The quasi xgamma distribution with application in bladder cancer data

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Abstract: For the purpose of generalizing or extending an existing probability distribution, incorporation of additional parameter to it is very common in the statistical distribution theory and practice. In fact, in most of the times, such extensions provide better fit to the real life situations compared to the existing ones. In this article, we propose and study a two-parameter probability distribution, called quasi xgamma distribution, as an extension or generalization of xgamma distribution (Sen et al. 2016) for modeling lifetime data. Important distributional properties along with survival characteristics and distributions of order statistics are studied in detail. Method of maximum likelihood and method of moments are proposed and described for parameter estimation. A data generation algorithm is proposed supported by a Monte-Carlo simulation study to describe the mean square errors of estimates for different sample sizes. A bladder cancer survival data is used to illustrate the application and suitability of the proposed distribution as a potential survival model.

Key words: Lifetime distributions, maximum likelihood estimation, order statistics, failure rate function.

1. Introduction

Modeling and analyzing lifetime data are crucial in many applied sciences such as biomedical sciences, reliability engineering and actuarial sciences, amongst others. Several lifetime distributions have been used to model such kinds of data. Relevant statistical methodologies and the quality of the procedures used in a statistical analysis depend heavily on the assumed probability model or distribution when parametric platform is utilized. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models.

Adding an extra parameter to an existing family of distributions is very common in the statistical distribution theory. Often introducing an extra parameter brings additional flexibility to a class of probability distributions, and, in turn, it can be very useful for data analysis purposes. Several authors in statistical literature have been introduced excellent methods in adding extra parameter(s) to existing distribution for added flexibility in terms of distributional properties, computations, statistical inferences and in describing uncertainties behind real world phenomena, see for more survey on methods of adding parameters to standard models Mudholkar and

Srivastava 1993, Azzalini 1995, Marshall and Olkin 1997, Eugene et al. 2002, Lee et al. 2013, Alzaatreh et al. 2013 and Jones 2014.

Therefore, introducing new probability distributions and/or extending (or generalizing) existing probability distributions by adding extra parameters into its form has become a time-honored device for obtaining more flexible new families of distributions.

In this article, we incorporate an extra parameter to the one parameter xgamma distribution, which serves as a useful lifetime model introduced and studied by Sen et al. 2016, for more flexibility in describing data that might follow situations.

A continuous random variable X is said to follow an xgamma distribution if its probability density function (PDF) is of the form

$$f(x) = \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}\,x^2\right)e^{-\theta x} \quad ; x > 0, \theta > 0 \tag{1}$$

and is denoted by $X \sim \text{xgamma}(\theta)$. The cumulative density function (CDF) of X is given by

$$F(x) = 1 - \frac{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)}{(1+\theta)}e^{-\theta x} \quad ; x > 0 \text{ and } \theta > 0 \tag{2}$$

2. The quasi xgamma distribution

A continuous random variable X, with support $(0, \infty)$, is said to follow a quasi xgamma distribution with parameters α and θ if its probability density function (PDF) is of the form

$$f(x) = \frac{\theta}{(1+\alpha)}\left(\alpha+\frac{\theta^2}{2}\,x^2\right)e^{-\theta x} \quad ; x > 0, \theta > 0 \text{ and } \alpha > 0 \tag{3}$$

and is denoted by $X \sim \text{QXD}(\alpha, \theta)$.
The cumulative distribution function (CDF) of quasi xgamma distribution is obtained as

$$F(x) = 1 - \frac{\left(1+\alpha+\theta x+\frac{\theta^2 x^2}{2}\right)}{(1+\alpha)}e^{-\theta x} \quad ; x > 0, \theta > 0 \text{ and } \alpha > 0 \tag{4}$$

It is to be noted that the quasi xgamma distribution such obtained includes xgamma distribution as a special case.
For particular values of α, from (3) we obtain the following special cases:

(i) When $\alpha = 0$, we have gamma distribution with shape parameter 3 and scale parameter θ, i.e., $X \sim G(\theta, 3)$.

(ii) When α = 1, we obtain a new class of distributions with PDF

$$f(x) = \frac{\theta}{2}\left(1 + \frac{\theta^2}{2}\, x^2\right) e^{-\theta x} \quad ; x > 0 \text{ and } \theta > 0$$

(iii) When α = θ, we have the xgamma distribution with PDF given in (1).

Note: The quasi xgamma distribution characterized by the PDF given in (3) is a special mixture of exponential distribution with mean 1/ θ and gamma distribution with shape parameter 3 and scale parameter θ, with mixing proportions α/(1+ α) and 1/(1+ α) respectively.

The xgamma distribution is also synthesized mixing exponential distribution with mean 1/ θ and gamma distribution with shape parameter 3 and scale parameter θ, with mixing proportions θ /(1+ θ) and 1/(1+ θ) respectively. The plots of the density functions of quasi xgamma distribution for fixed θ and varying α are shown in Fig. 1. The parameter α regulates the shape of the distribution for a fixed value of θ.

3. Moments and related measures

In this section we find moments and some related measures of quasi xgamma distribution with PDF given in (3).

The r^{th} order raw moments (about origin) of quasi xgamma distribution are

$$\mu_r{}' = E(X^r) = \int_0^\infty x^r \frac{\theta}{(1+\alpha)}\left(\alpha + \frac{\theta^2}{2}\, x^2\right) e^{-\theta x} dx$$

$$= \frac{r!}{\theta^r(1+\alpha)}\left[\alpha + \frac{1}{2}(r+1)(r+2)\right] \text{ for } r = 1, 2, 3, \ldots \tag{5}$$

In particular we have,

$$\mu_1{}' = E(X) = \frac{(3+\alpha)}{\theta(1+\alpha)} = \mu(\text{say}), \qquad \mu_2{}' = E(X^2) = \frac{2(6+\alpha)}{\theta^2(1+\alpha)}$$

$$\mu_3{}' = E(X^3) = \frac{6(10+\alpha)}{\theta^3(1+\alpha)}, \qquad \mu_4{}' = E(X^4) = \frac{24(15+\alpha)}{\theta^4(1+\alpha)}$$

The j^{th} order central (about μ) moments are given by

$$\mu_j = E\left[(X-\mu)^j\right] = \sum_{r=0}^{j} \binom{j}{r} \mu_r{}' (-\mu)^{j-r}, \qquad \text{for } j = 2, 3, 4, \ldots$$

In particular we have,

$$\mu_2 = Var(X) = \frac{\alpha^2 + 8\alpha + 3}{\theta^2(1+\alpha)^2} = \sigma^2(\text{say}), \qquad \mu_3 = \frac{2(\alpha^3 + 15\alpha^2 + 9\alpha + 3)}{\theta^3(1+\alpha)^3},$$

$$\mu_4 = \frac{3(\alpha^4 + 88\alpha^3 + 310\alpha^2 + 288\alpha + 177)}{\theta^4(1+\alpha)^4}$$

The coefficient of variation (γ), coefficient of skewness ($\sqrt{\beta_1}$) and coefficient of kurtosis (β_2) are given by

$$\gamma = \frac{\sqrt{\mathrm{Var}(X)}}{E(X)} = \frac{\sqrt{(\alpha^2 + 8\alpha + 3)}}{(3+\alpha)},$$

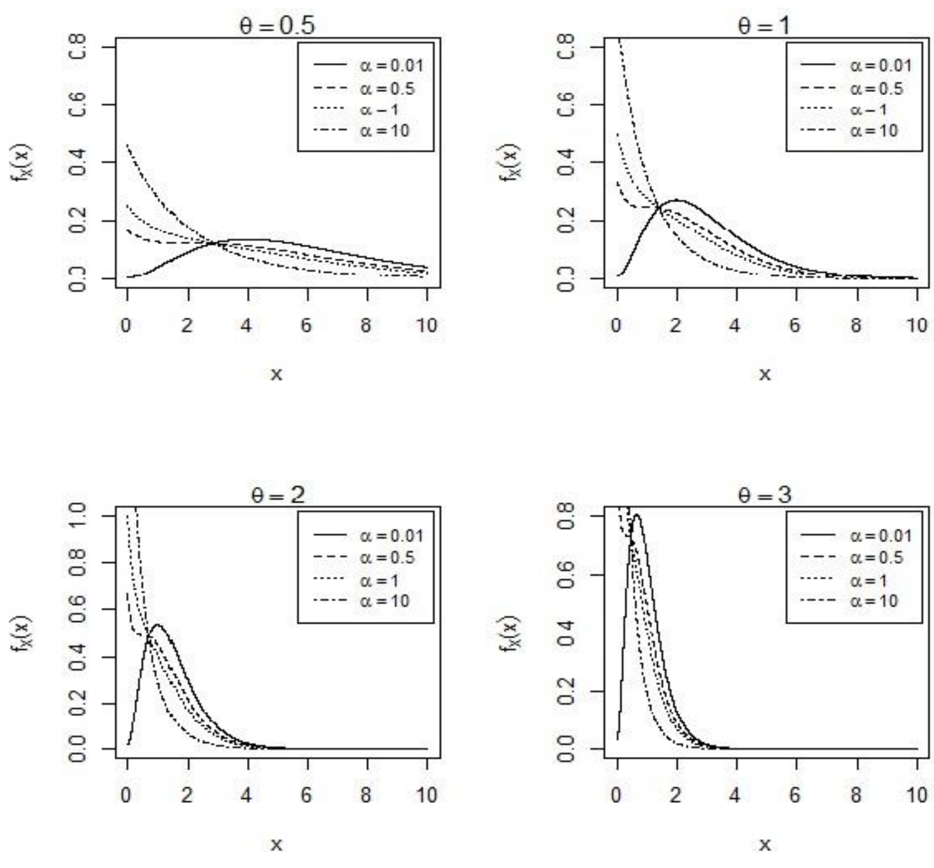

Figure 1: Density plots of quasi xgamma distribution for different parameter values

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{2(\alpha^3 + 15\alpha^2 + 9\alpha + 3)}{(\alpha^2 + 8\alpha + 3)^{\frac{3}{2}}} \quad \text{and}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{3(\alpha^4 + 88\alpha^3 + 310\alpha^2 + 288\alpha + 177)}{(\alpha^2 + 8\alpha + 3)^2}, \text{respectively.}$$

The moment generating function (MGF), characteristic function (CF) and cumulant generating function (CGF) of X~ QXD (α, θ) are obtained as

$$M_X(t) = E(e^{tX}) = \frac{1}{(1+\alpha)}\left[\alpha\left(1-\frac{t}{\theta}\right)^{-1} + \left(1-\frac{t}{\theta}\right)^{-3}\right]\ ;\ t \in \mathbb{R} \tag{6}$$

$$\phi_X(t) = E(e^{itX}) = \frac{1}{(1+\alpha)}\left[\alpha\left(1-i\frac{t}{\theta}\right)^{-1} + \left(1-i\frac{t}{\theta}\right)^{-3}\right]\ ;\ t \in \mathbb{R}, i = \sqrt{-1} \tag{7}$$

$$K_X(t) = \ln[M_X(t)] = \ln\frac{\theta}{(1+\alpha)(\theta-t)} + \ln\left[\alpha + \frac{\theta^2}{(\theta-t)^2}\right]\ ;\ t \in \mathbb{R} \tag{8}$$

respectively.

Theorem 1. For α > ½, the PDF, f(x) of X~ QXD (α, θ) , as given in (3), is decreasing in x.

Proof: We have from (3) the first derivative of f(x) with respect to x as

$$f'(x) = \frac{\theta^2}{(1+\alpha)}\left(\theta x - \alpha - \frac{1}{2}\theta^2 x^2\right)e^{-\theta x}$$

$f'(x)$ is negative in x when α > ½, and hence the proof.

So, we have from the above Theorem 1, for $\alpha \le \frac{1}{2}$, $\frac{d}{dx}$ f(x) = 0 implies that $\frac{1+\sqrt{1-2\alpha}}{\theta}$ is the unique critical point at which f(x) is maximized.

Hence, the mode of quasi xgamma distribution is given by

$$\text{Mode}(X) = \frac{1+\sqrt{1-2\alpha}}{\theta}\ ;\ \text{when } 0 < \alpha \le \frac{1}{2}, \quad \text{and is } 0, \quad \text{elsewhere.}$$

It is easy to show that, for X~ QXD (α, θ), Mode (X) < Median(X) < Mean(X), which also holds for xgamma distribution.

4. Entropies

The concept of information is too broad to be captured completely by a single definition. However, for any probability distribution, we define a quantity called the entropy, which has many properties that agree with the intuitive notion of what a measure of information should be. The concept of entropy was introduced in thermodynamics to provide a statement of the second law of thermodynamics. Later, statistical mechanics provided a connection between thermodynamic entropy and the logarithm of the number of microstates in a macro state of the system.

Therefore, entropy of a random variable X is a measure of variation of the uncertainty. A popular entropy measure is Renyi entropy (*Renyi 1961*). If non-negative random variable X has the probability density function f(x), then Renyi entropy is defined as

$$H_R(\gamma) = \frac{1}{1-\gamma}\ln\left[\int_0^\infty f^\gamma(x)dx\right]\ ;\ \gamma > 0 \textit{ and } \gamma \ne 1$$

When X~ QXD (α, θ), one can calculate

$$\int_0^\infty f^\gamma(x)dx = \frac{\alpha^\gamma\theta^{\gamma-1}}{(1+\alpha)^\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\Gamma(2j+1)}{2^j\alpha^j\gamma^{2j+1}}$$

to obtain Renyi entropy as

$$H_R(\gamma) = \frac{1}{1-\gamma}[\gamma\ln\alpha + (\gamma-1)\ln\theta - \gamma\ln(1+\alpha)] + \frac{1}{1-\gamma}\ln\left[\sum_{j=0}^{\gamma}\binom{\gamma}{j}\frac{\Gamma(2j+1)}{2^j\alpha^j\gamma^{2j+1}}\right] \tag{9}$$

Shannon measure of entropy, which is a special case of Renyi entropy, is defined as

$$H(f) = E[-\ln f(x)] = -\int_0^\infty \ln f(x)f(x)dx$$

If X~ QXD (α, θ), the Shannon entropy is obtained as

$$H(f) = \left(\frac{3+\alpha}{1+\alpha}\right) - \ln\left(\frac{\alpha\theta}{1+\alpha}\right) - \frac{1}{(1+\alpha)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j2^j\alpha^j}\left\{\alpha\Gamma(2j+1) + \frac{1}{2}\Gamma(2j+3)\right\} \tag{10}$$

5. Distributions of order statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n drawn from QXD (α, θ). Denote $X_{1:n}$, $X_{2:n}$, $\dots, X_{n:n}$ as n order statistics.

Then we have the PDF of j[th] order statistics, $X_{j:n}$, for QXD (α, θ) as

$$f_{X_{j:n}}(x) = \frac{n!}{(j-1)!\,(n-j)!}\left\{1\frac{\left(1+\alpha+\theta x+\frac{\theta^2x^2}{2}\right)}{(1+\alpha)}e^{-\theta x}\right\}^{j-1}$$
$$\left\{\frac{\left(1+\alpha+\theta x+\frac{\theta^2x^2}{2}\right)}{(1+\alpha)}e^{-\theta x}\right\}^{n-j}\frac{\theta}{(1+\alpha)}\left(\alpha+\frac{\theta^2}{2}x^2\right)e^{-\theta x}$$
$$for\ 0 < x < \infty\ \ ,\theta > 0\ and\ \alpha > 0 \tag{11}$$

In particular, the PDF of the largest order statistic, $X_{n:n} = \text{Max}(X_1, X_2, \dots, X_n)$, is given by

$$f_{X_{n:n}}(x) = \frac{n\theta}{(1+\alpha)^n}\left[(1+\alpha)(1-e^{-\theta x}) - \theta x\left(1+\frac{\theta x}{2}\right)e^{-\theta x}\right]^{n-1}\left(\alpha+\frac{\theta^2}{2}x^2\right)e^{-\theta x}$$
$$for\ 0 < x < \infty\ \ ,\theta > 0\ and\ \alpha > 0 \tag{12}$$

Similarly, the PDF of the smallest order statistic, $X_{1:n} = \text{Min}(X_1, X_2, \dots, X_n)$, is derived as

$$f_{X_{1:n}}(x) = \frac{n\theta}{(1+\alpha)^n}\left[1+\alpha+\theta x+\frac{\theta^2x^2}{2}\right]^{n-1}\left(\alpha+\frac{\theta^2}{2}\,x^2\right)e^{-n\theta x}$$

$$\text{for } 0 < x < \infty \;\; , \theta > 0 \text{ and } \alpha > 0 \qquad (13)$$

6. Survival characteristics

An important notion in reliability theory (or survival analysis) is the concept of 'aging' which is an inherent property of a unit or system, may be a living organism or a system of components. Aging is usually characterized by the failure rate function, mean residual life function, mean time to failure, etc.

In this section, we derive some such characteristics that are useful in the context of survival analysis and/or reliability analysis considering X as a random variable denoting lifetime (time-to an event) of a unit or a system. If X~ QXD (α, θ), the survival function (SF) of X is given by

$$S(x) = \frac{\left(1+\alpha+\theta x+\frac{\theta^2x^2}{2}\right)}{(1+\alpha)}e^{-\theta x} \quad ; x > 0\,, \theta > 0 \text{ and } \alpha > 0 \qquad (14)$$

The failure rate function or hazard rate function for a continuous distribution with PDF, f(x), CDF, F(x) and SF, S(x), is defined as

$$h(x) = \lim_{\Delta x\to 0}\frac{P(X < x+\Delta x|X > x)}{\Delta x} = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)}$$

For quasi xgamma distribution, the failure rate function is given by

$$h(x) = \frac{\theta\left(\alpha+\frac{1}{2}\theta^2x^2\right)}{\left(1+\alpha+\theta x+\frac{\theta^2x^2}{2}\right)} \qquad (15)$$

Note: h(x) obtained in (15) is bounded, i.e.,

$$\frac{\alpha\theta}{(1+\alpha)} < h(x) < \theta, \qquad \text{moreover,} \qquad h(0) = f(0) = \frac{\alpha\theta}{(1+\alpha)}.$$

For a continuous random variable X with PDF f(x) and CDF F(x), the mean residual life (MRL) function is defined as

$$m(x) = E(X-x|X > x) = \frac{1}{1-F(x)}\int_x^{\infty}[1-F(t)]dt$$

For quasi xgamma distribution MRL function is obtained as

$$m(x) = \frac{1}{\theta}+\frac{(2+\theta x)}{\theta\left(1+\alpha+\theta x+\frac{1}{2}\theta^2x^2\right)} \qquad (16)$$

The MRL function of quasi xgamma distribution given in (16) has the following properties

(i) $m(0) = E(X) = \frac{(3+\alpha)}{\theta(1+\alpha)}$

(ii) m(x) is decreasing in x with bounds $\frac{1}{\theta} < m(x) < \frac{(3+\alpha)}{\theta(1+\alpha)}$

The reverse hazard rate or reverse hazard function is defined as

$$r(x) = \lim_{\Delta x \to 0} \frac{P(X < x + \Delta x | X \le x)}{\Delta x} = \frac{f(x)}{F(x)}$$

It could be interpreted as the conditional probability at the state change happening in an infinitesimal interval preceding x, given that the state change takes place at x, or before x.
The reverse hazard rate for quasi xgamma lifetime distribution is obtained as

$$r(x) = \frac{\theta\left(\alpha + \frac{1}{2}\theta^2 x^2\right) e^{-\theta x}}{(1+\alpha)(1-e^{-\theta x}) - \theta x\left(1 + \frac{\theta x}{2}\right) e^{-\theta x}} \tag{17}$$

For a positive continuous random variable, stochastic ordering is an important tool for judging the comparative behavior. We recall some basic definitions:
A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order ($X \le_{st} Y$) if $F_X(x) \ge F_Y(x)$ for all x.
(ii) hazard rate order ($X \le_{hr} Y$) if $h_X(x) \ge h_Y(x)$ for all x.
(iii) mean residual life order ($X \le_{mrl} Y$) if $m_X(x) \le m_Y(x)$ for all x.
(iv) likelihood ratio order ($X \le_{lr} Y$) if $(f_X(x)/f_Y(x))$ decreases in x.

Theorem 2. Let X ~ QXD (α_1, θ_1) and Y ~ QXD (α_2, θ_2). If $\alpha_1 = \alpha_2$ and $\theta_1 \ge \theta_2$ (or, if $\theta_1 = \theta_2$ and $\alpha_1 \ge \alpha_2$), then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.
Proof: Let us denote the PDF of X as $f_X(x)$ and that of Y be $f_Y(x)$.
We have then the ratio

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1(1+\alpha_2)(2\alpha_1 + {\theta_1}^2 x^2)}{\theta_2(1+\alpha_1)(2\alpha_2 + {\theta_2}^2 x^2)} e^{-(\theta_1 - \theta_2)x}$$

So,

$$\ln\left(\frac{f_X(x)}{f_Y(x)}\right) = \ln\frac{\theta_1(1+\alpha_2)}{\theta_2(1+\alpha_1)} + \ln\left(2\alpha_1 + {\theta_1}^2 x^2\right) - \ln\left(2\alpha_2 + {\theta_2}^2 x^2\right) - (\theta_1 - \theta_2)x$$

The first derivative with respect to x gives

$$\frac{d}{dx}\ln\left(\frac{f_X(x)}{f_Y(x)}\right) = \frac{4x(\theta_1\alpha_2 - \theta_2\alpha_1)}{(2\alpha_1 + \theta_1 x^2)(2\alpha_2 + \theta_2 x^2)} - (\theta_1 - \theta_2)$$

Which is negative when $\alpha_1 = \alpha_2$ and $\theta_1 \ge \theta_2$ (or when $\theta_1 = \theta_2$ and $\alpha_1 \ge \alpha_2$) and so $X \le_{lr} Y$, rest of the orderings are well justified (see *Shaked and Shanthikumar 1994*). Hence the proof.

7. Estimation of parameters

In this section we discuss two classical methods of estimation, viz. method of moments (MM) and method of maximum likelihood (ML), in estimating the parameters of the quasi xgamma distribution under complete sample situation. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ be n observations or realizations on a random sample $X_1, X_2, \dots, X_n$ of size n drawn from the quasi xgamma distribution given in (3). We consider MM estimators first for estimating α and θ.
We equate

$$\mu_1{}' = \frac{(3+\alpha)}{\theta(1+\alpha)} = \text{Sample Mean} = m_1{}' = \overline{X}$$

$$\mu_2' = \frac{2(6+\alpha)}{\theta^2(1+\alpha)} = m_2' = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Let

$$b = \frac{\mu_2'}{\mu_1'^2} = \frac{2(6+\alpha)(1+\alpha)}{(3+\alpha)^2}$$

$$\Rightarrow (2-b)\alpha^2 + (14-6b)\alpha + (12-9b) = 0$$

Which is a quadratic in α and gives an estimate of α, say $\hat{\alpha}$.

The b can be easily estimated from the sample moments. The MM estimator of θ is then obtained as

$$\hat{\theta} = \frac{1}{\bar{\bar{X}}}\frac{(3+\hat{\alpha})}{(1+\hat{\alpha})}$$

Now, we obtain the maximum Likelihood estimators (MLEs) of the parameters. We have the likelihood function as

$$L(\alpha,\theta|\mathbf{x}) = \prod_{i=1}^{n} \frac{\theta}{(1+\alpha)}\left(\alpha + \frac{\theta^2}{2}\, x_i^2\right)\exp(-\theta x_i)$$

The log-likelihood function is given by

$$\ln L(\alpha,\theta|\mathbf{x}) = n\ln(\theta) - n\ln(1+\alpha) + \sum_{i=1}^{n}\ln\left(\alpha + \frac{\theta^2}{2}\, x_i^2\right) - \theta\sum_{i=1}^{n} x_i \qquad (18)$$

To find out the maximum likelihood estimators (MLEs) of α and θ, we have two likelihood equations as

$$\frac{\partial \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha} = \sum_{i=1}^{n}\frac{1}{\left(\alpha + \frac{\theta^2}{2}\, x_i^2\right)} - \frac{n}{(1+\alpha)} = 0 \qquad (19)$$

and

$$\frac{\partial \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\frac{\theta x_i^2}{\left(\alpha + \frac{\theta^2}{2}\, x_i^2\right)} - \sum_{i=1}^{n} x_i = 0 \qquad (20)$$

respectively.

Though the values of α and θ cannot be obtained analytically, we can utilize any numerical method, such as Newton-Raphson, for solving the non-linear equations (19) and (20) to obtain those.

Moreover, we can apply fisher's scoring method for getting the MLEs of α and θ. We have the second order derivatives as,

$$\frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha^2} = \frac{n}{(1+\alpha)^2} - \sum_{i=1}^{n}\frac{1}{\left(\alpha + \frac{\theta^2}{2}\, x_i^2\right)^2} \qquad (21)$$

$$\frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta^2} = \sum_{i=1}^{n}\frac{\alpha x_i^2 - \frac{\theta^2}{2} x_i^4}{\left(\alpha + \frac{\theta^2}{2}\, x_i^2\right)^2} - \frac{n}{\theta^2} \qquad (22)$$

$$\frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha\,\partial\theta} = \frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta\,\partial\alpha} = -\sum_{i=1}^{n} \frac{\theta x_i^{\,2}}{\left(\alpha + \frac{\theta^2}{2}\, x_i^{\,2}\right)^2} \tag{23}$$

Letting $\hat{\alpha}$ and $\hat{\theta}$ as the MLEs of α and θ, respectively, we solve the following equation

$$\begin{bmatrix} \frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta^2} & \frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta\,\partial\alpha} \\ \frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha\,\partial\theta} & \frac{\partial^2 \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha^2} \end{bmatrix}_{\hat{\theta}=\theta_0,\ \hat{\alpha}=\alpha_0} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \ln L(\alpha,\theta|\mathbf{x})}{\partial\theta} \\ \frac{\partial \ln L(\alpha,\theta|\mathbf{x})}{\partial\alpha} \end{bmatrix}_{\hat{\theta}=\theta_0,\ \hat{\alpha}=\alpha_0} \tag{24}$$

Method of successive iteration can be applied for initial values α_0 and θ_0 for α and θ, respectively.

8. Simulation study

In this section, we propose method of generating random samples of specific sizes from quasi xgamma distribution. A Monte-Carlo simulation study, which is also discussed in this section, was also conducted to assess the maximum likelihood estimates of the parameters in terms on mean square errors (MSEs) for different sample sizes.

The inversion method for generating random data from the quasi xgamma distribution fails because the equation F(x) = u, where u is an observation from the uniform distribution on (0, 1), cannot be explicitly solved in x. However, as already mentioned at the note in section 2, we can use the fact that the quasi xgamma distribution is a special mixture of exponential(θ) and gamma(3, θ) distributions with mixing proportions α/(1+ α) and 1/(1+ α) respectively.

To generate random data $X_i,\ i = 1, 2, \dots, n$ from quasi xgamma distribution with parameters α and θ, we can use the following algorithm:

1. Generate $U_i \sim \text{uniform}(0, 1),\ i = 1, 2, \dots, n.$
2. Generate $V_i \sim \text{exponential}(\theta),\ i = 1, 2, \dots, n.$
3. Generate $W_i \sim \text{gamma}(3, \theta),\ i = 1, 2, \dots, n.$
4. If $U_i \le \alpha/(1+\alpha)$, then set $X_i = V_i$, otherwise, set $X_i = W_i$.

A Monte-Carlo simulation study was carried out considering N=10, 000 times for selected values of n, α and θ. Samples of sizes 20, 30, 50, 80 and 100 were considered and values of (α , θ) were taken as (0.5,0.5), (1.5, 2.0) and (3.0, 4.0). The method of maximum likelihood was applied to obtain the estimates. Along with the maximum likelihood estimates the following measures were computed:

(i) Average mean square error (MSE) of the simulated estimates $\hat{\alpha}_i,\ i = 1, 2, \dots, N$:

$$\frac{1}{N}\sum_{i=1}^{N} (\hat{\alpha}_i - \alpha)^2.$$

(ii) Average mean square error (MSE) of the simulated estimates $\hat{\theta}_i,\ i = 1, 2, \dots, N$:

$$\frac{1}{N}\sum_{i=1}^{N} \left(\hat{\theta}_i - \theta\right)^2.$$

The results of the simulation study have been tabulated in Table 1 below.

It is clear from Table 1 that the MSEs for the estimates of α decrease as the sample size, n, increases and the estimate gets closer to the given value. The similar trend is observed in case of the estimates of θ and its MSE values for different sample sizes.

9. Application: Bladder cancer data

In this section, we use a real data set to show that the quasi xgamma distribution can be a better model than some recently developed models where the particular data are utilized. The data set, given in Table 2, represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in *Lee and Wang 2003*.

Table 1: Estimates of the parameters with corresponding MSE values

$\alpha = 0.5 ; \theta = 0.5$				
Sample Size	$\hat{\alpha}$	MSE of $\hat{\alpha}$	$\hat{\theta}$	MSE of $\hat{\theta}$
20	0.65563	0.64842	0.52574	0.01480
30	0.61249	0.41696	0.51593	0.00873
50	0.55139	0.24084	0.50952	0.00477
80	0.53788	0.14125	0.50112	0.00297
100	0.50544	0.12975	0.50135	0.00257
$\alpha = 1.5 ; \theta = 2.0$				
Sample Size	$\hat{\alpha}$	MSE of $\hat{\alpha}$	$\hat{\theta}$	MSE of $\hat{\theta}$
20	1.88951	1.53364	2.07156	0.24302
30	1.73208	1.47211	2.04003	0.18286
50	1.61201	1.18401	2.02723	0.12595
80	1.55745	1.07344	2.00678	0.07996
100	1.51757	0.73089	2.00165	0.06368
$\alpha = 3.0 ; \theta = 4.0$				
Sample Size	$\hat{\alpha}$	MSE of $\hat{\alpha}$	$\hat{\theta}$	MSE of $\hat{\theta}$
20	3.16671	1.27292	3.83684	0.13292
30	3.14370	1.02392	3.84705	0.12729
50	3.12033	0.96721	3.86446	0.09979
80	3.09858	0.78974	3.88473	0.08507
100	3.02271	0.72481	3.94853	0.07146

Table 2: Remission times (in months) of 128 bladder cancer patients

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 0.26, 0.31, 0.73, 0.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 11.98, 4.51, 2.07, 0.22, 13.8, 25.74, 0.50,2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 19.13, 6.54, 3.36, 0.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 1.76, 8.53, 6.93, 0.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 3.25, 12.03, 8.65, 0.39, 10.34, 14.83, 34.26, 0.90,2.69, 4.18, 5.34, 7.59, 10.66, 4.50, 20.28, 12.63, 0.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 6.25, 2.02, 22.69, 0.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 8.37, 3.36, 5.49, 0.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 12.02, 6.76, 0.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.1, 1.46, 4.40, 5.85, 2.02, 12.07.

The data set is positively skewed (skewness = 3.38) with mean remission time of 8.57 months, standard deviation of 10.56 months and unimodal (mode at 5 months).

We consider here gamma, log-normal among standard lifetime models, in addition, lindley (*Lindley 1958*), power lindley (PL) (*Ghitany et al. 2013*), transmuted lindley (TL) (*Merovci 2013*), exponentiated lindley (EL) (*Bakouch et al. 2012*), weighted lindley (WL) (*Ghitany et al. 2011*) and new generalized power lindley (NGPL) (*Mansour and Hamed 2015*) and xgamma (*Sen et al. 2016*) models among recently developed or popularized lifetime models, i.e., altogether nine lifetime models are considered to compare with quasi xgamma model for suitability of fit or goodness of fit for the above data.

In order to compare the two distribution models, we consider criteria like, -Log-likelihood, AIC (Akaike information criterion, see *Akaike 1974*) and BIC (Bayesian information criterion, see *Schwarz 1978*), for the data set. The better distribution corresponds to smaller -Log-likelihood, AIC and BIC values. We use maximum likelihood method of estimation (MLE) for estimating the model parameters. Table 3 shows the estimates of the model parameter(s) with standard error(s) of estimates in parenthesis and model selection criteria.

It is clear from Table 3 that the quasi xgamma distribution provides better fit to the bladder cancer data and, hence, the model acts as a strong competitor among the other models considered here for modeling such lifetime data.

According to model selection criterion, viz., AIC, the following order of best fit observed:

Best Quasi Xgamma → Gamma → Lognormal → New Generalized Power Lindley → Power Lindley → Transmuted Lindley → Exponentiated Lindley → Weighted Lindley → Lindley → Xgamma **Worst**

Table 3: MLEs of model parameters and model selection criteria for bladder cancer data

Model	Parameter estimates (Standard Error)	-Log-likelihood	AIC	BIC
Gamma(α, β)	$\hat{\alpha} = 0.9154(0.0910)$ $\hat{\beta} = 0.1069(0.0153)$	402.624	809.249	814.953
Lognormal(μ, σ)	$\hat{\mu} = 1.5109(0.1133)$ $\hat{\sigma} = 1.2819(0.0801)$	406.803	817.605	823.309
Lindley(θ)	$\hat{\theta} = 0.2129(0.0134)$	417.924	837.848	840.610
PL(θ, β)	$\hat{\theta} = 0.2943(0.0371)$ $\hat{\beta} = 0.8302(0.0472)$	413.353	830.707	836.410
TL(λ, θ)	$\hat{\lambda} = 0.6169(0.1688)$ $\hat{\theta} = 0.1557(0.0150)$	415.155	834.310	840.014
EL(α, θ)	$\hat{\alpha} = 0.1648(0.0166)$ $\hat{\theta} = 0.7330(0.0912)$	416.285	836.572	842.274
WL(α, θ)	$\hat{\alpha} = 0.1595(0.0172)$ $\hat{\theta} = 0.6827(0.1115)$	416.442	836.885	842.588
NGPL$(\lambda, \theta, \beta, \delta, \alpha)$	$\hat{\lambda} = -0.858(0.0938)$ $\hat{\theta} = 2.5044(1.6547)$ $\hat{\beta} = 0.3292(0.1341)$ $\hat{\delta} = 6.6798(2.6466)$ $\hat{\alpha} = 33.738(15.584)$	408.966	827.932	842.192
Xgamma(θ)	$\hat{\theta} = 0.2860(0.0159)$	425.169	852.338	855.190
Quasi Xgamma(α, θ)	$\hat{\alpha} = 16.827(2.0453)$ $\hat{\theta} = 0.1298(0.0179)$	402.320	808.640	814.344

Similarly, according to model selection criterion, viz., BIC, the following order of best fit observed:
Best Quasi Xgamma → Gamma → Lognormal → Power Lindley → Transmuted Lindley → Lindley → Exponentiated Lindley → New generalized Power Lindley → Weighted Lindley → Xgamma **Worst**

10. Concluding remarks

To facilitate better modeling of survival data there has been a great interest among statisticians and applied researchers in constructing flexible lifetime models. As a consequence, a significant progress has been made towards the generalization and/or extension of some well-known lifetime models and their successful application to data coming from diverse areas. In this paper, we introduce a new lifetime distribution, called quasi xgamma, as a generalization or extension of one parameter xgamma distribution that is recently developed by *Sen et al. 2016*.
We study different mathematical and statistical properties along with some important survival properties of the new model. Explicit expressions for the moments, distributions of order statistics are also obtained. The model parameters are estimated by maximum likelihood and method of moments. A sample generation algorithm is proposed and a Monte –Carlo simulation study is also carried out to understand the mean square errors of the estimates for different sample sizes. The new model is compared with some other newly developed models and it is shown that our model provides better fit than other lifetime models while considering real dataset. We understand that a new lifetime distribution is proposed in this article and it would be interesting to study the distribution considering some datasets in view of different censored mechanisms when specific interest comes into survival or reliability aspects. The article also opens a scope for studying Bayesian estimators of the parameters under different loss functions. The work in this direction is taken as a future research. We expect that the proposed distribution will serve as a potential alternative in modeling time-to-event in complete case situations and other types of data in comparison to the other models available in the literature.

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