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On Comparison of Renewal and Trend Renewal Processes with respect to Stochastic Orderings

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ABSTRACT
Trend renewal processes (TRPs) were introduced by Lindqvist (1993) as a time-transformed renewal process. After a brief introduction to TRPs with possibly some new results, independent ordinary renewal processes (ORPs) and TRPs are compared with respect to some stochastic orderings between the generating inter-arrival time random variables, like, the usual stochastic order, hazard rate order, likelihood ratio order and variability order, and on the basis of the trend function. Some illustrations are given.

KEYWORDS
Hazard rate ordering, Likelihood ratio ordering, Renewal process, Stochastic ordering, Trend renewal process, Variability ordering.

1. Introduction

Stochastic orders have been a topic of study in many areas of probability and statistics, like reliability theory, life-testing, actuarial science. Using properties associated with distribution functions (dfs), density functions, hazard rates and whatnot, many useful stochastic orders have been defined and these give insights into the relationships among classes of random variables (rvs) with specified properties. This has attracted a great amount of research problems in recent decades in the area of stochastic comparison of random variables. We have referred Shaked and Shanthikumar (2007), Nair, Sankaran and Balakrishnan (2018), Ravi and Prathibha (2012), Szekli (2012), and references therein for stochastic orders. Definitions used in this article are given in Appendix for ease of reference.

In this article, we look at the comparison of an ordinary renewal process (ORP) with a trend renewal process (TRP), and two independent TRPs on the basis of properties of the corresponding inter-arrival time distribution and the trend function. Section 1.1 is a brief introduction to TRP wherein some possibly new results are stated and proved. Section 2 contains the main results and Section 3 has some illustrative examples.

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1.1. Trend Renewal Process (TRP) - an Introduction

Let $\lambda(t), t \geq 0$, be a non-negative continuous function, $\Lambda(t) = \int_0^t \lambda(u)du < \infty, t \geq 0$, with $\Lambda(\infty) = \infty$. With $\{T_n, n \geq 0\}, T_0 = 0$, denoting the sequence of event/ failure/ arrival time, henceforth arrival time, and $N_{F,\lambda}(t) = \max\{n : T_n \leq t\}$, the counting process $\{N_{F,\lambda}(t), t \geq 0\}$ or the sequence $\{T_n, n \geq 0\}$ is called a trend renewal process (TRP) with renewal df F and trend function λ, if the time-transformed process $\{\Lambda(T_n), n \geq 0\}$ is an ORP with renewal df F, that is, if the random variables (rvs) $W_i = \Lambda(T_i) - \Lambda(T_{i-1}), i \geq 1$, known as transformed working times in reliability literature, are independent and identically distributed (iid) rvs with df F, with the assumption that $F(0) = 0$. If $\lambda(t) = 1, t \geq 0$, then $\Lambda(t) = t, t \geq 0$, and $\{N_{F,1}(t), t \geq 0\}$ is an ORP generated by the df F, which we write as $\{N_F(t), t \geq 0\}$. We denote TRP with $F = \text{Exp}(\mu)$, the exponential df with mean $\mu > 0$, as $\{N_{E,\lambda}(t), t \geq 0\}$. With $\lambda(t) = 1, t \geq 0$, $\{N_{E,1}(t), t \geq 0\}$ is a Poisson process (PP) with rate $\frac{1}{\mu}$, which we write as $\{N_E(t), t \geq 0\}$. Note that $\{N_{E,\lambda}(t), t \geq 0\}$ is a counting process with independent increments.

Lindqvist (1993) introduced and studied TRPs initially. Further, the TRP was investigated by Lindqvist et al. (2003) for statistical analysis of repairable systems. The challenge of calculating unknown trend parameters of a TRP in the scenario when its renewal distribution is unknown was examined by Jokiel-Rokita and Magiera (2012). Franz, Jokiel-Rokita, and Magiera (2014) investigated the issue of predicting a TRP's next failure time while the process's renewal distribution is unknown. A non-parametric estimation technique for TRP was devised by Saito and Dohi (2016) in cases where the failure rate function's form in the ORP is unknown.

Since $\Lambda(\cdot)$ is non-decreasing, we have

$$\begin{aligned} P\left(N_{F,\lambda}(t) \geq n\right) &= P\left(T_n \leq t\right) \\ &= P\left(\Lambda(T_n) \leq \Lambda(t)\right) \\ &= P\left(\sum_{i=1}^{n}(\Lambda(T_i) - \Lambda(T_{i-1})) \leq \Lambda(t)\right) \\ &= P\left(\sum_{i=1}^{n} W_i \leq \Lambda(t)\right) \\ &= F^{(n)}(\Lambda(t)), t \geq 0, \end{aligned}$$

where $F^{(n)}$ is the n-fold convolution of F with itself.
We now state and prove a few results some of which may be new. Here $\stackrel{d}{=}$ denotes equality in distribution.

Lemma 1.1. $N_{F,\lambda}(t) \stackrel{d}{=} N_F(\Lambda(t)), t \geq 0$.

Proof. For $t \geq 0$, we have $P\left(N_{F,\lambda}(t) \geq n\right) = F^{(n)}(\Lambda(t)) = P\left(N_F(\Lambda(t)) \geq n\right), n \geq 1$, so that $N_{F,\lambda}(t) \stackrel{d}{=} N_F(\Lambda(t)), t \geq 0$. □

Remark 1. More renewals occur in a TRP than in an ORP, generated by the same baseline df F, during $[0, t], t \geq 0$, when the trend function $\lambda(t) > 1, t \geq 0$. Because, the inequality $\lambda(t) > 1$, or equivalently, $\Lambda(t) > t$ implies that $\mathrm{P}\left(N_F\left(\Lambda(t)\right) \geq N_F(t)\right) = 1$, or equivalently, $\mathrm{P}\left(N_{F,\lambda}(t) \geq N_F(t)\right) = 1$ using Lemma 1.1.

Note that PP has independent and stationary increments and the following three results look at similar properties for a TRP with exponential baseline distribution.

Lemma 1.2. *The distribution of* $N_{E,\lambda}(t)$ *is given by* $P\left(N_{E,\lambda}(t)=n\right) = e^{-\frac{\Lambda(t)}{\mu}}\frac{\left(\frac{\Lambda(t)}{\mu}\right)^n}{n!}, n \geq 1, t \geq 0.$

Proof. Let $\{N_E(t), t \geq 0\}$ be a PP with rate $\frac{1}{\mu}$. Then $P\left(N_E(t)=n\right) = e^{-\frac{t}{\mu}}\frac{\left(\frac{t}{\mu}\right)^n}{n!}$. By Lemma 1.1, $N_{E,\lambda}(t) \stackrel{d}{=} N_E(\Lambda(t)), t \geq 0$. Therefore, $P\left(N_{E,\lambda}(t)=n\right) = P\left(N_E\left(\Lambda(t)\right)=n\right) = e^{-\frac{\Lambda(t)}{\mu}}\frac{\left(\frac{\Lambda(t)}{\mu}\right)^n}{n!}$. □

Lemma 1.3. $\{N_{E,\lambda}(t), t \geq 0\}$ *has independent increments.*

Proof. Let $0 = t_0 < t_1 < \ldots < t_n, n \geq 1$ be arbitrary time points so that $0 = \Lambda(t_0) < \Lambda(t_1) < \ldots < \Lambda(t_n)$, as $\Lambda(\cdot)$ is a non-decreasing function. Hence, by the properties of PP,

$$N_E(\Lambda(t_1)) - N_E(\Lambda(t_0)), N_E(\Lambda(t_2)) - N_E(\Lambda(t_1)), \ldots, N_E(\Lambda(t_n)) - N_E(\Lambda(t_{n-1}))$$

are independent rvs. By Lemma 1.2,

$$N_{E,\lambda}(t_1) - N_{E,\lambda}(t_0), N_{E,\lambda}(t_2) - N_{E,\lambda}(t_1), \ldots, N_{E,\lambda}(t_n) - N_{E,\lambda}(t_{n-1})$$

are independent rvs which implies that the process $\{N_{E,\lambda}(t), t \geq 0\}$ has independent increments. □

Lemma 1.4. $\{N_{E,\lambda}(t), t \geq 0\}$ *has stationary increments iff* $\Lambda(t) = \int_0^t \lambda(u)du$ *is a linear function.*

Proof. By Lemma 1.2 and the stationary increments property of PP, we have, for $0 < t_1 < t_2, 0 < s$,

$$\begin{aligned} P\left(N_{E,\lambda}(t_2) - N_{E,\lambda}(t_1) = n\right) &= P\left(N_E(\Lambda(t_2)) - N_E(\Lambda(t_1)) = n\right) \\ &= e^{-\frac{\Lambda(t_2)-\Lambda(t_1)}{\mu}}\frac{\left(\frac{\Lambda(t_2)-\Lambda(t_1)}{\mu}\right)^n}{n!}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} P\left(N_{E,\lambda}(t_2+s) - N_{E,\lambda}(t_1+s) = n\right) &= P\left(N_E(\Lambda(t_2+s)) - N_E(\Lambda(t_1+s)) = n\right) \\ &= e^{-\frac{\Lambda(t_2+s)-\Lambda(t_1+s)}{\mu}}\frac{\left(\frac{\Lambda(t_2+s)-\Lambda(t_1+s)}{\mu}\right)^n}{n!}. \end{aligned} \tag{2}$$

For stationary increment property to hold for the process $\{N_{E,\lambda}(t), t \geq 0\}$, it is enough to show that (1) = (2) or, equivalently,

$$\Lambda(t_2+s) - \Lambda(t_1+s) = \Lambda(t_2) - \Lambda(t_1), 0 < t_1 < t_2, 0 < s. \tag{3}$$

But (3) holds iff $\Lambda(t) = \int_0^t \lambda(u)du$ is a linear function, as shown below.
If $\Lambda(\cdot)$ is a linear function, then $\Lambda(t_2+s) - \Lambda(t_1+s) = \Lambda(t_2) + \Lambda(s) - \Lambda(t_1) - \Lambda(s) =$

$\Lambda(t_2)-\Lambda(t_1)$. Conversely, let (3) hold for all $0<t_1<t_2$ and $0<s$. If possible, let $\Lambda(\cdot)$ be a non-linear function so that $\Lambda(t+s)\neq\Lambda(t)+\Lambda(s)$ for some $t>0, s>0$. Then there exists $0<t_1<t_2$ and $0<s$ for which $\Lambda(t_2+s)-\Lambda(t_1+s)\neq(\Lambda(t_2)+\Lambda(s))-(\Lambda(t_1)+\Lambda(s))=\Lambda(t_2)-\Lambda(t_1)$, contradicting (3). Hence the proof. □

Remark 2. $\Lambda(\cdot)$ is a linear function iff $\{N_{E,\lambda}(t), t\geq 0\}$ is a PP.

Remark 3. We observe that a TRP with exponential baseline distribution $\{N_{E,\lambda}(t), t\geq 0\}$ is a non-homogeneous Poisson process with intensity function $\lambda(t)$ (Lindqvist et al., 2003).

2. Main Results

The following result characterizes comparison of transformed inter-arrival times of two independent TRPs based on stochastic ordering. Two corollaries to this give the same under hazard rate and likelihood ratio orderings.

Theorem 2.1. *If the transformed inter-arrival times of a TRP $\{N_{F,\lambda_1}(t), t\geq 0\}$ are larger (smaller) than those of another independent TRP $\{N_{G,\lambda_2}(t), t\geq 0\}$ in the usual stochastic order with $\lambda_1(t)\leq(\geq)\,\lambda_2(t), t\geq 0$, then $N_{F,\lambda_1}(t)\leq_{st}(\geq_{st})\,N_{G,\lambda_2}(t), t\geq 0$.*

Proof. If $\{N_{F,\lambda_1}(t), t\geq 0\}$ is a TRP with renewal df F and trend function $\lambda_1(\cdot)$, then $P\left(N_{F,\lambda_1}(t)\geq n\right)=F^{(n)}(\Lambda_1(t))$, where $\Lambda_1(t)=\int_0^t\lambda_1(u)du$. Similarly, $P\left(N_{G,\lambda_2}(t)\geq n\right)=G^{(n)}(\Lambda_2(t))$, where $\Lambda_2(t)=\int_0^t\lambda_2(u)du$, and $G^{(n)}$ is the n-fold convolution of G with itself. Let the transformed inter-arrival times of $\{N_{F,\lambda_1}(t), t\geq 0\}$ be larger (smaller) than those of another independent TRP $\{N_{G,\lambda_2}(t), t\geq 0\}$ in the usual stochastic order with $\lambda_1(t)\leq(\geq)\lambda_2(t), t\geq 0$. Then, for $x\geq 0$,

$$\begin{aligned}\bar{F}(x)\geq\bar{G}(x) &\Leftrightarrow F(x)\leq G(x)\\ &\Rightarrow F^{(n)}(x)\leq G^{(n)}(x), n\geq 1.\end{aligned}$$

Therefore, $F^{(n)}(\Lambda_1(t))\leq G^{(n)}(\Lambda_1(t)), n\geq 1, t\geq 0$. Now, for $t\geq 0$,

$$\begin{aligned}\lambda_1(t)\leq\lambda_2(t) &\Rightarrow \Lambda_1(t)\leq\Lambda_2(t)\\ &\Rightarrow G^{(n)}(\Lambda_1(t))\leq G^{(n)}(\Lambda_2(t)), n\geq 1\\ &\Rightarrow F^{(n)}(\Lambda_1(t))\leq G^{(n)}(\Lambda_1(t))\leq G^{(n)}(\Lambda_2(t)), n\geq 1\\ &\Rightarrow F^{(n)}(\Lambda_1(t))\leq G^{(n)}(\Lambda_2(t)), n\geq 1\\ &\Rightarrow P\left(N_{F,\lambda_1}(t)\geq n\right)\leq P\left(N_{G,\lambda_2}(t)\geq n\right), n\geq 1,\\ &\Rightarrow N_{F,\lambda_1}(t)\leq_{st}N_{G,\lambda_2}(t),\end{aligned}$$

completing the proof. □

Corollary 2.2. *The converse of Theorem 2.1 is not true.*

Proof. We have $\lambda_1(t)\leq\lambda_2(t), t\geq 0\Rightarrow\Lambda_1(t)\leq\Lambda_2(t), t\geq 0$, and,

$$N_{F,\lambda_1}(t)\leq_{st}N_{G,\lambda_2}(t), t\geq 0\Rightarrow \mathrm{P}\left(N_{F,\lambda_1}(t)\geq n\right)\leq\mathrm{P}\left(N_{G,\lambda_2}(t)\geq n\right), n\geq 0, t\geq 0,$$

so that with $n = 1$, we get

$$F(\Lambda_1(t)) \leq G(\Lambda_2(t)), t \geq 0.$$

However this does not necessarily imply that $F(x) \leq G(x), x \geq 0$. As there may exist dfs F and G such that $F(x) \geq G(x), x \geq 0$ as well as $F(\Lambda_1(t)) \leq G(\Lambda_2(t)), t \geq 0$. This is justified as $F(x) \geq G(x), x \geq 0 \Rightarrow F(\Lambda_1(t)) \geq G(\Lambda_1(t)), t \geq 0$, and, $\Lambda_2(t) \geq \Lambda_1(t) \Rightarrow G(\Lambda_2(t)) \geq G(\Lambda_1(t)), t \geq 0$. Hence there exist two possibilities:

(i) $F(\Lambda_1(t)) \geq G(\Lambda_2(t)) \geq G(\Lambda_1(t)), t \geq 0$, and
(ii) $G(\Lambda_2(t)) \geq F(\Lambda_1(t)) \geq G(\Lambda_1(t)), t \geq 0$.

□

Corollary 2.3. *If the transformed inter-arrival times of a TRP* $\{N_{F,\lambda_1}(t), t \geq 0\}$ *are larger (smaller) than those of another independent TRP* $\{N_{G,\lambda_2}(t), t \geq 0\}$ *in the hazard rate order with* $\lambda_1(t) \leq (\geq)\, \lambda_2(t), t \geq 0$, *then* $N_{F,\lambda_1}(t) \leq_{st} (\geq_{st})\, N_{G,\lambda_2}(t), t \geq 0$.

Proof. The proof follows using the stochastic ordering relationship: $X \geq_{lr} Y \Rightarrow X \geq_{hr} Y \Rightarrow X \geq_{st} Y$ and Theorem 2.1. □

Corollary 2.4. *If the transformed inter-arrival times of a TRP* $\{N_{F,\lambda_1}(t), t \geq 0\}$ *are larger (smaller) than the transformed inter-arrival times of another independent TRP* $\{N_{G,\lambda_2}(t), t \geq 0\}$ *in the likelihood ratio order with* $\lambda_1(t) \leq (\geq)\, \lambda_2(t), t \geq 0$, *then* $N_{F,\lambda_1}(t) \leq_{st} (\geq_{st})\, N_{G,\lambda_2}(t), t \geq 0$.

Proof. The proof follows using the stochastic ordering relationship: $X \geq_{lr} Y \Rightarrow X \geq_{hr} Y \Rightarrow X \geq_{st} Y$ and Theorem 2.1. □

The following result gives comparison of two independent TRPs with respect to stochastic ordering.

Theorem 2.5. *Consider two independent TRPs* $\{N_{F,\lambda_F}(t), t \geq 0\}$ *and* $\{N_{G,\lambda_G}(t), t \geq 0\}$ *such that the trend function is equal to the hazard function corresponding to the respective renewal distribution. Then the transformed inter-arrival times of* $\{N_{F,\lambda_F}(t), t \geq 0\}$ *are larger (smaller) than those of* $\{N_{G,\lambda_G}(t), t \geq 0\}$ *in the usual stochastic order iff* $N_{F,\lambda_F}(t) \leq_{st} (\geq_{st})\, N_{G,\lambda_G}(t), t \geq 0$.

Proof. We give the proof for stochastically larger relationship and the other proof is similar. For the TRP $\{N_{F,\lambda_F}(t), t \geq 0\}$, since the trend function λ_F is the hazard function corresponding to the renewal distribution F, we have

$$\lambda_F(t) = \frac{f(t)}{\bar{F}(t)} = \frac{d}{dt}(-\ln \bar{F}(t)) \Rightarrow \Lambda_F(t) = \int_0^t \lambda_F(u)du = -\ln \bar{F}(t), t \geq 0.$$

Similarly, for $\{N_{G,\lambda_G}(t), t \geq 0\}$, $\Lambda_G(t) = \int_0^t \lambda_G(u)du = -\ln \bar{G}(t), t \geq 0$. Therefore, $P\left(N_{F,\lambda_F}(t) \geq n\right) = F^{(n)}(\Lambda_F(t)) = F^{(n)}(-\ln \bar{F}(t))$ and $P\left(N_{G,\lambda_G}(t) \geq n\right) = G^{(n)}(\Lambda_G(t)) = G^{(n)}(-\ln \bar{G}(t)), n \geq 1, t \geq 0$. If the transformed inter-arrival times of $\{N_{F,\lambda_F}(t), t \geq 0\}$ are larger than those of $\{N_{G,\lambda_G}(t), t \geq 0\}$ in the usual stochastic order, then, for $x \geq 0$,

$$\bar{F}(x) \geq \bar{G}(x) \Leftrightarrow F(x) \leq G(x) \Rightarrow F^{(n)}(x) \leq G^{(n)}(x), n \geq 1,$$

and, for $t \geq 0$,

$$\bar{F}(t) \geq \bar{G}(t) \Rightarrow \ln \bar{F}(t) \geq \ln \bar{G}(t) \Rightarrow -\ln \bar{F}(t) \leq -\ln \bar{G}(t).$$

Therefore, $F^{(n)}(-\ln \bar{F}(t)) \leq G^{(n)}(-\ln \bar{F}(t)) \leq G^{(n)}(-\ln \bar{G}(t)), n \geq 1, t \geq 0$, which implies $P\left(N_{F,\lambda_F}(t) \geq n\right) \leq P\left(N_{G,\lambda_G}(t) \geq n\right), n \geq 1, t \geq 0 \Rightarrow N_{F,\lambda_F}(t) \leq_{st} N_{G,\lambda_G}(t), t \geq 0$.
To prove the converse, let

$$N_{F,\lambda_F}(t) \leq_{st} N_{G,\lambda_G}(t), t \geq 0 \Rightarrow P\left(N_{F,\lambda_F}(t) \geq n\right) \leq P\left(N_{G,\lambda_G}(t) \geq n\right), n \geq 1, t \geq 0.$$

For $n = 1$, we get

$$F(\Lambda_F(t)) \leq G(\Lambda_G(t)), t \geq 0 \Rightarrow F(t) \leq G(t) \text{ and } \Lambda_F(t) \leq \Lambda_G(t), t \geq 0.$$

Note that, by definition, $F(t) \leq G(t) \Leftrightarrow \Lambda_F(t) \leq \Lambda_G(t)$, so that $\bar{F}(t) \geq \bar{G}(t), t \geq 0$, completing the proof. □

The following results compare a TRP with that generated by the equilibrium distribution, TRPs generated by F and exponential df and a TRP and an ORP.

Corollary 2.6. $\tilde{N}_{F,\lambda}(t) \geq_{st} (\leq_{st}) N_{F,\lambda}(t)$ *iff F is NBUE (NWUE), where $\{N_{F,\lambda}(t), t \geq 0\}$ is a TRP with renewal df F and trend function $\lambda(\cdot)$ and $\{\tilde{N}_{F,\lambda}(t), t \geq 0\}$ is a TRP with renewal df as the equilibrium df of F and trend function $\lambda(\cdot)$.*

Using Theorem 2.7 of Ghosh and Ravi (2024) and Lemma 1.1 the corollary follows as $\Lambda(t)$ is a non-decreasing function of t.

Corollary 2.7. *If F is NBUE (NWUE), then* $N_{F,\lambda}(t) \leq_v (\geq_v) N_{E,\lambda}(t), t \geq 0$.

Proof. By Lemma 1.1 and 1.2, it is enough to show that if F is NBUE (NWUE), then $N_F(\Lambda(t)) \leq_v (\geq_v) N_E(\Lambda(t)), t \geq 0$, which follows from Remark 2.1 of Ghosh and Ravi (2024). □

Corollary 2.8. *If F is NBUE (NWUE), then* $N_{F,\lambda}(t) \leq_v (\geq_v) N_E(t), t \geq 0$ *provided* $\lambda(t) \leq (\geq) 1, t \geq 0$.

Proof. Enough to show that $N_{F,\lambda}(t) \leq_v (\geq_v) N_F(t), t \geq 0$ if $\lambda(t) \leq (\geq) 1, t \geq 0$ and the proof follows using Remark 2.1 of Ghosh and Ravi (2024). We have, for $n \geq 1$,

$$\begin{aligned}\sum_{i=n}^{\infty} P\left(N_{F,\lambda}(t) > i\right) &= \sum_{i=n}^{\infty} P\left(N_{F,\lambda}(t) \geq i+1\right) \\ &= \sum_{i=n}^{\infty} F_{i+1}(\Lambda(t)) \\ &\leq (\geq) \sum_{i=n}^{\infty} F_{i+1}(t) (\text{since } \lambda(t) \leq (\geq) 1 \Rightarrow \Lambda(t) \leq (\geq) t, t \geq 0) \\ &= \sum_{i=n}^{\infty} P\left(N_F(t) \geq i+1\right) \\ &= \sum_{i=n}^{\infty} P\left(N_F(t) > i\right).\end{aligned}$$

Hence $N_{F,\lambda}(t) \leq_v (\geq_v) N_F(t), t \geq 0$, and the proof is complete. □

The following result compares two independent TRPs under variability ordering.

Corollary 2.9. *If $\{N_{F,\lambda_1}(t), t \geq 0\}$ and $\{N_{G,\lambda_2}(t), t \geq 0\}$ are two independent TRPs with $\mu = \int_0^\infty \bar{F}(t)dt = \int_0^\infty \bar{G}(t)dt$, where $\lambda_1(t) \leq 1 \leq \lambda_2(t), t \geq 0$ and, F is NBUE and G is NWUE, then $N_{F,\lambda_1}(t) \leq_v N_{G,\lambda_2}(t), t \geq 0$.*

Proof. Let $\{N_E(t), t \geq 0\}$ be a PP with rate $\frac{1}{\mu}$. Then by Corollary 2.8, F NBUE and $\lambda_1(t) \leq 1, t \geq 0$ imply that $N_{F,\lambda_1}(t) \leq_v N_E(t), t \geq 0$, and, G NWUE and $\lambda_2(t) \geq 1, t \geq 0$ imply that $N_{G,\lambda_2}(t) \geq_v N_E(t), t \geq 0$. Hence the proof. □

The following result compares two independent TRPs with respect to the usual stochastic ordering on the basis of ageing properties and variability ordering of the generating rvs.

Corollary 2.10. *Let $\{N_{F,\lambda}(t), t \geq 0\}$ and $\{N_{G,\lambda}(t), t \geq 0\}$ be two independent TRPs with $\mu = \int_0^\infty \bar{F}(t)dt = \int_0^\infty \bar{G}(t)dt$ and $F \geq_v G$. Then $N_{F,\lambda}(t) \leq_{st} N_{G,\lambda}(t), t \geq 0$ iff F is NBUE and G is NWUE.*

Proof. By Corollary 2.6, $N_{F,\lambda}(t) \leq_{st} \tilde{N}_{F,\lambda}(t), t \geq 0$ iff F is NBUE and $\tilde{N}_{G,\lambda}(t) \leq_{st} N_{G,\lambda}(t), t \geq 0$ iff G is NWUE. By Theorem 2.9 of Ghosh and Ravi (2024), $F \geq_v G \Rightarrow \tilde{N}_F(\Lambda(t)) \leq_{st} \tilde{N}_G(\Lambda(t)), t \geq 0$. Therefore, by Lemma 1.1, $\tilde{N}_{F,\lambda}(t) \leq_{st} \tilde{N}_{G,\lambda}(t), t \geq 0$. Combining these, the proof is complete. □

3. Illustrations

The results are illustrated here with examples.

Example 3.1. Let $\{N_{F,\lambda}(t), t \geq 0\}$ be a TRP with df $F(t) = 1 - e^{-t^2}, t \geq 0$ and trend function $\lambda(t) = \frac{1}{1+t}, t \geq 0$, and let $\{N_E(t), t \geq 0\}$ be a PP with rate $\frac{1}{\mu}$ where $\mu = \int_0^\infty \bar{F}(t)dt = \frac{\sqrt{\pi}}{2}$. Then the pdf corresponding to df F is $f(t) = F'(t) = 2te^{-t^2}, t \geq 0$ and $(\ln f(t))'' = -\frac{1}{t^2} - 2 < 0, t \geq 0$ so that f is log-concave and hence F is also log-concave (Bagnoli and Bergstrom, 2005). Using the ageing class relationship, log-concave ⇒ IFR ⇒ NBU ⇒ NBUE, F is NBUE. Since $\lambda(t) \leq 1, t \geq 0$, and F is NBUE, by Corollary 2.8, we get $N_{F,\lambda}(t) \leq_v N_F(t) \leq_v N_E(t), t \geq 0$.

Example 3.2. Let $\{N_{F,\lambda}(t), t \geq 0\}$ be a TRP with df F and trend function $\lambda(t) = 1 + t, t \geq 0$. Since $\lambda(t) = 1 + t \geq 1, t \geq 0$, by Corollary 2.8, we get $N_{F,\lambda}(t) \geq_v N_F(t), t \geq 0$.

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Appendices

Appendix A. A few definitions:

Let X and Y be two nonnegative independent rvs with respective dfs $F(\cdot)$ and $G(\cdot)$, survival functions (sfs) $\bar{F}(\cdot)$ and $\bar{G}(\cdot)$, hazard rate functions $r_X(\cdot)$ and $r_Y(\cdot)$, reversed hazard rate functions $q_X(\cdot)$ and $q_Y(\cdot)$.

(1) For any continuous non-negative rv X, we define its hazard rate function and reversed hazard rate function as $r_X(t) = \frac{f(t)}{\bar{F}(t)}$ and $q_X(t) = \frac{f(t)}{F(t)}, t \geq 0$ respectively, where $f(\cdot)$ denotes the pdf of the rv X.

(2) X is said to be larger than Y, in the usual stochastic order denoted by $X \geq_{st} Y$, if $\bar{F}(t) \geq \bar{G}(t), t \geq 0$.

(3) X is said to be larger than Y in the hazard rate order, denoted by $X \geq_{hr} Y$, if $r_X(t) \leq r_Y(t), t \geq 0$, or equivalently if $\frac{\bar{F}(t)}{\bar{G}(t)}$ is non-decreasing in t.

(4) X is said to be larger than Y in the reverse hazard rate order, denoted by $X \geq_{rh} Y$ if $q_X(t) \geq q_Y(t), t \geq 0$, or equivalently if $\frac{F(t)}{G(t)}$ is non-decreasing in t.

(5) For two continuous non-negative independent rvs X and Y with respective pdfs f and g, X is larger than Y in the sense of likelihood ratio, denoted by $X \geq_{lr} Y$, if $\frac{f(t)}{g(t)} \uparrow t, t \geq 0$.

(6) For two continuous non-negative independent rvs X and Y, X is said to be stochastically less variable than Y, denoted as $X \leq_v Y$, if $\int_t^\infty \bar{F}(x)dx \leq \int_t^\infty \bar{G}(x)dx, t \geq 0$.

For two discrete non-negative independent rvs X and Y, X is said to be stochastically less variable than Y, denoted as $X \leq_v Y$, if $\sum_{k=n}^\infty \bar{F}(x) \leq \sum_{k=n}^\infty \bar{G}(x), n = 0, 1, \ldots$.

(7) X and the corresponding df F are said to be increasing failure rate (IFR) if $r_X(t) \uparrow t$.

(8) X and the corresponding df F are said to be New Better than used (NBU) if $\bar{F}(s+t) \leq \bar{F}(t)\bar{F})(s), t \geq 0, s \geq 0$.

(9) X and the corresponding df F are said to be New better than used in expectation (NBUE) if

(a) X has finite mean $\mu_F = \int_0^\infty \bar{F}(x)dx$,

(b) $\bar{F}(t) \geq \frac{1}{\mu_F} \int_t^\infty \bar{F}(x)dx, t \geq 0$.

(10) A real valued function f is said to be concave (convex) if for any $x, y \geq 0$ and for any $\alpha \in [0, 1]$,

$$f((1-\alpha)x + \alpha y) \geq (\leq)(1-\alpha)f(x) + \alpha f(y).$$

If f is twice-differentiable, then f is concave (convex) iff f'' is non-positive (non-negative).

(11) f is said to be log-concave (log-convex) if $\log f$ is concave (convex).

The dual stochastic orders/classes are defined by reversing the inequalities.

Appendix B. A few relationships among the stochastic orders:

(1) $X \geq_{hr} Y \Rightarrow X \geq_{st} Y$

(2) $X \geq_{lr} Y \Rightarrow X \geq_{hr} Y$ and $X \geq_{rh} Y$ (and therefore, $X \geq_{st} Y$)