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Vanishing of Witten L-functions

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Abstract. We study the vanishing of Witten L-functions for symmetric groups S_n and $SU(2)$. In this paper we discuss the order of the Witten L-function for S_n at $s = -2$. In addition, we investigate the relation between the products of conjugacy classes in $SU(s)$ and the special values of the Witten L-function for $SU(2)$.

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1. Introduction

Witten [12] discovered the Witten zeta function, and Kurokawa-Ochiai [9] introduced the Witten L-function as a generalization of the Witten zeta function. The constructions of the Witten zeta function and the Witten L-function are as follows. For a compact topological group G, the Witten zeta function is

$$\zeta_G^W(s) = \sum_{\rho \in \widehat{G}} (\deg \rho)^{-s}, \tag{1}$$

where $\widehat{G}$ is the unitary dual of G. For example,

$$\widehat{SU(2)} = \{\mathrm{Sym}^m \mid m = 0, 1, 2, \ldots\},$$

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where $\mathrm{Sym}^m : SU(2) \longrightarrow SU(m+1)$ is the symmetric tensor product representation. Hence,

$$\begin{aligned}\zeta^W_{SU(2)}(s) &= \sum_{m=0}^{\infty} \left(\deg(\mathrm{Sym}^m)\right)^{-s} \\ &= \sum_{m=0}^{\infty} (m+1)^{-s} \\ &= \sum_{n=1}^{\infty} n^{-s}\end{aligned}$$

is nothing but the Riemann zeta function $\zeta(s)$. Especially,

$$\zeta^W_{SU(2)}(s) = 0$$

for $s = -2, -4, -6, \ldots$ as shown by Euler.

Now the Witten L-function is constructed for $g \in G$ by

$$\zeta^W_G(s;\, g) = \sum_{\rho \in \widehat{G}} \frac{\chi(g)}{\deg \rho} (\deg \rho)^{-s}. \tag{2}$$

We notice that $\zeta^W_G(s;\, g)$ depends only on the conjugacy class $C = [g]$ of g. So, we use the notation $\zeta^W_G(s;\, C)$ for $C \in \mathrm{Conj}(G)$ also, where $\mathrm{Conj}(G)$ denotes the set of conjugacy classes of G.

Kurokawa-Ochiai [9] conjectured $\zeta^W_G(-2;\, g) = 0$ for each infinite group G. The typical example is Euler's result

$$\zeta^W_{SU(2)}(-2;\, I_2) = \zeta^W_{SU(2)}(-2) = \zeta(-2) = 0$$

as noticed above.

For a finite group G the orthogonality of characters implies

$$\zeta^W_G(-2;\, g) = \begin{cases} |G| & \text{if } g = e, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We proved $\zeta^W_{SU(3)}(-2;\, g) = 0$ in our previous paper [10]. González-Sánchez, Jaikin-Zapirain and Klopsch [5] proved $\zeta^W_G(-2) = 0$ when G is a FAb compact p-adic Lie group. Moreover, the actual order of zeros of Witten L-functions at $s = -2$ is known for certain cases. For example, $\zeta^W_{SU(2)}(s;\, g)$ has a simple zero at $s = -2$ for all $g \in SU(2)$ (Kurokawa-Ochiai [9]). In addition, $\zeta^W_{SU(3)}(s) = \zeta^W_{SU(3)}(s;\, I_3)$ has a zero of order 2 at $s = -2$ (Onodera [11]).

However, the order of zeros at $s = -2$ is not known for finite groups in general. In this paper, we discuss the order of the Witten L-function for finite group at $s = -2$. First of all, the order is not bounded:

Theorem 1. *Let $G_1, \ldots, G_n$ be finite groups. Then the Witten L-function for $G_1 \times \cdots \times G_n$ satisfies the following formula:*

$$\zeta^W_{G_1\times\cdots\times G_n}(s;\ (g_1,\ldots,g_n)) = \zeta^W_{G_1}(s;\ g_1)\cdots\zeta^W_{G_n}(s;\ g_n). \tag{4}$$

In particular, for a finite group G and $g \in G\backslash\{e\}$ the order of zeros of $\zeta^W_{G^n}(s;\ g)$ at $s = -2$ is not smaller than n, where G^n denotes the direct product $\underbrace{G \times \cdots \times G}_{n}$.

Example 1. If $G = S_3$, the Witten L-functions are calculated as follows: First, we give the character table of S_3.

Table 1. The character table of S_3

	(1)	(12)	(123)
Trivial	1	1	1
Sign	1	−1	1
Standard	2	0	−1

Then, we obtain the Witten L-functions of S_3;

$$\zeta^W_{S_3}(s;\ (123)) = 2 - 2^{-s-1},$$
$$\frac{d}{ds}\zeta^W_{S_3}(s;\ (123)) = 2^{-s-1}\log 2,$$
$$\zeta^W_{S_3}(s;\ (12)) = 0.$$

So, we see that the Witten L-function for S_3 has a simple zero at $s = -2$ if $g \neq (1)$ is even. In addition, we obtain the following Witten L-function and its differentiations for $S_3 \times S_3$:

$$\zeta^W_{S_3\times S_3}(s;\ ((123),\ (123))) = 4 - 2^{-s+1} + 2^{-2s-2} = (2 - 2^{-s-1})^2,$$
$$\frac{d}{ds}\zeta^W_{S_3\times S_3}(s;\ ((123),\ (123))) = 2^{-s+1}\log 2 - 2^{-2s-1}\log 2,$$
$$\frac{d^2}{ds^2}\zeta^W_{S_3\times S_3}(s;\ ((123),\ (123))) = -2^{-s+1}(\log 2)^2 + 2^{-2s}(\log 2)^2.$$

Hence, we see that the Witten L-function $\zeta^W_{S_3\times S_3}(s;\ ((123),\ (123)))$ has a zero of order 2 at $s = -2$.

Theorem 2. *If $g \in S_n$ is odd, the Witten L-function $\zeta^W_{S_n}(s;\ g)$ is constantly zero.*

Theorem 3. *If $g \in S_n$ is the n-cycle and $n = 2m+1$, $m \geq 1$, then $\zeta^W_{S_n}(s;\ g)$ has a simple zero at $s = -2$.*

By Theorems 2 and 3, we make the following conjecture.

Conjecture. The Witten L-function $\zeta_{S_n}^W(s;\ g)$ has a simple zero at $s = -2$ if and only if g is even.

Next we discuss the case $G = SU(2)$. In general, we introduce generalized Witten L-function

$$\zeta_G^W(s; C_1, \ldots, C_n) = \sum_{\rho \in \widehat{G}} \frac{\chi(C_1)}{\deg \rho} \cdots \frac{\chi(C_n)}{\deg \rho} (\deg \rho)^{-s},$$

where $C_1, \ldots, C_n \in \mathrm{Conj}(G)$, $\widehat{G}$ is the unitary dual of G and $\chi(C) = \mathrm{trace}(\rho(g))$ for $g \in C$. In our previous paper [10], we proved the following result.

Theorem A (Min [10]). *Let $C(\lambda)$ be the conjugacy class of the matrix*

$$\begin{pmatrix} e^{i\lambda} & 0 \\ 0 & e^{-i\lambda} \end{pmatrix} \in SU(2)$$

for $0 \leq \lambda \leq \pi$. Then we have

$$\zeta_{SU(2)}^W(-2;\ C(\lambda_1),\ C(\lambda_2),\ C(\lambda_3))$$
$$= \begin{cases} \frac{\pi}{4 \sin\lambda_1 \sin\lambda_2 \sin\lambda_3}, & \text{if } S_3^0(\{\lambda_i\}) < 2\pi \text{ and } S_3^2(\{\lambda_i\}) < 0, \\ \frac{\pi}{8 \sin\lambda_1 \sin\lambda_2 \sin\lambda_3}, & \text{if } S_3^0(\{\lambda_i\}) = 2\pi,\ S_3^2(\{\lambda_i\}) = 0 \\ & \qquad \text{with } 0 < \lambda_1,\ \lambda_2,\ \lambda_3 < \pi, \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where $S_n^m(\{\lambda_i\})$ is any sum of the type $\sum_{i=1}^n \pm\lambda_i$ which contains m minus signs.

We notice that

$$\zeta_{SU(2)}^W(-2;\ C(\lambda_1)) = 0$$

and

$$\zeta_{SU(2)}^W(-2;\ C(\lambda_1),\ C(\lambda_2)) = 0$$

by Kurokawa-Ochiai [9]. On the other hand, Jeffrey and Mare [8] proved the following result:

Theorem B (Jeffrey-Mare [8]). *Let $C(\lambda)$ be the conjugacy class of the matrix*

$$\begin{pmatrix} e^{i\lambda} & 0 \\ 0 & e^{-i\lambda} \end{pmatrix} \in SU(2)$$

for $0 \leq \lambda \leq \pi$.

Then, for each integer $n \geq 2$ and $0 \leq \lambda_1, \ldots, \lambda_n \leq \pi$, it holds that

$$C(\lambda_1) \cdots C(\lambda_n) \ni I \tag{6}$$

if and only if the following system of inequalities are satisfied:

i) For odd n:

$$S_n^0(\{\lambda_i\}) \le (n-1)\pi, \quad S_n^2(\{\lambda_i\}) \le (n-3)\pi, \ldots, S_n^{n-1}(\{\lambda_i\}) \le 0. \quad (7)$$

ii) For even n:

$$S_n^1(\{\lambda_i\}) \le (n-2)\pi, \quad S_n^3(\{\lambda_i\}) \le (n-4)\pi, \ldots, S_n^{n-1}(\{\lambda_i\}) \le 0. \quad (8)$$

We notice that the parity of m in $S_n^m(\{\lambda_i\})$ of (7) and (8) is determined to satisfy the following condition: Gromov-Witten invariant is not zero. For details, we refer to [1].

From Theorems A and B we get

$$\begin{aligned} &\{(\lambda_1, \lambda_2, \lambda_3) \mid C(\lambda_1)C(\lambda_2)C(\lambda_3) \not\ni I\} \\ &\quad \subset \{(\lambda_1, \lambda_2, \lambda_3) \mid \zeta^W_{SU(2)}(-2;\, C(\lambda_1),\, C(\lambda_2),\, C(\lambda_3)) = 0\}. \end{aligned} \quad (9)$$

We prove the following partial generalizations.

Theorem 4. *Let $n \ge 3$ be an odd integer. If*

$$C(\lambda_1)\cdots C(\lambda_n) \not\ni I,$$

it holds that

$$\zeta^W_{SU(2)}(-(n-1);\, C(\lambda_1), \ldots, C(\lambda_n)) = 0.$$

Theorem 5. *Let $n \ge 4$ be an even integer. If*

$$C(\lambda_1)\cdots C(\lambda_n) \not\ni I,$$

it holds that

$$\zeta^W_{SU(2)}(-(n-2);\, C(\lambda_1), \ldots, C(\lambda_n)) = 0.$$

From Theorems 4 and 5 we observe the following points:

(1) When n is an odd integer:
if $\zeta^W_{SU(2)}(-(n-1);\, C(\lambda_1), \ldots, C(\lambda_n)) \neq 0$, then it holds

$$C(\lambda_1)\cdots C(\lambda_n) \ni I.$$

(2) When n is an even integer:
if $\zeta^W_{SU(2)}(-(n-2);\, C(\lambda_1), \ldots, C(\lambda_n)) \neq 0$, it holds

$$C(\lambda_1)\cdots C(\lambda_n) \ni I.$$

We remark that the converse is not valid. For example, when $n = 5$ and $\lambda_1 = \lambda_2 = \lambda_3 = \frac{\pi}{4}$, $\lambda_4 = \lambda_5 = \frac{\pi}{12}$, the λ's satisfy the condition (7), but

$$\zeta^W_{SU(2)}(-4; C(\lambda_1), C(\lambda_2), C(\lambda_3), C(\lambda_4), C(\lambda_5)) = 0.$$

In addition, when $n = 4$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{\pi}{4}$, the λ's satisfy the condition (8), but

$$\zeta^W_{SU(2)}(-2; C(\lambda_1), C(\lambda_2), C(\lambda_3), C(\lambda_4)) = 0.$$

The needed calculations are supplied in the proof in Sections 5 and 6.

2. Proof of Theorem 1

First, if $g \neq 1$, we see that Witten L-function for a finite group G has a zero at $s = -2$ because of the orthogonality of the characters. That is to say,

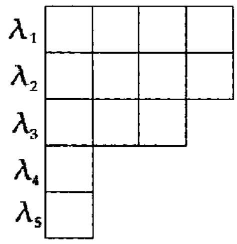
$$\begin{aligned}\zeta^W_G(-2;\ g) &= \sum_{\rho\in\widehat{G}} \frac{\chi(g)}{\deg\rho}(\deg\rho)^2 \\ &= \sum_{\rho\in\widehat{G}} \chi(g)\deg\rho \\ &= \sum \chi(g)\chi(1) \\ &= \sum \chi(g)\overline{\chi(1)} = 0. \end{aligned} \tag{10}$$

In addition, all the irreducible representations of $G_1 \times \cdots \times G_n$ arise as tensor products of irreducible representations of $G_1, \ldots, G_n$. Thus, the equation (4) holds. Hence, the order of $\zeta^W_{G_1\times\cdots\times G_n}(s;\ (g_1, \ldots, g_n))$ at $s = -2$ is equal to the sum of the order of $\zeta^W_{G_1}(s;\ g_1), \ldots, \zeta^W_{G_n}(s;\ g_n)$ at $s = -2$ for $g_1 \in G_1, \ldots, g_n \in G_n$. □

3. Irreducible representations of symmetric group S_n

Each conjugacy class of S_n is determined by its cycle type, a list of the lengths of the cycles. The identity has a cycle type (1^n) and a transposition has a cycle type $(2,\ 1^{n-2})$. Every irreducible representation of S_n is determined by its cycle type.

Set a cycle type $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_1 \geq \cdots \geq \lambda_k$, $\lambda_1 + \cdots + \lambda_k = n$. This kind of tuple is called a partition. For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, there is an associated Young diagram



with λ_i boxes in the i-th row, the rows of boxed lined up on the left.

The conjugate partition $\lambda' = (\lambda'_1, \ldots, \lambda'_l)$ of the partition λ is defined by reflecting the diagram in the 45° line. (See Figure 1)

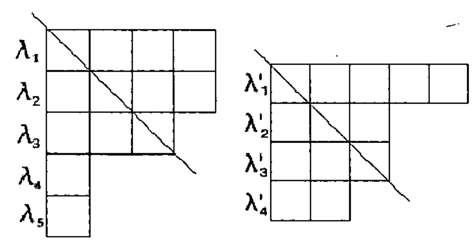


Figure 1. The conjugate partition

We denote by V_λ the representation corresponding to a cycle type λ. Then, the dimension $D_\lambda := \dim V_\lambda$ and the character χ_λ are determined as follows.

First, we recall the hook length of the boxes in Young diagram. We call the box in the i-th row and j-th column of λ ij-box. It is called the corner of the ij hook that consists of this box and all nodes to the right of it or below it (See Figure 2). The hook length h_{ij} is

$$h_{ij} = 1 + (\lambda_i - j) + (\lambda'_j - i).$$

Also, we denote by H_λ the product of all hook lengths of the partition λ.

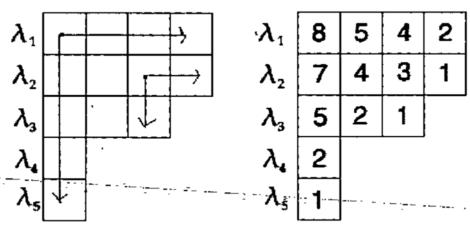


Figure 2. The hook lengths

Fact 1 (Hook length formula [4]). *The dimension of V_λ is given by*

$$D_\lambda = \frac{n!}{H_\lambda}.$$

Before we discuss the character associated to λ, we introduce some notations. We denote by $C_{\mathbf{i}}$ the conjugacy class in S_n determined by a sequence

$$\mathbf{i} = (i_1, \ldots, i_n) \text{ with} \sum \alpha i_\alpha = n,$$

where $C_{\mathbf{i}}$ consists of those permutations that have i_1 1-cycles, ..., and i_n n-cycles.

Also, we denote by $[f(x)]_{(l_1,\ldots,l_k)}$ the coefficient of $x_1^{l_1} \cdots x_k^{l_k}$ for a polynomial $f(x) = f(x_1, \ldots, x_k)$ and a k-tuple of non-negative integers $(l_1, \ldots, l_k)$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, set

$$l_1 = \lambda_1 + k - 1,\ l_2 = \lambda_2 + k - 2, \ldots, l_k = \lambda_k.$$

These l's are hook lengths of the first column of the Young diagram. Then, the character of V_λ on $g \in C_{\mathbf{i}}$ is as follows:

Fact 2 (Frobenius formula [3]).

$$\chi_\lambda(g) = \left[\prod_{1 \le i < j \le k} (x_i - x_j) \cdot \prod_{1 \le j \le n} \left(x_1^j + \cdots + x_k^j\right)^{i_j} \right]_{(l_1,\ldots,l_k)}.$$

Here, we give some examples of the dimension and the character.

Example 2. If $\lambda = (n-k, \underbrace{1, \ldots, 1}_{k})$, the dimension of V_λ is $\binom{n-1}{k}$.

Example 3. If g is a cycle of length n in S_n, $\chi_\lambda(g)$ is as follows;

$$\chi_\lambda(g) = \begin{cases} (-1)^k & \text{if } \lambda = (n-k, \underbrace{1, \ldots, 1}_{k}),\ 0 \le k \le n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

4. Proof of Theorem 2 and 3

Theorem 2. *If $g \in S_n$ is an odd cycle, the Witten L-function $\zeta_{S_n}^W(s; g)$ is constantly zero.*

Proof. First, we obtain $D_{\lambda'} = D_\lambda$ by the hook length formula. In addition, by $V_{\lambda'} \simeq \mathrm{sgn} \otimes V_\lambda$, we obtain

$$\chi_{\lambda'}(g) = \mathrm{sgn}(g)\chi_\lambda(g).$$

Thus, we see

$$\frac{\chi_{\lambda'}(g)}{\deg\rho}(\deg\rho)^{-s} = -\frac{\chi_\lambda(g)}{\deg\rho}(\deg\rho)^{-s}.$$

Thus, we obtain $\zeta^W_{S_n}(s;\ g) = 0$. □

Theorem 3. *If $g \in S_n$ is the n-cycle and $n = 2m+1,\ m \geq 1$, then $\zeta^W_{S_n}(s;\ g)$ has a simple zero at $s = -2$.*

Proof. First, the differentiation of the Witten L-function is written as follows;

$$\frac{d}{ds}\zeta^W_{S_n}(s;\ g) = -\sum_\lambda \chi_\lambda(g) D_\lambda^{-s-1} \log D_\lambda. \tag{12}$$

Thus, we obtain

$$\frac{d}{ds}\zeta^W_{S_n}(-2;\ g) = -\sum_\lambda \chi_\lambda(g) D_\lambda \log D_\lambda. \tag{13}$$

Here, recall Example 2 and Example 3. If $g = (1\,2\ldots n)$, the dimension D_λ and the character $\chi_\lambda(g)$ are given as follows;

$$\chi_\lambda(g) = \begin{cases} (-1)^k & \text{if } \lambda = (n-k, \underbrace{1,\ldots,1}_{k}),\ 0 \leq k \leq n-1, \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

and

$$D_\lambda = \binom{n-1}{k}. \tag{15}$$

Thus, we obtain

$$\frac{d}{ds}\zeta^W_{S_n}(-2;\ g) = \sum_{k=0}^{2m}(-1)^k \binom{2m}{k} \log\binom{2m}{k}. \tag{16}$$

We denote by $l(k)$ the maximal prime number not bigger than k. Generally, it holds that $\frac{k}{2} < l(k) \leq k$ by Bertrand-Chebyshev Theorem ([2]). Then, we obtain

$$l \mid \binom{2m}{i},\ l^2 \nmid \binom{2m}{i} \quad \text{for } 2m - l + 1 \leq i \leq l - 1.$$

Hence $\frac{d}{ds}\zeta^W_{S_n}(-2;\, g)$ is written as follows;

$$\frac{d}{ds}\zeta^W_{S_n}(-2;\, g) = -\sum_{k=2m-l+1}^{l-1}(-1)^k\binom{2m}{k}\log l + \sum \log\frac{q_2}{q_1} \tag{17}$$

with $q_1, q_2 \in \mathbb{N}$ satisfying $l \nmid q_1,\ q_2$ because all of the characters are integers.

Here, we only need to show that

$$-\sum_{k=2m-l+1}^{l-1}(-1)^k\binom{2m}{k} \neq 0.$$

Actually, we see that

$$-\sum_{k=2m-l+1}^{l-1}(-1)^k\binom{2m}{k} < 0$$

from

$$-\sum_{k=2m-l+1}^{l-1}(-1)^k\binom{2m}{k} = 2\sum_{k=0}^{2m-l}(-1)^k\binom{2m}{k}$$

and

$$\binom{2m}{i} < \binom{2m}{i+1} \quad \text{for } 0 \le i \le m-1.$$

Therefore, the order of $\zeta^W_{S_n}(s;\, g)$ at $s = -2$ is 1. □

Theorem 3a. *Let n be an odd prime. Suppose that $\sum_{\lambda:\text{hook}}\chi_\lambda(g)D_\lambda \neq 0$. Then the Witten L-function $\zeta^W_{S_n}(s;\, g)$ has a simple zero at $s = -2$.*

Proof. We obtain

$$\begin{cases} n \mid D_\lambda,\ n^2 \nmid D_\lambda & \text{if } \lambda \text{ is not a hook} \\ n \nmid D_\lambda & \text{if } \lambda \text{ is a hook} \end{cases} \tag{18}$$

by the hook length formula.

Here, we may rewrite $\frac{d}{ds}\zeta^W_{S_n}(-2;\, g)$ as follows:

$$\begin{aligned}\frac{d}{ds}\zeta^W_{S_n}(-2;\, g) &= -\sum_{\lambda:\text{hook}}\chi_\lambda(g)D_\lambda\log D_\lambda - \sum_{\lambda:\text{not hook}}\chi_\lambda(g)D_\lambda\log n \\ &\quad - \sum_{\lambda:\text{not hook}}\chi_\lambda(g)D_\lambda\log\frac{D_\lambda}{n} \\ &= \sum_{\lambda:\text{hook}}\chi_\lambda(g)D_\lambda\log n + \sum\log\frac{r_1}{r_2},\end{aligned} \tag{19}$$

where r_1, r_2 are integers which are not divisible by n. By a similar argument to Theorem 3, we see that

$$\frac{d}{ds}\zeta^{W}_{S_n}(-2;\, g) \neq 0 \tag{20}$$

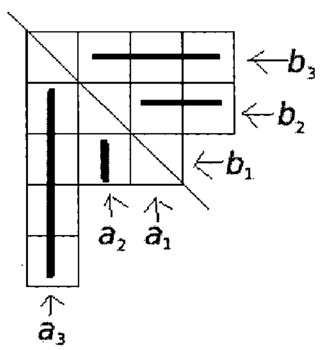
if

$$\sum_{\lambda\,:\,\mathrm{hook}} \chi_\lambda(g) D_\lambda \neq 0.$$

□

We give some examples for Theorem 3a. For this purpose, we introduce a notation.

Notation 1 (Frobenius characteristics). Let r be the length of the diagonal of a partition λ and a_i, b_i be the number of boxes below and to the right of the i-th box of the diagonal, reading from lower right to upper left. We write $(b_r, \dots, b_1 \mid a_r, \dots, a_1)$ for such a partition λ. Frobenius called $(b_r, \dots, b_1 \mid a_r, \dots, a_1)$ the characteristics of the partition. For example, for the partition $\lambda = (4, 4, 3, 1, 1)$, Frobenius characteristics are $(3, 2, 0 \mid 4, 1, 0)$.



Example 4. *If* $g = (1\,2\,3)$,

$$\chi_\lambda(g) = \frac{D_\lambda (n-3)!}{n!}\left(\frac{M_3}{2} - \frac{3}{2}n(n-1)\right), \tag{21}$$

where $M_3 = \sum_{j=1}^{r}[b_j(b_j+1)(2b_j+1) + a_j(a_j+1)(2a_j+1)]$.

Then,

$$\sum_{\lambda\,:\,\mathrm{hook}} \chi_\lambda(g) D_\lambda = \frac{1}{n(n-1)(2n-1)}\binom{2n-2}{n-1}\left(-\frac{n^3}{4} - n^2 + \frac{9}{4}n\right). \tag{22}$$

Hence it is non-zero for every odd prime $n \geq 5$.

Example 5. *If $g = (1\,2)(3\,4)$,*

$$\chi_\lambda(g) = \frac{D_\lambda(n-4)!}{n!}(M_2^2 - 2M_3 + 4n(n-1)), \tag{23}$$

where $M_2 = \sum_{j=1}^{r}[b_j(b_j+1) - a_j(a_j+1)]$ and M_3 is same as in Example 4. Then,

$$\sum_{\lambda:hook} \chi_\lambda(g)D_\lambda = \frac{1}{n(n-2)}\binom{2n-2}{n-1}(n^2-2n-1). \tag{24}$$

Hence it is non-zero for every odd prime $n \geq 5$.

Now we give a generalization of Theorem 3a for all $n \in \mathbb{N}$.

Theorem 3b. *The Witten L-function $\zeta_{S_n}^W(s;\, g)$ has a simple zero at $s = -2$ if $\sum_\lambda \chi_\lambda(g)D_\lambda \neq 0$, where $\sum_\lambda$ is the sum over λ's satisfying $l(n) \nmid D_\lambda$.*

Proof. By the hook length formula, if $l(n) \mid D_\lambda$, then $l(n)^2 \nmid D_\lambda$. Thus, we may prove our claim by a similar way to the proof of Theorem 3a. □

Lastly, we discuss the reason why we notice the characters whose degree is non-divisible by $l(n)$. In fact, if the number of the characters whose degree is non-divisible by $l(n)$ is small we calculate the order more easily.

Proposition. *Let $p(n)$ be the number of the partition of n. We denote by $q(n)$ the number of the irreducible characters with the degree non-divisible by $l(n)$. Then,*

$$\lim_{n\to\infty} \frac{q(n)}{p(n)} = 0.$$

Proof. If n is a prime number, n itself is $l(n)$ and $q(n) = n$. On the other hand, Hardy-Ramanujan ([6]) proved that

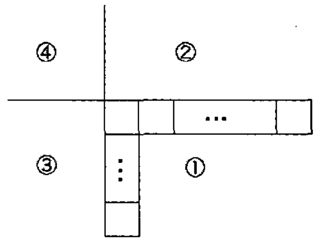
$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty. \tag{25}$$

Thus, we obtain

$$\lim_{\substack{n\to\infty \\ n:\text{prime}}} \frac{q(n)}{p(n)} = \lim_{\substack{n\to\infty \\ n:\text{prime}}} 4n^2\sqrt{3}\exp\left(-\pi\sqrt{\frac{2n}{3}}\right) = 0.$$

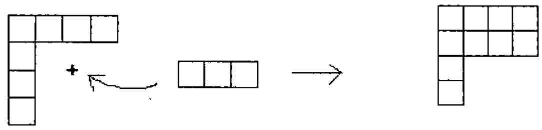
If n is not a prime number, we count the number of the characters of which degree is non-divisible by $l(n)$, i.e. the partition which includes a hook of length $l(n)$.

We show that there exist exactly $l(n)p(n-l(n))$ partitions including a hook of length $l(n)$ in the following manner. Firstly, we add $n - l(n)$ boxes around the hook of length $l(n)$.

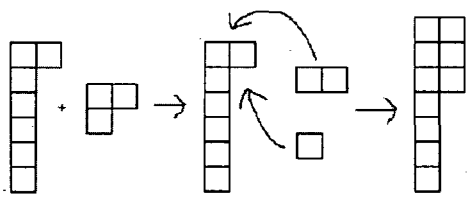


We add boxes in ①, ②, ③ and ④. However, if we add a box in ④, we need add at least $l(n)$ boxes in ② and ③. So we may add boxes only in ①, ② and ③.

We denote by k the length of the column in the $l(n)$-hook. If $n - l(n) + 1 \leq k \leq 2l(n) - n$ (here we may take such k because there exists a prime number p satisfying $\frac{2m}{3} < p \leq m$ for all integers $m \geq 6$), we may add $n - l(n)$ boxes only in ①. For each hook, we construct a partition of n which includes a hook of length $l(n)$ by making a partition of $n - l(n)$ and sticking that. Thus, we construct $p(n - l(n))$ partitions for each hook. Therefore, we obtain $(3l(n) - 2n)p(n - l(n))$ partitions if $n - l(n) + 1 \leq k \leq 2l(n) - n$.



Next, if $2l(n) - n < k \leq n$, we also construct a partition of n which includes a hook of length $l(n)$ by making a partition of $n - l(n)$. In this case, we construct the partition as follows. we denote by $\alpha = (\alpha_1, \ldots, \alpha_r)$ the partition of $n - l(n)$. We stick the rows of $\alpha_j \geq n - k + 1$ in ② and stick the other rows in ①.



Finally, if $1 \leq k \leq n - l(n)$, the partition of n satisfying our assumption is a conjugate partition of $2n - l(n) < k \leq n$.

Consequently, we get $l(n)p(n - l(n))$ partitions which includes a hook of length $l(n)$. Hence, we see that

$$\lim_{n\to\infty} \frac{q(n)}{p(n)} = \lim_{n\to\infty} \frac{l(n)p(n - l(n))}{p(n)}$$

$$= \lim_{n\to\infty} l(n) \exp\left[-\pi\left(\sqrt{\frac{2}{3}} - \sqrt{\frac{2(n-l(n))}{3n}}\right) n^{\frac{1}{2}}\right]$$

$$= 0. \tag{26}$$

□

By Proposition, we see that counting the number of characters whose degree is non-divisible by $l(n)$ is more effective than counting all partitions.

5. Proof of Theorem 4

Proof. We see that $C(\lambda_1)\cdots C(\lambda_n) \ni I$ holds if and only if $(\lambda_1, \ldots, \lambda_n)$ is the point of outside of the "polygon" (7) by Theorem B.

Assume that one of a sum $\sum_{i=1}^{n} \pm\lambda_i$ satisfies the following inequality:

$$(n-1-2j)\pi < S_n^{2j}(\{\lambda_i\}) \ (\leq (n-2j)\pi).$$

Here, we may assume

$$-\lambda_1 - \cdots - \lambda_{2j} + \lambda_{2j+1} + \cdots + \lambda_n > (n-1-2j)\pi$$

without loss of generality. Then the inequalities are determined only by the number t of minus signs from λ_{2j+1} to λ_n. Then the inequailties are determined as follows:

$$(n-1-2j-2t)\pi < \underbrace{\sum \pm\lambda_i}_{2j} + \underbrace{\sum \pm\lambda_i}_{n-2j} < (n+1-2j-2t)\pi, \tag{27}$$

where $\underbrace{\sum \pm\lambda_i}_{n-2j}$ contains t minus signs.

Moreover, we obtain

$$\zeta_{SU(2)}^{W}(-(n-1);\ C(\lambda_1), \ldots, C(\lambda_n))$$

$$= -\frac{\pi}{(2\sqrt{-1})^{n-1}\sin\lambda_1\cdots\sin\lambda_r}\left(\sum B_1\left(\left\langle\frac{S_n^{2k}(\{\lambda_i\})}{2\pi}\right\rangle\right)\right)$$

$$= -\frac{\pi}{(2\sqrt{-1})^{n-1}\sin\lambda_1\cdots\sin\lambda_r}\left(-2^{n-2} - \sum\left[\frac{S_n^{2k}(\{\lambda_i\})}{2\pi}\right]\right), \tag{28}$$

where $\langle x\rangle = x - [x]$, $B_1(x)$ denotes the Bernoulli polynomial of degree 1 and the sum $\sum$ consists of all of the sum $S_n^{2k}(\{\lambda_i\})$. Therefore, we see that

$$\zeta_{SU(2)}^{W}(-(n-1);\ C(\lambda_1), \ldots, C(\lambda_n)) = 0$$

by applying (27) to (28). □

6. Proof of Theorem 5

Proof. First, we obtain

$$\zeta^W_{SU(2)}(-(n-2);\ C(\lambda_1),\ldots,C(\lambda_n))$$
$$= \frac{\pi^2}{(2\sqrt{-1})^{n-2}\sin\lambda_1\cdots\sin\lambda_n}$$
$$\times\left(\sum{}'B_2\left(\left\langle\frac{S^{2k}(\{\lambda_i\})}{2\pi}\right\rangle\right)-\sum{}''B_2\left(\left\langle\frac{S^{2k+1}(\{\lambda_i\})}{2\pi}\right\rangle\right)\right.$$
$$\left.+(-1)^{\frac{n}{2}}\sum^{*}B_2\left(\left\langle\frac{S^{n/2}(\{\lambda_i\})}{2\pi}\right\rangle\right)\right), \tag{29}$$

where $\sum'$ consists of the sum S_n^{2k} for $2k < \frac{n}{2}$, $\sum''$ consists of the sum S_n^{2k+1} for $2k+1 < \frac{n}{2}$ and $\sum^*$ consists of the sum $S_n^{n/2}$ starting from the positive sign. Here, $B_2(x)$ denotes the Bernoulli polynomial of degree 2. Moreover, we may assume

$$-\lambda_1-\cdots-\lambda_{2j-1}+\lambda_{2j}+\cdots+\lambda_n > (n-2j)\pi \tag{30}$$

as in the proof of Theorem 4. Then the inequalities of λ's are determined as follows:

$$(n-2j-2t)\pi < \underbrace{\sum\pm\lambda_i}_{2j-1}+\underbrace{\sum\pm\lambda_i}_{n-2j+1} < (n+2-2j-2t)\pi, \tag{31}$$

where $\underbrace{\sum\pm\lambda_i}_{n-2j+1}$ contains t minus signs. Therefore, in a similar way to the proof of Theorem 4, we obtain

$$\zeta^W_{SU(2)}(-(n-2);\ C(\lambda_1),\ldots,C(\lambda_n)) = 0$$

when n is an even integer. □

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