(3,4)$, $(3,5)$, $(4,5)$ and $(5,5)$ edges. Specifically, there are $(9b+9)$ number of $(3,3)$ edges, 6 number of $(3,4)$, $(3b-3)$ number of $(3,5)$, 2 number of $(4,5)$ and $(b-2)$ number of $(5,5)$. Hence, the total number of edges are $13b+12$. Then, the Harmonic index is

$$H(G_8) = \left[\frac{9b+9}{2}\right]\left[(\frac{2}{6})\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \quad (2.5)$$

From the cases (i), (ii) and (iii) we conclude in general form of $H(G_n)$ is

$$H(G_n) = \left[\frac{(3b+3)(n-2)}{2}\right]\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \quad (2.6)$$

□

3 Cubic graphs attached with bridges and relations among them

In this section, the different kinds of cubic graphs attached with bridges relations of Harmonic index are established. Also, some useful remarks are advertised.

Theorem 3.1. *The Harmonic index of a bridge graph formed with the bridges for the cubic graph of $n = 4$ vertices followed by 6 vertices is*

$$H(G_{4,6}) = (b-3)\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \quad (3.1)$$

Proof. A bridge graph $B(G_4, G_6) = B(G_4, G_6; v_1, v_2)$ of a cubic graph of $G_{4,6}$ is attached by a bridge in the same sequence. Then the bridge graph will have the types $(3,3)$, $(3,4)$, $(3,5)$, $(4,5)$ and $(5,5)$ of edges. The number of $(3,3)$ edges is $(b-3)$, $(3,4)$ edges is 6, $(3,5)$ edges is $(3b-3)$, $(4,5)$ edges is 2 and $(5,5)$ edges is $(b-2)$. Hence, the resultant Harmonic index is

$$H(G_{4,6}) = (b-3)\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \quad (3.2)$$

□

Theorem 3.2. *The Harmonic index of a bridge graph formed by cubic graph of* 6 *vertices followed by* 4 *vertices for even and odd number of bridges respectively is*

$$H(G_{6,4}) = \left[\frac{9b+9}{2}\right]\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \quad (3.3)$$

Proof. The sequence of $B(G_6, G_4) = B(G_6, G_4; v_1, v_2)$ of a cubic graph of $G_{6,4}$ vertices are attached by a bridge. Then, the bridge graph of the types $(3,3)$, $(3,4)$, $(3,5)$, $(4,5)$ and $(5,5)$ of edges. The number of $(3,3)$ edges is $(9b+9)/2$, $(3,4)$ edges is 6, $(3,5)$ edges is $(3b-3)$, $(4,5)$ edges is 2 and $(5,5)$ edges is $(b-2)$. Therefore, the Harmonic index results in the following equation.

$$H(G_{6,4}) = \left[\frac{9b+9}{2}\right]\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \quad (3.4)$$

$\square$

Remark 1: The Harmonic index of the graph obtained by $n = 4$ and $n = 6$ vertices are attached by a bridge in a sequence. This sequence is followed by another sequence of $n = 6$ and $n = 4$ vertices. The two sequences are attached by a bridge are compared. The following relation is found.

$$H(G_{6,4}) = H(G_{4,6}) + \left[\frac{7b+15}{2}\right]. \quad (3.5)$$

Theorem 3.3. *The Harmonic index of a bridge graph formed by cubic graph of* $n = 4, 6, 8$ *vertices is*

$$H(G_{4,6,8}) = (6b+6)\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \quad (3.6)$$

Proof. A bridge graph formed by considering $4, 6$ and 8 vertices as $B(G_4, G_6, G_8) = B(G_4, G_6, G_8; v_1, v_2, v_3)$. The vertices of $n = 4, 6, 8$ are attached by two bridges in a sequence. Such a bridge graph will have the types $(3,3)$, $(3,4)$, $(3,5)$, $(4,4)$, $(4,5)$, and $(5,5)$ of edges. The number of $(3,3)$ edges are $(6b+6)$, $(3,4)$ edges are 6, $(3,5)$ edges are $(3b-3)$, $(4,5)$ edges are 2 and $(5,5)$ is equal to $(b-2)$. Hence, the resultant Harmonic index is

$$H(G_{4,6,8}) = (6b+6)\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \quad (3.7)$$

$\square$

Theorem 3.4. *The Harmonic index of a bridge graph formed by cubic graph formed by vertices $G_{8,6,4}$ is*

$$H(G_{8,6,4}) = (6b+6)\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \quad (3.8)$$

Proof. A bridge graph formed by considering the arrangement as $B(G_8, G_6, G_4) = B(G_8, G_6, G_4; v_1, v_2, v_3)$ are attached by two bridges in a sequence. This bridge graph will be similar to the arrangement as $B(G_4, G_6, G_8) = B(G_4, G_6, G_8; v_1, v_2, v_3)$. Hence, the Harmonic index in such an arrangement as

$$H(G_{8,6,4}) = (6b+6)\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+(3b-3)\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+(b-2)\left[\frac{2}{10}\right]. \quad (3.9)$$

□

Remark 2: The Harmonic index is found to be equal for a bridge graph formed by cubic graphs with vertices $G_{4,6,8}$ attached by two bridges in a sequence and the bridge graph formed by cubic graphs with vertices in the reverse order, i.e., $G_{8,6,4}$ and attached by two bridges in a sequence.

$$H(G_{8,6,4}) = H(G_{4,6,8}). \quad (3.10)$$

Theorem 3.5. *A bridge graph of a cubic graph is formed with vertices $n = 4, 6, 8$ in a sequence. And this bridge graph is attached to another bridge graph in the reverse sequence with $n = 8, 6, 4$. This combined sequence is repeated for varying number of bridges. The Harmonic index of such a combined bridge graph in a sequence is*

$$H(G_{4,6,8,8,6,4,4,6,8,8,6,4}) = (6b+6)\left[\frac{2}{6}\right]+6\left[\frac{2}{7}\right]+\left[\frac{(3b+9)}{2}\right]\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+\left[\frac{(b+1)}{2}\right]\left[\frac{2}{10}\right]. \quad (3.11)$$

Proof. A bridge graph formed by considering $4, 6$ and 8 vertices as $B(G_4, G_6, G_8) = B(G_4, G_6, G_8; v_1, v_2, v_3)$ of a cubic graph are in the order of $4, 6, 8, 8, 6, 4, 4, 6, 8, 8, 6, 4$ vertices. A minimum of 5 bridges is necessary to form such a combination. Such a bridge graph of the types $(3,3)$, $(3,4)$, $(3,5)$, $(4,4)$, $(4,5)$ and $(5,5)$ edges. The number of $(3,3)$ edges are $(6b+6)$, $(3,4)$ edges are 6, $(3,5)$ edges are $(3b+9)/2$, $(4,5)$ edges are 2 and $(5,5)$ is equal to $(b+1)/2$. Hence, the Harmonic index is

$$H(G_{4,6,8,8,6,4,4,6,8,8,6,4}) = (6b+6)\left[\frac{2}{6}\right]+(6)\left[\frac{2}{7}\right]+\left[\frac{(3b+9)}{2}\right]\left[\frac{2}{8}\right]+(2)\left[\frac{2}{9}\right]+\left[\frac{(b+1)}{2}\right]\left[\frac{2}{10}\right]. \quad (3.12)$$

□

Theorem 3.6. *The Harmonic index of a bridge graph formed by cubic graph with vertices* $G_{4,6,8,10}$ *is*

$$H(G_{4,6,8,10}) = \left[\frac{(15b+15)}{2}\right]\left[\frac{2}{6}\right] + (6)\left[\frac{2}{7}\right] + (3b-3)\left[\frac{2}{8}\right] + (2)\left[\frac{2}{9}\right] + (b-2)\left[\frac{2}{10}\right]. \tag{3.13}$$

Proof. A bridge graph formed by considering $G_{4,6,8,10}$ vertices as $B(G_4, G_6, G_8, G_{10}) = B(G_4, G_6, G_8, G_{10}; v_1, v_2, v_3, v_4)$ with $G_{4,6,8,10}$ vertices obtained. Such a bridge graph of the types $(3,3)$, $(3,4)$, $(3,5)$, $(4,5)$ and $(5,5)$ edges. The number of $(3,3)$ edges are $(15b+15)/2$, $(3,4)$ edges are 6, $(3,5)$ edges are $(3b-3)$, $(4,5)$ edges are 2 and $(5,5)$ is equal to $(b-2)$. Hence, the Harmonic index is

$$H(G_{4,6,8,10}) = \left[\frac{(15b+15)}{2}\right]\left[\frac{2}{6}\right] + (6)\left[\frac{2}{7}\right] + (3b-3)\left[\frac{2}{8}\right] + (2)\left[\frac{2}{9}\right] + (b-2)\left[\frac{2}{10}\right]. \tag{3.14}$$

□

Theorem 3.7. *The Harmonic index of a bridge graph formed by cubic graph of* $G_{10,8,6,4}$ *is*

$$H(G_{10,8,6,4}) = (3b+3)\left[\frac{2}{7}\right] + \left[\frac{(15b+15)}{2}\right]\left[\frac{2}{6}\right] + b\left[\frac{2}{8}\right]. \tag{3.15}$$

Proof. A bridge graph $B(G_{10}, G_8, G_6, G_4) = B(G_{10}, G_8, G_6, G_4; v_1, v_2, v_3, v_4)$ of a cubic graph with $G_{10,8,6,4}$ vertices is obtained by three bridges in a sequence. Then, the bridge graph of the types $(3,3)$, $(3,4)$ and $(4,4)$ edges. The number of $(3,4)$ edges is $(3b+3)$, $(3,3)$ edges is $(15b+15)/2$, and $(4,4)$ edges is equal to the number of bridges attached. Then, the Harmonic index is

$$H(G_{10,8,6,4}) = (3b+3)\left[\frac{2}{7}\right] + \left[\frac{(15b+15)}{2}\right]\left[\frac{2}{6}\right] + b\left[\frac{2}{8}\right]. \tag{3.16}$$

□

Remark 3: The Harmonic index of a bridge graph, formed by cubic graph with vertices $G_{10,8,6,4}$ vertices in a sequence, is compared with the Harmonic index of bridge graph formed by cubic graph with $G_{10,8,6,4}$ vertices by three bridges in a sequence and both are found to be equal.

$$H(G_{4,6,8,10}) = H(G_{10,8,6,4}). \tag{3.17}$$

Remark 4: The Harmonic index of the bridge graph formed by cubic graph with $G_{4,6}$ vertices is compared with a bridge graph formed by cubic graph with $G_{4,6,8}$ vertices. The resultant equality as shown

$$H(G_{4,6,8}) = H(G_{4,6}) + \left[\frac{5b+9}{3}\right]. \tag{3.18}$$

Remark 5: The relationship between Harmonic index of bridge graph formed by cubic graph with $G_{4,6,8,10}$ and $G_{4,6,8}$ as shown

$$H(G_{4,6,8,10}) = H(G_{4,6,8}) + \left[\frac{b+1}{2}\right]. \tag{3.19}$$

Remark 6: The relationship between Harmonic index of bridge graph formed by cubic graph with $G_{4,6}$ vertices is compared with the bridge graph formed by cubic graph with $G_{4,6,8,10}$ vertices as shown

$$H(G_{4,6,8,10}) = H(G_{4,6}) + \left[\frac{13b+21}{6}\right]. \tag{3.20}$$

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References

[1] V. Lokesha and P. S. Ranjini, *Eccentric Connectivity Index, Hyper and Reverse-Wiener Indices of the Subdivision Graph*, General Mathematics Notes, 2(2),(2011), Art.14.

[2] M. A. Rajan, V. Lokesha and P. S. Ranjini, *On New Vertex Naturally Labeled Graphs*, Int. e-Journal of Mathematics and Engineering, (IeJMAE), 2(1)(2011), pp. 992-999.

[3] M. A. Rajan, V. Lokesha and K. M. Niranjan, *On Study of Vertex Labeling Graph Operations*, J. Sci. Res. 3(2)(2011), pp. 291-301. http://dx.doi.org/10.3329/jsr.v3i2.6222

[4] M. A. Rajan, V. Lokesha and P. S. Ranjini, *On some Properties of Splitting of Graphs*, Antarctica Journal of Mathematics, 8(2)(2011), pp. 103-108.

[5] P. S. Ranjini, V. Lokesha and Usha A., *Relation between Phenylene and Hexagonal Squeeze using Harmonic Index*, International Journal of Graph Theory, 1(4)(2013), pp. 116-121.

[6] P. S. Ranjini, V. Lokesha and M. A. Rajan, *On Wiener Polynomial of the Subdivision Graphs*, Chineese Journal of Engineering mathematics, (3)(2011), pp. 411-418.

[7] P. S. Ranjini, V. Lokesha and Naci Cangul, *On the Zagreb Indices of the Line Graphs of the Subdivision Graphs*, Applied Mathematics and Computation, 218(3)(2011), pp. 699-702.
http://dx.doi.org/10.1016/j.amc.2011.03.125

[8] P. S. Ranjini, V. Lokesha, M. A. Rajan and M. Phani Raju,*On the Shultz Index of the Subdivision Graphs*, Adv. Stud. Contemp. Math., Kyungshang 21(3)(2011), pp. 279-290.

[9] B. Shwetha Shetty, V. Lokesha and P. S. Ranjini, *On the Harmonic Index of Graph Operations*, Transactions on Combinatorics, (2015), in Press.

[10] R. Todeschini, V. Consonni, *Handbook of Molecular Descriptors*, Wiley-VCH, Weinheim(2000). http://dx.doi.org/10.1002/9783527613106

[11] Xinli Xu, *Relationships between Harmonic Index and other Topological Indices*, Applied Mathematical Sciences, 6(41),(2012), pp. 2013-2018.

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