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Tamagawa torsors of an Abelian variety

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Abstract. For an abelian variety A/K over a number field K, we define the set of *Tamagawa torsors* of A at a prime v of K to be the set of principal homogeneous spaces of A over the completion K_v of K at v that are split by an unramified extension of K_v. In this paper, we study some of the arithmetic properties of the Tamagawa torsors. We also give, following Mazur's theory of visibility, conditions under which non-trivial elements of the Tamagawa torsors of A may be interpreted as rational points on another abelian variety B.

Introduction

Let A/K be an abelian variety defined over a number field K. The Shafarevich-Tate group of A, denoted by $Ш(A/K)$, is the set of isomorphism classes of principal homogeneous spaces of A (also called A-torsors) defined over K that are split by every completion K_v of K, where v is a prime of K. In other words, the set of non-trivial A-torsors[1] in $Ш(A/K)$ have a K_v-rational point for every prime v but do not have any K-rational points. The group $Ш(A/K)$ is a fundamental arithmetic invariant of A/K and its conjectural finiteness can be potentially used in effectively determining the set of rational points $A(K)$. In this paper, we define a new arithmetic invariant of A. Specifically, we consider the set of A-torsors defined over K_v that are split by an unramified extension of K_v. We call this the set of *Tamagawa torsors* of A at v and denote it by $\mathrm{TT}(A/K_v)$. The terminology is rather prosaically explained by the fact (proved in Section 2) that $\mathrm{TT}(A/K_v)$ is a finite set of order $c_{A,v}$, the usual *Tamagawa number* of A at v.[2]

As we explain presently, this is a useful way of interpreting the Tamagawa number of A at v, particularly in the context of the Birch and Swinnerton-Dyer

[1]The *trivial* A-torsors are isomorphic to A over K.

[2]We do welcome suggestions for a different terminology and/or notation.

(BSD) Conjecture. In this paper, we study some of the arithmetic properties of $\mathrm{TT}(A/K_v)$ including its duality properties as well as its relationship to other invariants of A/K such as the Selmer group of A as well as $Ш(A/K)$. We also show how to construct non-trivial elements of $\mathrm{TT}(A/K_v)$ via the rational points on another variety B, also defined over K.

Tamagawa torsors correspond to locally unramified cohomology classes and these have been fairly well-studied in the literature. Consequently, most of the theorems discussed in this paper are simple reinterpretations of known results. In particular, the idea of interpreting the Tamagawa number in terms of torsors is evident in [Maz72,Mil86] and directly stated in [Ste04]. The splitting property of these torsors is mentioned explicitly in [Gon]. Our aim has been to unify the somewhat diverse results by interpreting the Tamagawa number in terms of principal homogeneous spaces.

We now briefly describe the contents and organization of this paper. In Section 1, we present some definitions and results involving the Néron model of A. In Section 2, we define the group of Tamagawa torsors and prove its basic finiteness property. In Section 3, we study the duality properties of Tamagawa torsors. In Section 4, we relate Tamagawa torsors to Selmer groups. Finally, in Section 5, we study visibility properties of Tamagawa torsors. We first show that every Tamagawa torsor is 'visible' in some ambient variety. We then prove an extension of a theorem of Mazur by means of which Tamagawa torsors of A can be interpreted as rational points on another abelian variety B also defined over K.

1. Preliminaries

Let K be a number field with ring of integers $\mathcal{O}_K$ and let A/K be an abelian variety over K. Let $\mathcal{A}$ denote the *Néron model* of A/K over $X = \operatorname{Spec} \mathcal{O}_K$ [BLR90]. Thus $\mathcal{A}$ is separated and of finite type over X with generic fiber A/K, and satisfies the *Néron mapping property*: for each smooth X-scheme S with generic fiber S_K, the restriction map

$$\mathrm{Hom}_X(S, \mathcal{A}) \to \mathrm{Hom}_K(S_K, A)$$

is bijective. Alternatively, recalling [Mil80, §II.1] that A defines a sheaf (which we also denote as A) for the étale (or the flat *fpqf*) topology over $\operatorname{Spec} K$, we consider the direct-image sheaf $j_*(A)$ for the étale (or *fpqf*) topology over $X = \operatorname{Spec} \mathcal{O}_K$, where $j : \operatorname{Spec} K \hookrightarrow X$ is the inclusion of the generic point. When the sheaf $j_*(A)$ is *representable*, the smooth scheme $\mathcal{A} \to X$ representing $j_*(A)$ will be called the Néron model of A.[3] Abusing notation, we identify the functor $j_*(A)$ itself as the Néron model of

[3] It is a deep theorem of Néron that $\mathcal{A}$ exists, up to isomorphism.

A and write $\mathcal{A} = j_*(A)$. Over the flat topology for X, we have a short exact sequence of group schemes (or sheaves):

$$0 \to \mathcal{A}^0 \to \mathcal{A} \to \Phi_A \to 0 \tag{1.1}$$

where $\mathcal{A}^0$ is the largest open subgroup scheme of $\mathcal{A}$ with connected fibers (also called the *identity component*) and $\Phi_A \cong \mathcal{A}/\mathcal{A}^0$ is the *component group* of A. If we regard Φ_A as an étale sheaf over X and denote by $\Phi_{A,v}$ its stalk at a prime v, then $\Phi_{A,v}$ can be considered as a finite, étale group scheme over $\operatorname{Spec} k_v$. Equivalently, $\Phi_{A,v}$ is a finite abelian group equipped with a continuous action of $\mathrm{Gal}(\overline{k}_v/k_v)$, where k_v is the residue field of K_v. Over $\operatorname{Spec} k_v$, we thus have an exact sequence of group schemes

$$0 \to \mathcal{A}_v^0 \to \mathcal{A}_v \to \Phi_{A,v} \to 0$$

The group scheme $\Phi_{A,v} = \mathcal{A}_v/\mathcal{A}_v^0$ of connected components is called the *component group* of A at v. The finite group $\Phi_{A,v}(k_v)$ is called the *arithmetic component group* of A at v and $c_{A,v} = \#\Phi_{A,v}(k_v)$ is called the *Tamagawa number* of A at v. Considering the natural closed immersion $i_v : \operatorname{Spec} k_v \hookrightarrow X$, we find that

$$\Phi_A = \bigoplus_v (i_v)_* \Phi_{A,v} \tag{1.2}$$

where the direct sum is over all v or equivalently, over the finite set of v where A has bad reduction. The short exact sequence (1.1) of étale (or, flat) sheaves over X induces a long exact sequence of étale (or flat) cohomology groups:

$$\begin{aligned} 0 &\to \mathcal{A}^0(X) \to \mathcal{A}(X) \to \Phi_A(X) \\ &\to H^1(X, \mathcal{A}^0) \to H^1(X, \mathcal{A}) \to H^1(X, \Phi_A) \end{aligned} \tag{1.3}$$

where we can write, for all i,

$$H^i(X, \Phi_A) = \bigoplus_v H^i(\operatorname{Spec} k_v, \Phi_{A,v}) \tag{1.4}$$

which follows from (1.2). The Néron mapping property implies that

$$\mathcal{A}(X) \cong A(K)$$

Furthermore, it is also known (see the Appendix to [Maz72]) that, staying away from 2-primary components,

$$Ш(A/K) \cong \mathrm{im}[H^1(X, \mathcal{A}^0) \longrightarrow H^1(X, \mathcal{A})]$$

where Ш(A/K) is the Shafarevich-Tate group of A. Thus, the exact sequence (1.3) becomes

$$0 \to \mathcal{A}^0(X) \to A(K) \to \bigoplus_v \Phi_{A,v}(k_v) \to H^1(X, \mathcal{A}^0) \to \text{Ш}(A/K) \to 0 \tag{1.5}$$

In particular, staying away from 2-primary components, we note that the group Ш(A/K) may be expressed in two ways. First, the exact sequence

$$\bigoplus_v \Phi_{A,v}(k_v) \to H^1(X, \mathcal{A}^0) \to \text{Ш}(A/K) \to 0 \tag{1.6}$$

identifies Ш(A/K) as a cokernel. Secondly, the exact sequence

$$0 \to \text{Ш}(A/K) \to H^1(X, \mathcal{A}) \to \bigoplus_v H^1(\operatorname{Spec} k_v, \Phi_{A,v}) \tag{1.7}$$

identifies Ш(A/K) as a kernel.

2. Tamagawa torsors

Let A/K_v be an abelian variety defined over the completion K_v of K at a prime v. Let K_v^{ur} be the maximal unramified extension of K_v. The inclusion $\mathrm{Gal}(\overline{K_v}/K_v^{\mathrm{ur}}) \subset \mathrm{Gal}(\overline{K_v}/K_v)$ induces a map

$$H^1(K_v, A) \longrightarrow H^1(K_v^{\mathrm{ur}}, A)$$

whose kernel corresponds to the *unramified* subgroup of $H^1(K_v, A)$. The map may also be given as

$$\mathrm{WC}(A/K_v) \longrightarrow \mathrm{WC}(A/K_v^{\mathrm{ur}})$$

where $\mathrm{WC}(A/K_v) \cong H^1(K_v, A)$ denotes the Weil-Châtelet group of A over K_v. In this situation, the kernel represents the set of principal homogeneous spaces of A (or A-torsors) defined over K_v that are split by K_v^{ur}. Since K_v^{ur} is the directed union of finite unramified Galois extensions of K_v, it follows that the kernel corresponds to non-trivial A-torsors over K_v that do not have a K_v-rational point but have a L_i-rational point for some finite unramified Galois extension L_i of K_v. We denote this kernel by $\mathrm{TT}(A/K_v)$ and call it the group of *Tamagawa torsors* of A at v. In particular, $\mathrm{TT}(A/K_v)$ is the subgroup of unramified cohomology classes in $H^1(K_v, A)$. In order to analyze the group $\mathrm{TT}(A/K_v)$, we begin with the following proposition.

Proposition 2.1. *Let L_i be a finite, unramified Galois extension of K_v with residue field l_i, and let A/K_v be an abelian variety over K_v. Then*

$$H^1(L_i/K_v, A(L_i)) \cong H^1(l_i/k_v, \Phi_{A,v}(l_i))$$

Proof. Let $\mathcal{O}_{L_i}$ be the ring of integers of L_i and $\mathfrak{m}_{L_i}$ its maximal ideal so that $l_i = \mathcal{O}_{L_i}/\mathfrak{m}_{L_i}$. In particular, l_i is a finite Galois extension of the finite field k_v and there is a canonical isomorphism $\mathrm{Gal}(L_i/K_v) \cong \mathrm{Gal}(l_i/k_v)$. Let us denote this group as G_i. Let $\mathcal{A}_{X_v} = \mathcal{A} \times_X X_v$, where $X_v = \operatorname{Spec} \mathcal{O}_{K_v}$. In particular, $\mathcal{A}_{X_v}$ is the Néron model of A/K_v over X_v. Since $\mathcal{A}_{X_v}$ is smooth over X_v, it follows that the reduction map

$$\mathcal{A}_{X_v}(\mathcal{O}_{L_i}) \to \mathcal{A}_v(l_i)$$

is surjective [Mil80, §I.4.13]. Thus there is an exact sequence of G_i-modules

$$0 \to \mathcal{A}^0_{X_v}(\mathcal{O}_{L_i}) \to \mathcal{A}_{X_v}(\mathcal{O}_{L_i}) \to \Phi_{A,v}(l_i) \to 0$$

We also find that $\mathcal{A}_{X_v} \otimes_{X_v} \operatorname{Spec} \mathcal{O}_{L_i}$ is the Néron model of $A \otimes_{K_v} L_i$ so that in particular $\mathcal{A}_{X_v}(\mathcal{O}_{L_i}) \cong A(L_i)$. The above sequence now becomes

$$0 \to \mathcal{A}^0_{X_v}(\mathcal{O}_{L_i}) \to A(L_i) \to \Phi_{A,v}(l_i) \to 0$$

It therefore suffices to show that $H^n(G_i, \mathcal{A}^0_{X_v}(\mathcal{O}_{L_i})) = 0$ for $n = 1, 2$. Let $[\xi] \in H^1(G_i, \mathcal{A}^0_{X_v}(\mathcal{O}_{L_i}))$ be represented by an $\mathcal{A}^0_{X_v}$-torsor C. Since $\mathcal{A}^0_v$ is a connected algebraic group over the finite field k_v, it follows from Lang's Theorem that the $\mathcal{A}^0_v$-torsor $C \otimes_{\mathcal{O}_{K_v}} k_v$ is trivial so that $C(k_v) \neq \varnothing$. It now follows from Hensel's lemma that $C(\mathcal{O}_{K_v}) \neq \varnothing$, and hence $[\xi] = 0$.

On the other hand, we find that $H^2(G_i, \mathcal{A}^0_{X_v}(\mathcal{O}_{L_i}/\mathfrak{m}^r_{L_i})) = 0$ for all r, since G_i has cohomological dimension 1. It follows [Ser79, XII §3 Lemma 3] that $H^2(G_i, \mathcal{A}^0_{X_v}(\mathcal{O}_{L_i})) = 0$. □

Remark. The above proof, which follows that of Prop 3.8 in [Mil86], contains a subtle error which is explained and corrected in the errata for [Mil86], available at www.jmilne.org.

Theorem 2.2. *The set* $\mathrm{TT}(A/K_v)$ *is finite and* $\#\mathrm{TT}(A/K_v) = c_{A,v}$.

Proof. The inflation-restriction sequence

$$0 \longrightarrow H^1(K_v^{\mathrm{ur}}/K_v, A(K_v^{\mathrm{ur}})) \longrightarrow H^1(K_v, A) \longrightarrow H^1(K_v^{\mathrm{ur}}, A)$$

identifies the set of Tamagawa torsors with the injective image of the group $H^1(K_v^{\mathrm{ur}}/K_v, A(K_v^{\mathrm{ur}}))$ in $H^1(K_v, A)$. There is an isomorphism

$$H^1(K_v^{\mathrm{ur}}/K_v, A(K_v^{\mathrm{ur}})) \cong \varinjlim H^1(L_i/K_v, A(L_i))$$

where the direct limit is over all finite, unramified Galois extensions L_i of K_v. It now follows from Proposition 2.1 above that

$$H^1(K_v^{\mathrm{ur}}/K_v, A(K_v^{\mathrm{ur}})) \cong \varinjlim H^1(l_i/k_v, \Phi_{A,v}(l_i)) \cong H^1(k_v, \Phi_{A,v})$$

But $\Phi_{A,v}$ is a finite Galois module over the finite field k_v so that its Herbrand quotient is 1. This implies that

$$\#H^1(\operatorname{Spec} k_v, \Phi_{A,v}) = \#H^0(\operatorname{Spec} k_v, \Phi_{A,v}) = c_{A,v} \qquad \square$$

Remark 2.3. The proof above shows that the set $\mathrm{TT}(A/K_v)$ is trivial when A has *good reduction* at v. It also follows from the theorem that the direct sum $\bigoplus_v \mathrm{TT}(A/K_v)$ is the set of Tamagawa torsors of A over all v and has order $\prod_v c_{A,v}$.

3. Local duality of Tamagawa torsors

Let $A^\vee$ be the abelian variety dual to A over K_v. For any abelian group M, we denote by $M^* := \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$ the *Pontryagin dual* of M or the character group of M. The main theorem in this section identifies the character group of $\mathrm{TT}(A/K_v)$ as the arithmetic component group $\Phi_{A^\vee,v}(k_v)$.

Theorem 3.1. *There is a canonical pairing*

$$\Phi_{A^\vee,v}(k_v) \times \mathrm{TT}(A/K_v) \to \mathbf{Q}/\mathbf{Z}$$

which induces an isomorphism

$$\mathrm{TT}(A/K_v)^* \cong \Phi_{A^\vee,v}(k_v)$$

Theorem 3.1 will follow as a consequence of two lemmas that we now establish. To begin with, there is a canonical, non-degenerate pairing due to Tate [Mil86, Cor I.3.4], [Tat57]

$$A^\vee(K_v) \times H^1(K_v, A) \to \mathbf{Q}/\mathbf{Z}$$

which induces an isomorphism $A^\vee(K_v) \cong H^1(K_v, A)^*$ of locally compact groups.

Lemma 3.2. *Let L_i be a finite, unramified Galois extension of K_v. Then there is a perfect pairing*

$$A^\vee(K_v)/\mathrm{Nm}(A^\vee(L_i)) \times H^1(L_i/K_v, A(L_i)) \to \mathbf{Q}/\mathbf{Z}$$

where $\mathrm{Nm}(A^\vee(L_i)) \subset A^\vee(K_v)$ *is the image of* $A^\vee(L_i)$ *in* $A^\vee(K_v)$ *under the norm mapping corresponding to* $\mathrm{Gal}(L_i/K_v)$.

Proof. The restriction map

$$H^1(K_v, A) \to H^1(L_i, A)$$

fits into a commutative diagram

$$\begin{array}{ccc} H^1(K_v, A) & \longrightarrow & H^1(L_i, A) \\ \downarrow & & \downarrow \\ A^\vee(K_v)^* & \longrightarrow & A^\vee(L_i)^* \end{array}$$

where the vertical arrows are isomorphisms by Tate's local duality. The bottom horizontal map is dual [Tat57] to the norm map

$$\mathrm{Nm}_{L_i/K_v} : A^\vee(L_i) \to A^\vee(K_v)$$

so that there are isomorphisms [Tat57, §8, Cor 1]

$$\begin{aligned} H^1(L_i/K_v, A(L_i)) &\cong \ker(H^1(K_v, A) \to H^1(L_i, A)) \\ &\cong \ker(A^\vee(K_v)^* \to A^\vee(L_i)^*) \\ &\cong (\mathrm{coker}(A^\vee(L_i) \xrightarrow{\mathrm{Nm}} A^\vee(K_v)))^* \\ &\cong (A^\vee(K_v)/\mathrm{Nm}(A^\vee(L_i)))^* \end{aligned}$$

In particular, $\mathrm{Nm}(A^\vee(L_i)) \subset A^\vee(K_v)$ is the exact annihilator of the subgroup $H^1(L_i/K_v, A(L_i)) \subset H^1(K_v, A)$ under the Tate pairing described above. □

Since $H^1(L_i/K_v, A(L_i)) \cong H^1(l_i/k_v, \Phi_{A,v})$ by Proposition 2.1 and since the latter group is finite, we conclude that $A^\vee(K_v)/\mathrm{Nm}(A^\vee(L_i))$ is finite as well. Subgroups of $A^\vee(K_v)$ of the form $\mathrm{Nm}(A^\vee(L_i))$ for any finite extension L_i will be called the *norm subgroups* of $A^\vee(K_v)$. We define the group of *universal norms* on $A^\vee(K_v)$ from K_v^{ur} to be

$$\mathrm{Nm}(A^\vee(K_v^{\mathrm{ur}})) = \bigcap_{L_i} \mathrm{Nm}(A^\vee(L_i))$$

where the intersection is over all finite, unramified extensions L_i of K_v. It follows from Lemma 3.2 that $\mathrm{TT}(A/K_v) = \varinjlim_{L_i} H^1(L_i/K_v, A(L_i))$ is the Pontryagin dual of $A^\vee(K_v)/\mathrm{Nm}(A^\vee(K_v^{\mathrm{ur}}))$. To prove Theorem 3.1, it therefore suffices to show that

Lemma 3.3. *For an abelian variety A/K_v, the arithmetic component group $\Phi_{A,v}(k_v)$ is isomorphic to $A(K_v)$ modulo the subgroup of universal norms from K_v^{ur}, i.e.*

$$\Phi_{A,v}(k_v) \cong A(K_v)/\mathrm{Nm}(A(K_v^{\mathrm{ur}}))$$

Proof. The norm map $A(L_i) \to A(K_v)$ may be given as a map

$$\mathcal{A}_{X_v}(\mathcal{O}_{L_i}) \to \mathcal{A}_{X_v}(\mathcal{O}_{K_v})$$

where $\mathcal{A}_{X_v}$ is as defined in the proof of Proposition 2.1. Choosing the finite, unramified extension L_i such that $[L_i : K_v] = [l_i : k_v]$ is coprime to $c_{A,v} = \#\Phi_{A,v}(k_v)$, we find that there is an induced norm map

$$\mathrm{Nm}^0 : \mathcal{A}^0_{X_v}(\mathcal{O}_{L_i}) \to \mathcal{A}^0_{X_v}(\mathcal{O}_{K_v})$$

whose kernel $G = \ker(\mathrm{Nm}^0)$ is a smooth group scheme with connected fibers. Let $x : \operatorname{Spec} \mathcal{O}_{K_v} \to \mathcal{A}^0_{X_v}$ be any $\mathcal{O}_{K_v}$-point of $\mathcal{A}^0_{X_v}$. The Nm^0-pullback of x is a smooth $\mathcal{O}_{K_v}$-scheme Y that is also a G-torsor for the étale topology. The special fiber $Y_v = Y \times_{\operatorname{Spec} \mathcal{O}_{K_v}} \operatorname{Spec} k_v$ is a torsor for a smooth, connected group scheme over the finite field k_v. By Lang's Theorem, Y_v has a k_v-rational point. Since Y is smooth over $\mathcal{O}_{K_v}$, the reduction map

$$Y(\mathcal{O}_{K_v}) \to Y_v(k_v)$$

is surjective and thus, Y has an $\mathcal{O}_{K_v}$-point. We conclude that Nm^0 is surjective. Now consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A}^0_{X_v}(\mathcal{O}_{L_i}) & \longrightarrow & \mathcal{A}_{X_v}(\mathcal{O}_{L_i}) \cong A(L_i) & \longrightarrow & \Phi_{A,v}(l_i) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Nm}^0} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}^0_{X_v}(\mathcal{O}_{K_v}) & \longrightarrow & \mathcal{A}_{X_v}(\mathcal{O}_{K_v}) \cong A(K_v) & \longrightarrow & \Phi_{A,v}(k_v) & \longrightarrow & 0
\end{array}$$

The left vertical map is surjective while the right vertical map is the 0 map. It follows that $A(K_v)/\bigcap_{L_i} \mathrm{Nm}(A(L_i)) \cong \Phi_{A,v}(k_v)$. □

Remark 3.4. The pairing in Theorem 3.1 also follows directly from the canonical pairing $\Phi_{A^\vee,v} \times \Phi_{A,v} \to \mathbf{Q}/\mathbf{Z}$ defined by Grothendieck [Mil86, III C.13].

Theorem 3.5. *There are exact sequences*

$$0 \longrightarrow \mathcal{A}^{\vee 0}(X) \longrightarrow A^\vee(K) \longrightarrow \textstyle\bigoplus_v \Phi_{A^\vee,v}(k_v)$$

$$0 \longrightarrow Ш(A/K) \longrightarrow H^1(X, \mathcal{A}) \longrightarrow \textstyle\bigoplus_v \mathrm{TT}(A/K_v)$$

where $\mathcal{A}^\vee$ is the Néron model of $A^\vee$. Assuming that $Ш(A/K)$ is finite, the images of the two-right hand maps are orthogonal complements of each other under the pairing described in Theorem 3.1.

Proof. The first exact sequence is 1.5 applied to $A^\vee$ while the second exact sequence follows from 1.7 and Theorem 2.2. The second part of the theorem follows from the main result in [Gon]. □

Remark 3.6. The second exact sequence in the theorem above also shows that the index $[H^1(X, \mathcal{A}) : Ш(A/K)]$ divides $\prod_v c_{A,v}$, the product of the Tamagawa numbers of A.

Remark 3.7. According to the discussion in [Gon], the cokernel of the map $A^\vee(K) \to \bigoplus_v \Phi_{A^\vee,v}(k_v)$ may be defined as the *Néron class group* of $A^\vee/K$ and denoted by $C_{A^\vee,K}$. If the image of the map $H^1(X, \mathcal{A}) \to \bigoplus_v \mathrm{TT}(A/K_v)$ is denoted by $C^1_{A,K}$, then the above theorem states that, assuming $Ш(A/K)$ is finite, there is a perfect pairing

$$C_{A^\vee,K} \times C^1_{A,K} \to \mathbf{Q}/\mathbf{Z}$$

It also follows that the short exact sequence

$$0 \to C_{A^\vee,K} \to H^1(X, \mathcal{A}^{\vee 0}) \to Ш(A^\vee/K) \to 0$$

has, as its dual, the short exact sequence

$$0 \to Ш(A/K) \to H^1(X, \mathcal{A}) \to C^1_{A,K} \to 0$$

4. Selmer groups and Tamagawa torsors

In this section, we relate the Tamagawa torsors to Selmer groups. We begin with the local case. Consider the Kummer exact sequence

$$0 \to A(K_v) \otimes \mathbf{Z}/n\mathbf{Z} \to H^1(K_v, A[n]) \to H^1(K_v, A)[n] \to 0$$

over K_v for any integer n. Passage to direct limits yields the exact sequence

$$0 \to A(K_v) \otimes \mathbf{Q}/\mathbf{Z} \to H^1(K_v, A_{\mathrm{tor}}) \to H^1(K_v, A) \to 0$$

Let $H^1_{\mathrm{TT}}(K_v, A_{\mathrm{tor}}) \subset H^1(K_v, A_{\mathrm{tor}})$ be the inverse image of $\mathrm{TT}(A/K_v) \subset H^1(K_v, A)$ under the surjection $H^1(K_v, A_{\mathrm{tor}}) \to H^1(K_v, A)$. We then have an exact sequence

$$0 \to A(K_v) \otimes \mathbf{Q}/\mathbf{Z} \to H^1_{\mathrm{TT}}(K_v, A_{\mathrm{tor}}) \to \mathrm{TT}(A/K_v) \to 0$$

Identifying the injective image of $A(K_v) \otimes \mathbf{Q}/\mathbf{Z}$ in $H^1(K_v, A_{\mathrm{tor}})$ (as well as in $H^1_{\mathrm{TT}}(K_v, A_{\mathrm{tor}})$) as the *local Selmer group* $\mathrm{Sel}(A/K_v)$, we have thus proved that

Proposition 4.1. *Let $H^1_{\mathrm{TT}}(K_v, A_{\mathrm{tor}})$ be defined as above. Then there is an exact sequence*

$$0 \to \mathrm{Sel}(A/K_v) \to H^1_{\mathrm{TT}}(K_v, A_{\mathrm{tor}}) \to \mathrm{TT}(A/K_v) \to 0$$

In particular, the index of $\mathrm{Sel}(A/K_v)$ *in* $H^1_{\mathrm{TT}}(K_v, A_{\mathrm{tor}})$ *is equal to the Tamagawa number of A at v.*

The global Kummer exact sequence is related to the local one by means of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(K)\otimes \mathbf{Z}/n\mathbf{Z} & \longrightarrow & H^1(K, A[n]) & \longrightarrow & H^1(K, A)[n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & \searrow & \downarrow & & \\ 0 & \longrightarrow & \prod_v A(K_v)\otimes \mathbf{Z}/n\mathbf{Z} & \longrightarrow & \prod_v H^1(K_v, A[n]) & \longrightarrow & \prod_v H^1(K_v, A)[n] & \longrightarrow & 0 \end{array}$$

where the vertical arrows are induced by the inclusions $\mathrm{Gal}(\overline{K}/K) \subset \mathrm{Gal}(\overline{K_v}/K_v)$ and $A(\overline{K}) \subset A(\overline{K_v})$ for every v. The *global n-Selmer group,* denoted by $\mathrm{Sel}_n(A/K)$, is the kernel of the diagonal map in the above diagram so that there is an exact sequence

$$0 \to \mathrm{Sel}_n(A/K) \to H^1(K, A[n]) \to \prod_v H^1(K_v, A)[n]$$

Let $H^1_{\mathrm{TT}}(K, A[n])$ be the subgroup of $H^1(K, A[n])$ that maps to $\bigoplus_v \mathrm{TT}(A/K_v)[n] \subseteq \prod_v H^1(K_v, A)[n]$ under the map above. There is thus an exact sequence

$$0 \to H^1_{\mathrm{TT}}(K, A[n]) \to H^1(K, A[n]) \to \prod_v H^1(K_v^{\mathrm{ur}}, A)[n]$$

Proposition 4.2. *There is an exact sequence*

$$0 \to \mathrm{Sel}_n(A/K) \to H^1_{\mathrm{TT}}(K, A[n]) \to \bigoplus_v \mathrm{TT}(A/K_v)[n]$$

Proof. Apply the kernel-cokernel exact sequence [Mil86, I 0.24] to the pair of maps

$$H^1(K, A[n]) \to \prod_v H^1(K_v, A)[n] \to \prod_v H^1(K_v^{\mathrm{ur}}, A)[n]$$

□

Let $H^1_{\mathrm{TT}}(K, A_{\mathrm{tor}})$ be the direct sum of the groups $H^1_{\mathrm{TT}}(K, A[n])$ and the Selmer group $\mathrm{Sel}(A/K)$ be that of the groups $\mathrm{Sel}_n(A/K)$ over all n. We find that

Corollary 4.3. *The index of the Selmer group* $\mathrm{Sel}(A/K)$ *in* $H^1_{\mathrm{TT}}(K, A_{\mathrm{tor}})$ *divides* $\prod_v c_{A,v}$.

Proof. Pass on to the direct limit of the exact sequence in Proposition 4.2.

□

Remark 4.4. The results proved so far show that there is a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A(K)\otimes \mathbf{Q}/\mathbf{Z} & \longrightarrow & \mathrm{Sel}(A/K) & \longrightarrow & Ш(A/K) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A(K)\otimes \mathbf{Q}/\mathbf{Z} & \longrightarrow & H^1_{\mathrm{TT}}(K, A_{\mathrm{tor}}) & \longrightarrow & H^1_{\mathrm{ur}}(K, A) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \bigoplus_v \mathrm{TT}(A/K_v) & = & \bigoplus_v \mathrm{TT}(A/K_v) & & \end{array}$$

that relates the Mordell-Weil group, the Selmer group, the Shafarevich-Tate group and the Tamagawa torsors of A. Here $H^1_{\mathrm{ur}}(K, A) \cong H^1(X, \mathcal{A})$ is the subgroup of $H^1(K, A)$ that maps to $H^1(K_v^{\mathrm{ur}}/K_v, A(K_v^{\mathrm{ur}})) \subseteq H^1(K_v, A)$ for every v. The diagram shows, in particular, that index of $\mathrm{Sel}(A/K)$ in $H^1_{\mathrm{TT}}(K, A_{\mathrm{tor}})$ as well as that of $Ш(A/K)$ in $H^1_{\mathrm{ur}}(K, A)$ divide $\prod_v c_{A,v}$, the product of the Tamagawa numbers of A.

5. Visibility of Tamagawa torsors

Let $\iota : A \hookrightarrow J$ be an embedding[4] of abelian varieties over K_v. The kernel of the induced map $H^1(K_v, A) \to H^1(K_v, J)$ may be defined, following [CM00], as the *visible* subgroup of $H^1(K_v, A)$ with respect to the embedding ι and denoted by $\mathrm{Vis}_J\, H^1(K_v, A)$[5] i.e.

$$\mathrm{Vis}_J\, H^1(K_v, A) := \ker(H^1(K_v, A) \to H^1(K_v, J))$$

The terminology can be explained by noting that if $C = J/A$ then, over K_v, there is a short exact sequence

$$0 \to A \to J \xrightarrow{\pi} C \to 0$$

of $\mathrm{Gal}(\overline{K_v}/K_v)$-modules which induces a long exact sequence of cohomology groups

$$0 \to A(K_v) \to J(K_v) \to C(K_v) \to H^1(K_v, A) \to H^1(K_v, J) \to \cdots$$

which can be truncated to the exact sequence

$$0 \to J(K_v)/A(K_v) \to C(K_v) \to \mathrm{Vis}_J\, H^1(K_v, A) \to 0$$

[4]i.e. a morphism that is also a closed immersion.

[5]Although the visible subgroup depends on the choice of embedding, it is usually clear from the context and is therefore omitted from the notation.

Let $\xi \in \mathrm{Vis}_J H^1(K_v, A)$ be the image of $P \in C(K_v)$. Then $\pi^{-1}(P)$ is a coset of A in J, and thus is a torsor under A. This explains how elements in $\mathrm{Vis}_J H^1(K_v, A)$ are 'visible' in $J(\overline{K_v})$.[6] Clearly we can define $\mathrm{Vis}_J \mathrm{TT}(A/K_v)$, the visible part of the Tamagawa torsors, as

$$\mathrm{Vis}_J \mathrm{TT}(A/K_v) := \mathrm{Vis}_J H^1(K_v, A) \cap \mathrm{TT}(A/K_v)$$

Let L_i be a finite, unramified Galois extension of K_v and let $\mathrm{Res}_{L_i/K_v}(A_{L_i})$ be the *restriction of scalars* of A_{L_i} from L_i to K_v. In particular, it is an abelian variety over K_v of dimension $[L_i : K_v].\dim(A)$

Proposition 5.1. *Every element in* $\mathrm{TT}(A/K_v)$ *is visible in* $\mathrm{Res}_{L_i/K_v}(A_{L_i})$ *for some finite, unramified Galois extension* L_i *of* K_v.

Proof. There is a canonical embedding $A \hookrightarrow \mathrm{Res}_{L_i/K_v}(A_{L_i})$ of abelian varieties over K_v which induces a map

$$H^1(K_v, A) \to H^1(K_v, \mathrm{Res}_{L_i/K_v}(A_{L_i}))$$

We thus have an exact sequence

$$0 \to \mathrm{Vis}_{\mathrm{Res}_{L_i/K_v}(A_{L_i})}(H^1(K_v, A)) \to H^1(K_v, A) \to H^1(K_v, \mathrm{Res}_{L_i/K_v}(A_{L_i}))$$

On the other hand, the inflation-restriction sequence with respect to the extension L_i/K_v is

$$0 \to H^1(L_i/K_v, A(L_i)) \to H^1(K_v, A) \to H^1(L_i, A)$$

A straightforward application of Shapiro's lemma [Ser97, I §2, Prop 10] implies that there is an isomorphism

$$H^1(K_v, \mathrm{Res}_{L_i/K_v}(A_{L_i})) \cong H^1(L_i, A)$$

It follows that we have an isomorphism

$$\mathrm{Vis}_{\mathrm{Res}_{L_i/K_v}(A_{L_i})}(H^1(K_v, A)) \cong H^1(L_i/K_v, A(L_i))$$

Upon passage to the direct limit over such L_is, we obtain isomorphisms

$$\begin{aligned} \varinjlim \mathrm{Vis}_{\mathrm{Res}_{L_i/K_v}(A_{L_i})}(H^1(K_v, A)) &\cong \varinjlim H^1(L_i/K_v, A(L_i)) \\ &\cong H^1(K_v^{\mathrm{ur}}/K_v, A(K_v^{\mathrm{ur}})) \\ &\cong \mathrm{TT}(A/K_v) \end{aligned}$$

where the last isomorphism follows from the proof of Theorem 2.2. □

[6] Considering the embedding $A \hookrightarrow J$ over the number field K, it can be shown [CM00] that $\mathrm{Vis}_J H^1(K, A)$ is finite.

Having described the ambient variety (i.e. $\mathrm{Res}_{L_i/K_v}(A_{L_i})$) in which any Tamagawa torsor is always visible, we now give a method by means of which Tamagawa torsors of A may be interpreted as K-rational points on another variety B with which it shares a certain p-congruence. The following is a variation of the main theorem proved in [AB13].

Theorem 5.2. *Let A and B be abelian varieties of the same dimension over a number field K, having ranks $r_A = 0$ and $r_B > 0$ respectively and such that B has semistable reduction over K. Let N be an integer divisible by the residue characteristics of the primes of bad reduction for both A and B. Let p be an odd prime such that $e_p < p - 1$, where e_p is the largest ramification index of any prime of K lying over p, and such that*

$$gcd\left(p,\ N \cdot \#A(K)_{\mathrm{tor}} \cdot \prod_v c_{B,v}\right) = 1$$

Suppose further that $B[p] \cong A[p]$ over K. Assuming that $Ш(A/K)$ has trivial p-primary components, there is an injection

$$B(K)/pB(K) \hookrightarrow \bigoplus_v \mathrm{TT}(A/K_v)[p]$$

Proof. We briefly sketch the proof, referring to [AB13] for details. The isomorphism $A[p] \cong B[p]$ over K induces an isomorphism $\mathcal{A}[p] \cong \mathcal{B}[p]$ over $X = \mathrm{Spec}\, \mathcal{O}_K$, where $\mathcal{A}$ and $\mathcal{B}$ are the corresponding Néron models (this is the heart of the proof as given in [AB13]). It then follows, given the conditions of the theorem, that we have a diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & Ш(A/K)[p] \\ & & \downarrow \\ B(K)/pB(K) & \xrightarrow{\varphi} & H^1(X, \mathcal{A})[p] \\ & & \downarrow \\ & & \bigoplus_v H^1(\mathrm{Spec}\, k_v, \Phi_{A,v})[p] \end{array}$$

such that $\ker(\varphi) = 0$. However, $Ш(A/K)[p]$ is trivial by the conditions of the theorem and $H^1(\mathrm{Spec}\, k_v, \Phi_{A,v})[p] \cong \mathrm{TT}(A/K_v)[p]$ (see the proof of Theorem 2.2). The desired result now follows. □

We may also say, in the language of [CM00], that $\mathrm{TT}(A/K_v)$ is 'explained' by $B(K)$. We now reinterpret the example discussed in [AB13], which was first discovered in [Ste04]. Consider the optimal elliptic curves $A = 114C1$ and $B = 57A1$. The data in [Cre97] shows that A has rank 0, $A(\mathbf{Q})_{\text{tor}} \cong \mathbf{Z}/4\mathbf{Z}$ and $Ш(A/\mathbf{Q})$ has trivial conjectural order. On the other hand, we find that $B(\mathbf{Q}) \cong \mathbf{Z}$ and $\prod_l c_{B,l} = 2$. Furthermore, we have $A[5] \cong B[5]$ over $\mathbf{Q}$. Thus, the triple $(A, B, 5)$ satisfies the hypothesis in Theorem 5.2 and we conclude that there is an injection $\mathbf{Z}/5\mathbf{Z} \hookrightarrow \bigoplus_p \mathrm{TT}(A/\mathbf{Q}_p)$ i.e. A has a Tamagawa torsor of order 5. This agrees with the available data, according to which $\prod_l c_{A,l} = 20$.

Remark 5.3. For an optimal elliptic curve $A/\mathbf{Q}$ of rank 0, the second part of the BSD Conjecture states that

$$\frac{L_{A,\mathbf{Q}}(1)}{\Omega_A} \cdot (\#A(\mathbf{Q})_{\text{tor}})^2 \stackrel{?}{=} \#Ш(A/\mathbf{Q}) \cdot \prod_p c_{A,p}$$

where $L_{A,\mathbf{Q}}(s)$ is the *L-function* associated to $A/\mathbf{Q}$, Ω_A is the *real volume* of A computed using a Néron differential. Under the conditions of Theorem 5.2, Agashe has shown [Aga10, Prop 1.5] that p divides the left-hand side of the BSD formula. Since $Ш(A/\mathbf{Q})$ is assumed to have trivial p-torsion, it follows that p must divide the Tamagawa numbers of A. With our interpretation of the Tamagawa number as the number of Tamagawa torsors, this implies that A must have a Tamagawa torsor of order p which is precisely what the theorem confirms. Thus one may also view Theorem 5.2 as providing theoretical evidence for the BSD Conjecture.

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