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On generalized graph ideals of complete bipartite graphs

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Abstract. Let $S = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring in two sets of variables over a field K. Using the notion of linear quotients, we investigate significative classes of graph ideals of S that have a linear resolution, namely the generalized graph ideals, in order to compute standard algebraic invariants of S modulo such ideals. Moreover we are able to determine the structure of the ideals of vertex covers for such generalized graph ideals.

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Introduction

The present work focuses on the study of monomial ideals of mixed products that arise from a simple graph, the so-called generalized graph ideals ([16]). They are generated by square-free monomials of fixed finite degree $q \geqslant 2$ associated to the paths of length $q-1$ of the graph, the $(q-1)$-paths. Let $\mathcal{G}$ be a graph on vertex set $[n] = \{v_1, \dots, v_n\}$ and $R = K[X_1, \dots, X_n]$ be the polynomial ring over a field K, with one variable X_i for each vertex v_i. The generalized graph ideal of $\mathcal{G}$ is the ideal of R generated by all the square-free monomials $X_{i_1} \cdots X_{i_q}$ of degree q such that the vertex v_{i_j} is adjacent to $v_{i_{j+1}}$, for all $1 \leqslant j \leqslant q-1$. It is denoted by $I_q(\mathcal{G})$. The monomial generators of $I_q(\mathcal{G})$ correspond to the $(q-1)$-paths of $\mathcal{G}$.

In [17] there are various results about monomial ideals of R associated to the edges of $\mathcal{G}$. Some problems arise when we will study good properties for monomial ideals and for some algebras related to such ideals ([8,9,14]). In the last years, monomial ideals of the polynomial ring $S = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ in two sets of variables over a field K were considered and some algebraic properties of them were studied ([12,14]). We are interested to investigate bipartite graphs because these graphs determine monomial ideals in S. More precisely, we say that a graph $\mathcal{G}$ is bipartite if its vertex set $[n+m]$ can be partitioned into two disjoint subsets $[n] = \{x_1, \ldots, x_n\}$ and $[m] = \{y_1, \ldots, y_m\}$ such that any edge joins a vertex of $[n]$ with a vertex of $[m]$. In [16], the ideals of mixed products that describe the generalized graph ideals of complete bipartite graphs are considered. In particular, when $\mathcal{G}$ is a bipartite complete graph, it is: $I_q(\mathcal{G}) = I_\ell J_{\ell+1} + I_{\ell+1} J_\ell$ if $q = \ell + 1$, $\ell \in \mathbb{N}^*$, and $I_q(\mathcal{G}) = I_\ell J_\ell$ if $q = 2\ell$, $\ell \in \mathbb{N}^*$, where I_ℓ (resp. J_ℓ) is the monomial ideal of S generated by all the square-free monomials of degree ℓ in the variables $X_1, \ldots, X_n$ (resp. $Y_1, \ldots, Y_m$). In [15], the authors study the Rees algebra of $I_q(\mathcal{G})$ and they examine when $I_q(\mathcal{G})$ is of linear type.

In this note we prove that the generalized graph ideals $I_q(\mathcal{G})$ of a compete bipartite graph $\mathcal{G}$ have linear resolution, by using the technique of studying the linear quotients of such ideals as previously employed in [10,12]. We also give formulae for standard invariants of $S/I_q(\mathcal{G})$ such as dimension, projective dimension, depth, and Castelnuovo-Mumford regularity. Moreover, we establish under what conditions $I_q(\mathcal{G})$ is Cohen-Macaulay. Lastly, we determine the generators of the ideals of vertex covers of $I_q(\mathcal{G})$.

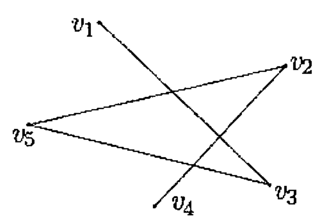
The paper is organized in the following way. Section 1 contains notations and terminology on graphs and algebraic theory associated with them. In section 2, according to results that characterize monomial ideals with linear quotients ([3,5]) it is proved that the ideals $I_q(\mathcal{G})$ have linear resolution. As a consequence of this, together with the computation of integers connected to $I_q(\mathcal{G})$, we are able to determine standard algebraic invariants of such ideals. Some of these will be useful for showing in what cases the ideals $I_q(\mathcal{G})$ are Cohen-Macaulay. In section 3 we consider algebraic aspects linked to a generalization of the notion of minimal vertex covers that holds for complete bipartite graphs. Let $I \subset S$ be a monomial ideal. The ideal of (minimal) covers of a monomial ideal I of S, denoted by I_c, is generated by all monomials $X_{i_1} \cdots X_{i_k} Y_{j_1} \cdots Y_{j_l}$ such that $(X_{i_1}, \ldots, X_{i_k}, Y_{j_1}, \ldots, Y_{j_l})$ is an associated (minimal) prime of I. The ideal of vertex covers of $I_q(\mathcal{G})$ is denoted by $(I_q)_c(\mathcal{G})$. When $\mathcal{G}$ is a complete bipartite graph, the structure of $(I_q)_c(\mathcal{G})$ is fully described.

1. Preliminary notions

Let $\mathcal{G}$ be a graph with vertices $v_1, \ldots, v_n$. Let $R = K[X_1, \ldots, X_n]$ be the polynomial ring over a field K with one variable X_i for each vertex v_i.

Definition 1.1. *The* generalized graph ideal *of $\mathcal{G}$, denoted by $I_q(\mathcal{G})$, is the ideal of $K[X_1, \ldots, X_n]$ generated by all the square-free monomials $X_{i_1} \cdots X_{i_q}$ of degree q such that the vertex v_{i_j} is adjacent to $v_{i_{j+1}}$ for all $1 \leqslant j \leqslant q-1$.*

Example 1.1. Let $\mathcal{G}$ be the graph on vertex set $\{v_1, \ldots, v_5\}$



$I_3(\mathcal{G}) = (X_1X_3X_5, X_2X_5X_3, X_4X_2X_5), I_4(\mathcal{G}) = (X_1X_3X_5X_2, X_3X_5X_2X_4).$

Definition 1.2. *A* path of length $q-1$ *in $\mathcal{G}$, or* $(q-1)$-path, *is an alternating sequence of vertices and edges $\{v_1, z_1, v_2, \ldots, v_{q-1}, z_{q-1}, v_q\}$, where $z_i = \{v_i, v_{i+1}\}$ is the edge joining v_i and v_{i+1}, and all the vertices are distinct.*

Remark 1.1. Two paths are equal if they consist of the same elements, independently of the order.

Remark 1.2. In general $I_q(\mathcal{G})$ is associated to the paths of length $q-1$ in $\mathcal{G}$. More precisely, the generators of $I_q(\mathcal{G})$ correspond to the $(q-1)$-paths in $\mathcal{G}$.

Remark 1.3. For $q = 2$, $I_2(\mathcal{G})$ is the generalized graph ideal generated by square-free monomials of degree 2 corresponding to the edges of $\mathcal{G}$. $I_2(\mathcal{G})$ is the so-called *edge ideal* of $\mathcal{G}$, and simply denoted by $I(\mathcal{G})$.

We are interested to consider generalized graph ideals associated to bipartite graphs.

Definition 1.3. *A graph $\mathcal{G}$ is said to be* bipartite *if its vertex set $[n+m]$ can be partitioned into two disjoint subsets $[n] = \{x_1, \ldots, x_n\}$ and $[m] = \{y_1, \ldots, y_m\}$ such that every edge of $\mathcal{G}$ joins $[n]$ with $[m]$.*

Definition 1.4. *A graph $\mathcal{G}$ is* complete bipartite *if it is bipartite and contains every edge that joins $[n]$ with $[m]$. Such a graph is denoted by $\mathcal{K}_{n,m}$.*

If $\mathcal{G}$ is a complete bipartite graph, the generalized graph ideal $I_q(\mathcal{G})$ is a well determined ideal of mixed products.

Let $S = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring over a field K in two sets of variables with $\deg(X_i) = \deg(Y_j) = 1$, for all $i = 1, \dots, n$, $j = 1, \dots, m$. Given the non negative integers k, r, s, t such that $k+r = s+t$, in [16] the authors define the square-free monomial ideals of S:

$$L = I_k J_r + I_s J_t,$$

where I_k (resp. J_r) is the monomial ideal of S generated by all the square-free monomials of degree k (resp. r) in the variables $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_m$).

These ideals are called *ideals of mixed products*. Setting $I_0 = J_0 = S$, the following cases occur:

1) $L = I_k + J_k$, with $1 \leqslant k \leqslant \inf\{n, m\}$
2) $L = I_k J_r$, with $1 \leqslant k \leqslant n$, $1 \leqslant r \leqslant m$
3) $L = I_k J_r + I_{k+1} J_{r-1}$, with $1 \leqslant k \leqslant n$, $2 \leqslant r \leqslant m$
4) $L = J_r + I_s J_t$, with $r = s + t$, $1 \leqslant s \leqslant n$, $1 \leqslant r \leqslant m$, $t \geqslant 1$
5) $L = I_k J_r + I_s J_t$, with $k + r = s + t$, $1 \leqslant k \leqslant n$, $1 \leqslant r \leqslant m$.

Example 1.2.

1) $S = K[X_1, X_2, X_3; Y_1, Y_2]$
   $L = I_2 J_1 = (X_1X_2Y_1, X_1X_3Y_1, X_2X_3Y_1, X_1X_2Y_2, X_1X_3Y_2, X_2X_3Y_2)$.
2) $S = K[X_1, X_2; Y_1, Y_2, Y_3]$
   $L = I_1 J_2 + I_2 J_1 = (X_1Y_1Y_2, X_1Y_1Y_3, X_1Y_2Y_3, X_2Y_1Y_2, X_2Y_1Y_3, X_2Y_2Y_3, X_1X_2Y_1, X_1X_2Y_2, X_1X_2Y_3)$.

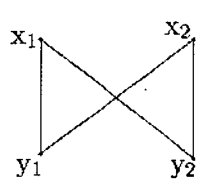
Let $\mathcal{G}$ be a complete bipartite graph with vertices $x_1, \dots, x_n; y_1, \dots, y_m$.

The generalized graph ideal $I_q(\mathcal{G})$ is the ideal of S generated by all the square-free monomials of degree q corresponding to the $(q-1)$-paths of $\mathcal{G}$.

More precisely, $I_q(\mathcal{G})$ is an ideal of mixed products of the form:

$$I_q(\mathcal{G}) = \begin{cases} I_\ell J_{\ell+1} + I_{\ell+1} J_\ell & \text{if } q = 2\ell + 1, \ell \in \mathbb{N}^* \\ I_\ell J_\ell & \text{if } q = 2\ell, \ell \in \mathbb{N}^* \end{cases}$$

Example 1.3. Let $\mathcal{G} = \mathcal{K}_{2,2}$, the complete bipartite graph on vertex set $\{x_1, x_2; y_1, y_2\}$



In $S = K[X_1, X_2; Y_1, Y_2]$ one has:

$$I_3(\mathcal{G}) = (X_1Y_1Y_2, X_2Y_1Y_2, X_1X_2Y_1, X_1X_2Y_2) = I_1J_2 + I_2J_1$$
$$I_4(\mathcal{G}) = (X_1Y_1X_2Y_2) = I_2J_2.$$

The generators of $I_3(\mathcal{G})$ correspond to the paths of length 2.
The generator of $I_4(\mathcal{G})$ corresponds to the path of length 3.

2. Linear resolutions and invariants

Throughout this section, $\mathcal{G}$ will be a complete bipartite graph $\mathcal{K}_{n,m}$.

Here we illustrate some algebraic aspects of the generalized graph ideal $L = I_q(\mathcal{G})$ generated in degree $q \geqslant 2$ which arises from the paths of $\mathcal{G}$.

We prove that this ideal admits linear quotients and has a linear resolution.

We also compute standard algebraic invariants for $I_q(\mathcal{G})$ such as dimension, projective dimension, depth, Castelnuovo-Mumford regularity; and finally, we establish suitable conditions for which $I_q(\mathcal{G})$ is a Cohen-Macaulay ideal.

Let $S = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$. For a monomial ideal $L \subset S$ we denote by $G(L)$ its unique set of minimal generators.

Definition 2.1. *A monomial ideal $L \subset S$ is said to have* linear quotients *if there is an ordering $u_1, \dots, u_t$ of monomials belonging to $G(L)$ such that the colon ideal $(u_1, \dots, u_{j-1}) : (u_j)$ is generated by a subset of $\{X_1, \dots, X_n; Y_1, \dots, Y_m\}$, for all $j = 2, \dots, t$.*

Remark 2.1. A monomial ideal L of S generated in one degree that has linear quotients admits a linear resolution ([3], Lemma 4.1).

For a monomial ideal L of S having linear quotients with respect to the ordering $u_1, \dots, u_t$ of the monomials of $G(L)$, let $\mathfrak{q}_j(L)$ denote the number of the variables which is required to generate the ideal $(u_1, \dots, u_{j-1}) : (u_j)$, and set $\mathfrak{q}(L) = \max_{2\leqslant j\leqslant t} \mathfrak{q}_j(L)$.

Remark 2.2. The integer $\mathfrak{q}(L)$ is independent of the choice of the ordering of the generators that gives linear quotients ([6]).

In order to study the property of the ideal L of S of having linear quotients, we premise the following

Definition 2.2 (cfr. [13]). *A monomial ideal L of S generated in one degree is called* bi-polymatroidal *if the following condition is satisfied: for all monomials $u = X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_m^{b_m}$ and $v = X_1^{c_1} \cdots X_n^{c_n} Y_1^{d_1} \cdots Y_m^{d_m}$ in $G(L)$ and for each i with $a_i > c_i$ or k with $b_k > d_k$ one has $j \in \{1, \dots, n\}$ with $a_j < c_j$ or $l \in \{1, \dots, m\}$ with $b_l < d_l$ such that $X_ju/X_i \in G(L)$ or $Y_lu/Y_k \in G(L)$.*

Proposition 2.1. *The ideals $I_q(\mathcal{G})$ are bi-polymatroidal ideals.*

Proof.

a) Let $I_q(\mathcal{G}) = I_\ell J_\ell$, $q = 2\ell$. The set of the minimal generators of $I_q(\mathcal{G})$ is given by all the paths in $\mathcal{G}$, namely $\{X_{i_1}\cdots X_{i_\ell}Y_{j_1}\cdots Y_{j_\ell} \mid 1 \leqslant i_1 < \cdots < i_\ell \leqslant n, 1 \leqslant j_1 < \cdots < j_\ell \leqslant m\}$.
Let $u = X_1^{a_1}\cdots X_n^{a_n}Y_1^{b_1}\cdots Y_m^{b_m}$, $v = X_1^{c_1}\cdots X_n^{c_n}Y_1^{d_1}\cdots Y_m^{d_m} \in G(I_q(\mathcal{G}))$ with $0 \leqslant a_i, b_j \leqslant 1$, then $X_1^{a_1}\cdots X_n^{a_n} \in I_\ell$ and $Y_1^{b_1}\cdots Y_m^{b_m} \in J_\ell$ such that $\sum_{i=1}^n a_i + \sum_{j=1}^m b_j = q$. Thus it easily follows by the structure of $G(I_q(\mathcal{G}))$ that for each i with $a_i > c_i$ or k with $b_k > d_k$ one has j with $a_j < c_j$ or l with $b_l < d_l$ such that $X_j u/X_i \in G(I_q(\mathcal{G}))$ or $Y_l u/Y_k \in G(I_q(\mathcal{G}))$.

b) Let $I_q(\mathcal{G}) = I_\ell J_{\ell+1} + I_{\ell+1}J_\ell$, $q = 2\ell+1$. The set of the minimal generators of $I_q(\mathcal{G})$ is given by $\{X_{i_1}\cdots X_{i_\ell}Y_{j_1}\cdots Y_{j_{\ell+1}}, X_{i_1}\cdots X_{i_{\ell+1}}Y_{j_1}\cdots Y_{j_\ell} \mid 1 \leqslant i_1 < \cdots < i_{\ell+1} \leqslant n, 1 \leqslant j_1 < \cdots < j_\ell \leqslant m\}$, corresponding to all the paths in $\mathcal{G}$.
Let $u = X_1^{a_1}\cdots X_n^{a_n}Y_1^{b_1}\cdots Y_m^{b_m}$, $v = X_1^{c_1}\cdots X_n^{c_n}Y_1^{d_1}\cdots Y_m^{d_m} \in G(I_q(\mathcal{G}))$ with $0 \leqslant a_i, b_j \leqslant 1$, then either $X_1^{a_1}\cdots X_n^{a_n} \in I_\ell$ and $Y_1^{b_1}\cdots Y_m^{b_m} \in J_{\ell+1}$ or $X_1^{a_1}\cdots X_n^{a_n} \in I_{\ell+1}$ and $Y_1^{b_1}\cdots Y_m^{b_m} \in J_\ell$ such that $\sum_{i=1}^n a_i + \sum_{j=1}^m b_j = q$. Thus it easily follows by the structure of $G(I_q(\mathcal{G}))$ that for each i with $a_i > c_i$ or k with $b_k > d_k$ one has j with $a_j < c_j$ or l with $b_l < d_l$ such that $X_j u/X_i \in G(I_q(\mathcal{G}))$ or $Y_l u/Y_k \in G(I_q(\mathcal{G}))$. □

Theorem 2.1. *The ideals $I_q(\mathcal{G})$ have linear quotients.*

Proof. Let $u \in G(I_q(\mathcal{G}))$. Set $N = (v \in G(I_q(\mathcal{G})) \mid v \prec u)$ with $\prec$ the lexicographical order on $X_1, \ldots, X_n; Y_1, \ldots, Y_m$ induced by $X_1 \succ X_2 \succ \cdots \succ X_n \succ Y_1 \succ Y_2 \succ \cdots \succ Y_m$. Then we prove that $N : u = (v/\mathrm{GCD}(u, v) \mid v \in N)$ is generated by monomials of degree one, that is a subset of $\{X_1, \ldots, X_n; Y_1, \ldots, Y_m\}$. Therefore we have to prove that for all $v \prec u$ there exists a variable of S in $N : u$ that divides $v/\mathrm{GCD}(u, v)$. Let $u = X_1^{a_1}\cdots X_n^{a_n}Y_1^{b_1}\cdots Y_m^{b_m}$ and $v = X_1^{c_1}\cdots X_n^{c_n}Y_1^{d_1}\cdots Y_m^{d_m}$ in $G(I_q(\mathcal{G}))$. Since $v \prec u$ there exists an integer i with $a_i > c_i$ and $a_k = c_k$ for $k = 1, \ldots, i-1$. Hence by definition of bi-polymatroidal ideal there exists an integer j with $c_j > a_j$ such that $w = X_j(u/X_i) \in G(I_q(\mathcal{G}))$. Since $i < j$, it follows that $w \in N$ and $w = X_j(u/X_i) \in G(I_q(\mathcal{G}))$ implies $wX_i = X_j u$, that is $X_j \in N : u$. Since the j-th component of the vector exponent of $v/\mathrm{GCD}(u, v)$ is given by $c_j - \min\{c_j, a_j\} = c_j - a_j > 0$, then X_j divides $v/\mathrm{GCD}(u, v)$ as required. If we suppose that $a_k = c_k$ for all $k = 1, \ldots, n$, $b_i > d_i$ and $b_l = d_l$ for all $l = 1, \ldots, i-1$, $i \in \{1, \ldots, m\}$ then we obtain $Y_j \in N : u$ and Y_j divides $v/\mathrm{GCD}(u, v)$. So the assertion follows. □

Corollary 2.1. *The ideals $I_q(\mathcal{G})$ have a linear resolution.*

Proof. The statement descends from Theorem 2.1 and Remark 2.1. □

We will now investigate standard algebraic invariants of $S/I_q(\mathcal{G})$. Recall the following

Definition 2.3. *A* vertex cover *of $I_q(\mathcal{G})$ is a subset W of $\{X_1, \ldots, X_n; Y_1, \ldots, Y_m\}$ such that each $u \in G(I_q(\mathcal{G}))$ is divided by some variables of W.*

Let $h(I_q(\mathcal{G}))$ denote the minimal cardinality of the vertex covers of $I_q(\mathcal{G})$.

Lemma 2.1. *Let $S = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ and $I_q(\mathcal{G}) \subset S$. Then:*

$$h(I_q(\mathcal{G})) = \min\{n, m\} - \ell + 1,$$

where $\ell = \frac{q}{2}$ if q is even and $\ell = \frac{q-1}{2}$ if q is odd.

Proof.

a) Let $I_q(\mathcal{G}) = I_\ell J_\ell$, $q = 2\ell$. The set of minimal generators of $I_q(\mathcal{G})$ derives from all the paths in $\mathcal{G}$, $\{X_{i_1} \cdots X_{i_\ell} Y_{j_1} \cdots Y_{j_\ell} \mid 1 \leqslant i_1 < \cdots < i_\ell \leqslant n, 1 \leqslant j_1 < \cdots < j_\ell \leqslant m\}$. Being I_ℓ (resp. J_ℓ) generated by all the monomials of degree ℓ in the variables $X_1, \ldots, X_n$ (resp. $Y_1, \ldots, Y_m$), by the structure of $I_q(\mathcal{G}) = I_\ell J_\ell$, one has:

   for $q = 2$, $h(I_q(\mathcal{G})) = \min\{n, m\}$
   for $q = 4$, $h(I_q(\mathcal{G})) = \min\{n, m\} - 1$
   for $q = 6$, $h(I_q(\mathcal{G})) = \min\{n, m\} - 2$
   ....
   for $q = 2\ell$, $h(I_q(\mathcal{G})) = \min\{n, m\} - \ell + 1$.

b) Let $I_q(\mathcal{G}) = I_\ell J_{\ell+1} + I_{\ell+1} J_\ell$, $q = 2\ell + 1$. The set of the minimal generators of $I_q(\mathcal{G})$ derives from all the paths in $\mathcal{G}$, $\{X_{i_1} \cdots X_{i_\ell} Y_{j_1} \cdots Y_{j_{\ell+1}}, X_{i_1} \cdots X_{i_{\ell+1}} Y_{j_1} \cdots Y_{j_\ell} \mid 1 \leqslant i_1 < \cdots < i_{\ell+1} \leqslant n, 1 \leqslant j_1 < \cdots < j_\ell \leqslant m\}$. Being I_ℓ (resp. $I_{\ell+1}$) generated by all the monomials of degree ℓ (resp. $\ell + 1$) in the variables $X_1, \ldots, X_n$ and J_ℓ (resp. $J_{\ell+1}$) generated by all the monomials of degree ℓ (resp. $\ell + 1$) in the variables $Y_1, \ldots, Y_m$, by the structure of $I_q(\mathcal{G}) = I_\ell J_{\ell+1} + I_{\ell+1} J_\ell$, one has:

   for $q = 3$, $h(I_q(\mathcal{G})) = \min\{n, m\}$
   for $q = 5$, $h(I_q(\mathcal{G})) = \min\{n, m\} - 1$
   for $q = 7$, $h(I_q(\mathcal{G})) = \min\{n, m\} - 2$
   ....
   for $q = 2\ell + 1$, $h(I_q(\mathcal{G})) = \min\{n, m\} - \ell + 1$.

In conclusion, for $q \leqslant \min\{n + m, 2n + 1, 2m + 1\}$,

$$h(I_q(\mathcal{G})) = \min\{n, m\} - \ell + 1,$$

with $\ell = \frac{q}{2}$ if q is even and $\ell = \frac{q-1}{2}$ if q is odd. □

Lemma 2.2. *Let $S = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ and $I_q(\mathcal{G}) \subset S$. Then:*

$$\mathrm{q}\,(I_q(\mathcal{G})) = n + m - q.$$

Proof.

a) Let $I_q(\mathcal{G}) = I_\ell J_\ell$, $q = 2\ell$.
For $q = 2$, $I_q(\mathcal{G}) = I_1 J_1 = (\{X_r Y_s \mid 1 \leqslant r \leqslant n, 1 \leqslant s \leqslant m\})$. The maximum number of the variables which is required to generate the linear quotients of the ideal $I_2(\mathcal{G})$ is given by the subset $\{X_{i_1}, \dots, X_{i_{n-1}}; Y_{j_1}, \dots, Y_{j_{m-1}}\} \subset S$. Hence $\mathrm{q}\,(I_2(\mathcal{G})) = n + m - 2$.
For $q = 4$, $I_q(\mathcal{G}) = I_2 J_2$ and the maximum number of the variables which is required to generate the linear quotients is given by the subset of variables $\{X_{i_1}, \dots, X_{i_{n-2}}; Y_{j_1}, \dots, Y_{j_{m-2}}\}$. Hence $\mathrm{q}\,(I_4(\mathcal{G})) = n + m - 4$.
For $q = 6$, $I_q(\mathcal{G}) = I_3 J_3$ and the maximum number of the variables which is required to generate the linear quotients is given by the subset of variables $\{X_{i_1}, \dots, X_{i_{n-3}}; Y_{j_1}, \dots, Y_{j_{m-3}}\}$. Hence $\mathrm{q}\,(I_6(\mathcal{G})) = n + m - 6$.
Thus, when $I_q(\mathcal{G}) = I_\ell J_\ell$, the maximum number of the variables which is required to generate the linear quotients is given by the subset $\{X_{i_1}, \dots, X_{i_{n-\ell}}; Y_{j_1}, \dots, Y_{j_{m-\ell}}\} \subset S$. Hence $\mathrm{q}\,(I_{2\ell}(\mathcal{G})) = n + m - 2\ell$.

b) Let $I_q(\mathcal{G}) = I_\ell J_{\ell+1} + I_{\ell+1} J_\ell$, $q = 2\ell + 1$.
For $q = 3$, $I_q(\mathcal{G}) = I_1 J_2 + I_2 J_1 = (\{X_r Y_s Y_\sigma, X_r X_\rho Y_s \mid 1 \leqslant r < \rho \leqslant n, 1 \leqslant s < \sigma \leqslant m\})$. The maximum number of the variables which is required to generate the linear quotients of $I_3(\mathcal{G})$ is given by the subset of variables $\{X_{i_1}, \dots, X_{i_{n-1}}; Y_{j_1}, \dots, Y_{j_{m-2}}\} \subset S$ or by $\{X_{i_1}, \dots, X_{i_{n-2}}; Y_{j_1}, \dots, Y_{j_{m-1}}\} \subset S$. In any case it follows that $\mathrm{q}\,(I_3(\mathcal{G})) = n + m - 3$.
For $q = 5$, $I_q(\mathcal{G}) = I_2 J_3 + I_3 J_2$ and the maximum number of the variables which is required to generate the linear quotients is given by the subset $\{X_{i_1}, \dots, X_{i_{n-2}}; Y_{j_1}, \dots, Y_{j_{m-3}}\}$ or by $\{X_{i_1}, \dots, X_{i_{n-3}}; Y_{j_1}, \dots, Y_{j_{m-2}}\}$. In any case it follows that $\mathrm{q}\,(I_5(\mathcal{G})) = n + m - 5$.
For $q = 7$, $I_q(\mathcal{G}) = I_3 J_4 + I_4 J_3$ and the maximum number of the variables which is required to generate the linear quotients is given by the subset of variables $\{X_{i_1}, \dots, X_{i_{n-3}}; Y_{j_1}, \dots, Y_{j_{m-4}}\}$ or by $\{X_{i_1}, \dots, X_{i_{n-4}}; Y_{j_1}, \dots, Y_{j_{m-3}}\}$. In any case it follows that $\mathrm{q}\,(I_7(\mathcal{G})) = n + m - 7$.
Thus, when $I_q(\mathcal{G}) = I_\ell J_{\ell+1} + I_{\ell+1} J_\ell$, the maximum number of the variables which is required to generate the linear quotients is given by the subset of variables $\{X_{i_1}, \dots, X_{i_{n-\ell}}; Y_{j_1}, \dots, Y_{j_{m-\ell-1}}\} \subset S$ or by $\{X_{i_1}, \dots, X_{i_{n-\ell-1}}; Y_{j_1}, \dots, Y_{j_{m-\ell}}\} \subset S$. In any case it follows that $\mathrm{q}\,(I_{2\ell+1}(\mathcal{G})) = n + m - 2\ell - 1$.
In conclusion, $\mathrm{q}\,(I_q(\mathcal{G})) = n+m-q$, for $q \leqslant \min\{n+m, 2n+1, 2m+1\}$.

□

Theorem 2.2. *Let $S = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ and $I_q(\mathcal{G}) \subset R$. Then:*

1) $\dim_S(S/I_q(\mathcal{G})) = n + m - \min\{n, m\} + \ell - 1$, *where $\ell = \frac{q}{2}$ if q is even and $\ell = \frac{q-1}{2}$ if q is odd.*
2) $\mathrm{pd}_S(S/I_q(\mathcal{G})) = n + m - q + 1$.
3) $\mathrm{depth}_S(S/I_q(\mathcal{G})) = q - 1$.
4) $\mathrm{reg}_S(S/I_q(\mathcal{G})) = 1$.

Proof.

1) One has $\dim_S(S/I_q(\mathcal{G})) = \dim_S S - h(I_q(\mathcal{G}))$ (see [4]). Hence, by Lemma 2.1, $\dim_S(S/I_q(\mathcal{G})) = n + m - \min\{n, m\} + \ell - 1$, where $\ell = \frac{q}{2}$ if q is even and $\ell = \frac{q-1}{2}$ if q is odd.
2) The length of the minimal free resolution of $S/I_q(\mathcal{G})$ over S is equal to $\mathrm{q}(I_q(\mathcal{G})) + 1$ ([6], Corollary 1.6). Then $\mathrm{pd}_S(S/I_q(\mathcal{G})) = n + m - q + 1$.
3) As a consequence of 2), by Auslander-Buchsbaum formula, one has $\mathrm{depth}_S(S/I_q(\mathcal{G})) = n + m - \mathrm{pd}_S(S/I_q(\mathcal{G})) = n + m - (n + m - q + 1) = q - 1$.
4) $I_q(\mathcal{G})$ has a linear resolution, then $\mathrm{reg}_S(S/I_q(\mathcal{G})) = 1$. □

Remark 2.3. The computation of the algebraic invariants for mixed product ideals was made in [11] using different techniques with respect to the above theorem.

The following results explain conditions for which $I_q(\mathcal{G})$ is a Cohen-Macaulay ideal.

Proposition 2.2. *Let $I_q(\mathcal{G}) = I_\ell J_\ell \subset S = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$, $q = 2\ell$. $I_q(\mathcal{G})$ is Cohen Macaulay if and only if $\ell = n + m - \min\{n, m\}$.*

Proof. By Theorem 2.2 one has $\dim_S(S/I_q(\mathcal{G})) = n + m - \min\{n, m\} + \ell - 1$ and $\mathrm{depth}_S(S/I_q(\mathcal{G})) = q - 1$. $I_q(\mathcal{G})$ is Cohen Macaulay if and only if $\dim_S(S/I_q(\mathcal{G})) = \mathrm{depth}_S(S/I_q(\mathcal{G}))$. Hence the equality holds if and only if $n + m - \min\{n, m\} + \ell - 1 = q - 1$, $q = 2\ell \Leftrightarrow 2\ell = n + m - \min\{n, m\} + \ell \Leftrightarrow \ell = n + m - \min\{n, m\}$. □

Proposition 2.3. *Let $I_q(\mathcal{G}) = I_\ell J_{\ell+1} + I_{\ell+1} J_\ell \subset S = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$, $q = 2\ell + 1$. $I_q(\mathcal{G})$ is Cohen Macaulay if and only if $\ell = \ell = n + m - \min\{n, m\} - 1$.*

Proof. By Theorem 2.2 one has $\dim_S(S/I_q(\mathcal{G})) = n + m - \min\{n, m\} + \ell - 1$ and $\mathrm{depth}_S(S/I_q(\mathcal{G})) = q - 1$. $I_q(\mathcal{G})$ is Cohen Macaulay if and only if $\dim_S(S/I_q(\mathcal{G})) = \mathrm{depth}_S(S/I_q(\mathcal{G}))$. Hence the equality holds if and only if $n + m - \min\{n, m\} + \ell - 1 = q - 1, q = 2\ell + 1 \Leftrightarrow 2\ell = n + m - \min\{n, m\} + \ell - 1 \Leftrightarrow \ell = n + m - \min\{n, m\} - 1$. □

3. Ideals of vertex covers for the generalized graph ideals of a complete bipartite graph

Definition 3.1. *Let $\mathcal{G}$ be a graph on vertex set $[n] = \{v_1, \ldots, v_n\}$. A subset C of $[n]$ is said a* generalized vertex cover *of $\mathcal{G}$ if every path of $\mathcal{G}$ is incident with one vertex in C. C is said* minimal *if no proper subset of C is a generalized vertex cover of $\mathcal{G}$.*

Remark 3.1. There exists a one to one correspondence between generalized vertex covers of $\mathcal{G}$ and prime ideals of $I_q(\mathcal{G})$ that preserves the minimality. In fact, $\wp$ is a minimal prime ideal of $I_q(\mathcal{G})$ if and only if $\wp = (C)$, for some minimal generalized vertex cover C of $\mathcal{G}$. Thus $I_q(\mathcal{G})$ has primary decomposition $(C_1) \cap \cdots \cap (C_r)$, where $C_1, \ldots, C_r$ are the minimal generalized vertex covers of $\mathcal{G}$.

An algebraic aspect linked to the generalized vertex covers of $\mathcal{G}$ is the notion of ideal of vertex covers for the generalized graph ideals of $\mathcal{G}$.

Definition 3.2. *The* ideal of vertex covers *for the generalized graph ideal $I_q(\mathcal{G})$, denoted by $(I_q)_c(\mathcal{G})$, is the ideal of R generated by all monomials $X_{i_1} \cdots X_{i_r}$ such that $(X_{i_1}, \ldots, X_{i_r})$ is an associated prime ideal of $I_q(\mathcal{G})$.*

Hence $(I_q)_c(\mathcal{G}) = (\{X_{i_1} \cdots X_{i_r} \mid \{v_{i_1}, \ldots, v_{i_r}\}$ is a generalized vertex cover of $\mathcal{G}\})$ and the minimal generators of $(I_q)_c(\mathcal{G})$ correspond to the minimal generalized vertex covers.

The following generalizes to $(I_q)_c(\mathcal{G})$ the characterization of the ideal of vertex covers for the edge ideal of any graph $\mathcal{G}$ given in [17].

Property 3.1. $(I_q)_c(\mathcal{G}) = \left(\bigcap_{\{v_{i_1}, z_{i_1}, \ldots, v_{i_q}\} \text{ path in } \mathcal{G}} (X_{i_1}, \ldots, X_{i_q})\right), \ \forall q \geqslant 2.$

From now on, let $S = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$, $n \geqslant m$, be the polynomial ring in two sets of variables over a field K and $\mathcal{K}_{n,m}$ be a complete bipartite graph on vertex set $[n+m] = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$.

Let $I_q(\mathcal{K}_{n,m})$ denote the generalized graph ideals of $\mathcal{K}_{n,m}$, where $2 \leqslant q \leqslant \min\{n+m, 2m+1\}$. It is known that the generators of $I_q(\mathcal{K}_{n,m})$ correspond to the $(q-1)$-paths in $\mathcal{K}_{n,m}$.

In [7] the author illustrates a method for determining in degree q the number of $(q-1)$-paths in any connected graph $\mathcal{G}$, by using only the incidence matrix of $\mathcal{G}$. The composition of such paths and the generators of the generalized graph ideals $I_q(\mathcal{G})$ are also studied.

Let $(I_q)_c(\mathcal{K}_{n,m})$ indicate the ideal of vertex covers of $I_q(\mathcal{K}_{n,m})$, $\forall q$.

The following result establishes the structure of $(I_q)_c(\mathcal{K}_{n,m})$.

Theorem 3.1. *Let $(I_q)_c(\mathcal{K}_{n,m})$ be the ideal of vertex covers for the generalized graph ideal associated to the complete bipartite graph $\mathcal{K}_{n,m}$, $n \geqslant m, 2 \leqslant q \leqslant \min\{n+m, 2m+1\}$. It is structured as follows:*

– *if* $q = 2p$, $(I_q)_c(\mathcal{K}_{n,m})$ *has* $\binom{n}{p-1} + \binom{m}{p-1}$ *generators,*

$$X_1 \cdots X_{n-p+1}, \ldots, X_p \cdots X_n, Y_1 \cdots Y_{m-p+1}, \ldots, Y_p \cdots Y_m;$$

– *if* $q = 2p+1$, $p \neq m$, $(I_q)_c(\mathcal{K}_{n,m})$ *has* $\binom{n}{p-1} + \binom{m}{p-1} + \binom{n}{p}\binom{m}{p}$ *generators,*

$$\begin{gathered} X_1 \cdots X_{n-p+1}, \ldots, X_p \cdots X_n, Y_1 \cdots Y_{m-p+1}, \ldots, Y_p \cdots Y_m, \\ X_1 \cdots X_{n-p} Y_1 \cdots Y_{m-p}, \ldots, X_1 \cdots X_{n-p} Y_{p+1} \cdots Y_m, \ldots\ldots, \\ X_{p+1} \cdots X_n Y_1 \cdots Y_{m-p}, \ldots, X_{p+1} \cdots X_n Y_{p+1} \cdots Y_m; \end{gathered}$$

– *if* $q = 2m+1$, $(I_q)_c(\mathcal{K}_{n,m})$ *has* $\binom{m}{m-1} + \binom{n}{m}$ *generators,*

$$X_1 \cdots X_{n-m}, \ldots, X_{m+1} \cdots X_n, Y_1, \ldots, Y_m.$$

Proof. Let's calculate for any $q \geqslant 2$ the generators and their number for the ideals of vertex covers $(I_q)_c(\mathcal{K}_{n,m})$.

– $(I_2)_c(\mathcal{K}_{n,m})$

Its generators are $X_1 \cdots X_n, Y_1 \cdots Y_m$; their number is $\binom{n}{0} + \binom{m}{0} = 2$.

– $(I_3)_c(\mathcal{K}_{n,m})$

Its generators are $X_1 \cdots X_n, Y_1 \cdots Y_m$,

$$\begin{aligned} &X_1 \cdots X_{n-1} Y_1 \cdots Y_{m-1}, X_1 \cdots X_{n-1} Y_1 \cdots Y_{m-2} Y_m, \ldots, X_1 \cdots X_{n-1} Y_2 \cdots Y_m, \\ &X_1 \cdots X_{n-2} X_n Y_1 \cdots Y_{m-1}, X_1 \cdots X_{n-2} X_n Y_1 \cdots Y_{m-2} Y_m, \ldots, \\ &X_1 \cdots X_{n-2} X_n Y_2 \cdots Y_m, \ldots\ldots\ldots\ldots\ldots\ldots, \\ &X_2 \cdots X_n Y_1 \cdots Y_{m-1}, X_2 \cdots X_n Y_1 \cdots Y_{m-2} Y_m, \ldots, X_2 \cdots X_n Y_2 \cdots Y_m; \end{aligned}$$

their number is $\binom{n}{0} + \binom{m}{0} + \binom{n}{1}\binom{m}{1} = 2 + nm$.

– $(I_4)_c(\mathcal{K}_{n,m})$

Its generators are:

$$\begin{gathered} X_1 \cdots X_{n-1}, X_1 \cdots X_{n-2} X_n, X_1 \cdots X_{n-3} X_{n-1} X_n, \ldots, X_2 \cdots X_n, \\ Y_1 \cdots Y_{m-1}, Y_1 \cdots Y_{m-2} Y_m, Y_1 \cdots Y_{m-3} Y_{m-1} Y_m, \ldots, Y_2 \cdots Y_m; \end{gathered}$$

their number is $\binom{n}{1} + \binom{m}{1} = n + m$.

– $(I_5)_c(\mathcal{K}_{n,m})$

Its generators are:

$$X_1\cdots X_{n-1},\ X_1\cdots X_{n-2}X_n,\ \dots,\ X_1X_3\cdots X_n,\ X_2\cdots X_n,$$
$$Y_1\cdots Y_{m-1},\ Y_1\cdots Y_{m-2}Y_m,\ \dots,\ Y_1Y_3\cdots Y_m,\ Y_2\cdots Y_m,$$
$$X_1\cdots X_{n-2}Y_1\cdots Y_{m-2},\ X_1\cdots X_{n-2}Y_1\cdots Y_{m-3}Y_{m-1},\ \dots,\ X_1\cdots X_{n-2}Y_3\cdots Y_m,$$
$$X_1\cdots X_{n-3}X_{n-1}Y_1\cdots Y_{m-2},\ X_1\cdots X_{n-3}X_{n-1}Y_1\cdots Y_{m-3}Y_{m-1},\ \dots,$$
$$X_1\cdots X_{n-3}X_{n-1}Y_3\cdots Y_m,\ \dots\dots\dots\dots,$$
$$X_3\cdots X_nY_1\cdots Y_{m-2},\ X_3\cdots X_nY_1\cdots Y_{m-3}Y_{m-1},\ \dots,\ X_3\cdots X_nY_3\cdots Y_m;$$

their number is $\binom{n}{1}+\binom{m}{1}+\binom{n}{2}\binom{m}{2}=n+m+\frac{n(n-1)}{2}\frac{m(m-1)}{2}$.

– $(I_6)_c(\mathcal{K}_{n,m})$

Its generators are:

$$X_1\cdots X_{n-2},\ X_1\cdots X_{n-3}X_{n-1},\ X_1\cdots X_{n-3}X_n,\ \dots,\ X_3\cdots X_n,$$
$$Y_1\cdots Y_{m-2},\ Y_1\cdots Y_{m-3}Y_{m-1},\ Y_1\cdots Y_{m-3}Y_m,\ \dots,\ Y_3\cdots Y_m;$$

their number is $\binom{n}{2}+\binom{m}{2}=\frac{n(n-1)}{2}\frac{m(m-1)}{2}$.

– $(I_7)_c(\mathcal{K}_{n,m})$

Its generators are:

$$X_1\cdots X_{n-2},\ X_1\cdots X_{n-3}X_{n-1},\ \dots,\ X_2X_4\cdots X_n,\ X_3\cdots X_n,$$
$$Y_1\cdots Y_{m-2},\ Y_1\cdots Y_{m-3}Y_{m-1},\ \dots,\ Y_2Y_4\cdots Y_m,\ Y_3\cdots Y_m,$$
$$X_1\cdots X_{n-3}Y_1\cdots Y_{m-3},\ X_1\cdots X_{n-3}Y_1\cdots Y_{m-4}Y_{m-2},\ \dots,\ X_1\cdots X_{n-3}Y_4\cdots Y_m,$$
$$X_1\cdots X_{n-4}X_{n-2}Y_1\cdots Y_{m-3},\ X_1\cdots X_{n-4}X_{n-2}Y_1\cdots Y_{m-4}Y_{m-2},\ \dots,$$
$$X_1\cdots X_{n-4}X_{n-2}Y_4\cdots Y_m,\ \dots\dots\dots\dots,$$
$$X_4\cdots X_nY_1\cdots Y_{m-3},\ X_4\cdots X_nY_1\cdots Y_{m-4}Y_{m-2},\ \dots,\ X_4\cdots X_nY_4\cdots Y_m;$$

their number is $\binom{n}{2}+\binom{m}{2}+\binom{n}{3}\binom{m}{3}=\frac{n(n-1)}{2}+\frac{m(m-1)}{2}+\frac{n(n-1)(n-2)}{6}\frac{m(m-1)(m-2)}{6}$; and so on, until $q=\min\{n+m,2m+1\}$. It results:

- $\min\{n+m,2m+1\}=n+m$ if and only if $n=m$, hence $q=2m$. $(I_{2m})_c(\mathcal{K}_{n,m})=(X_1,\dots,X_m,Y_1,\dots,Y_m)$; it has $2\binom{m}{m-1}$ generators.
- $\min\{n+m,2m+1\}=2m+1$ if and only if $n-m\geqslant 1$ (if $n=m+1$ then $2m+1=n+m$); hence $q=2m+1$. $(I_{2m+1})_c(\mathcal{K}_{n,m})=(X_1\cdots X_{n-m},\dots,X_{m+1}\cdots X_{n-1}X_n,Y_1,\dots,Y_m)$; it has $\binom{m}{m-1}+\binom{n}{m}$ generators. □

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