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On a conjecture of Bateman about $r_5(n)$

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Abstract. Let $r_5(n)$ be the number of ways of writing n as a sum of five integer squares. In his study of this function, Bateman was led to formulate a conjecture regarding the sum

$$\sum_{|j|\le\sqrt{n}} \sigma(n-j^2)$$

where $\sigma(n)$ is the sum of positive divisors of n. We give a proof of Bateman's conjecture in the case n is square-free and congruent to 1 (mod 4).

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1. Introduction

Let $r_s(n)$ denote the number of solutions to the Diophantine equation

$$x_1^2 + \cdots x_s^2 = n \quad (x_i \in \mathbb{Z}, 1 \le i \le s). \tag{1.1}$$

We have the generating function for $r_s(n)$:

$$\sum_{n=0}^{\infty} r_s(n)q^n = \left(\sum_{j=-\infty}^{\infty} q^{j^2}\right)^s. \tag{1.2}$$

Thus, we see

$$r_{s+1}(n) = \sum_{|j|\le\sqrt{n}} r_s(n-j^2). \tag{1.3}$$

The study of $r_s(n)$ has a long history. If s is even, the identity (1.2) leads to the application of integral weight modular forms because

$$\sum_{j=-\infty}^{\infty} q^{j^2}$$

is the classical θ-function which is a modular form of weight $\frac{1}{2}$ for $\Gamma_o(4)$.

If s is odd, one can use the theory of half-integral weight modular forms to analyse $r_s(n)$. Alternatively, one can also use the circle method of Ramanujan (that was developed by Hardy and Ramanujan in their study of the partition function and later by Hardy and Littlewood in their work on Waring's problem) to derive asymptotic formulas. Indeed, using the circle method, Hardy [4] showed

$$r_s(n) = \frac{\pi^{s/2}}{\Gamma(s/2)} n^{s/2-1} \sum_{k=1}^{\infty} \sum_{\substack{h \pmod k \\ (h,k)=1}} \left(\frac{G(h,k)}{k}\right)^s e^{\frac{-2\pi inh}{k}} + O(n^{s/4}) \tag{1.4}$$

where $\Gamma(s)$ is the Gamma function and

$$G(h,k) = \sum_{j=1}^{k} e^{\frac{2\pi ihj^2}{k}}.$$

See Chapter 5 of [6] for an introduction to the circle method. In particular, for $s = 5$,

$$r_5(n) = \frac{4}{3}\pi^2 n^{3/2} \sum_{k=1}^{\infty} A_k(n) + O(n^{5/4}) \tag{1.5}$$

where

$$A_k(n) = \sum_{\substack{h \pmod k \\ (h,k)=1}} \left(\frac{G(h,k)}{k}\right)^5 e^{\frac{-2\pi inh}{k}}.$$

In his interesting paper [1], Bateman used an elementary method to study $r_5(n)$. His method allowed him to improve Hardy's error term in (1.5) from $O(n^{5/4})$ to $O(n^{1+\epsilon})$. His "naive" approach begins with the formula

$$r_5(n) = \sum_{|j| \le \sqrt{n}} r_4(n - j^2)$$

and then uses the classical formula of Jacobi for $r_4(n)$. Let us recall this formula.

Let $\sigma(n)$ denote the sum

$$\sigma(n) = \sum_{d|n} d.$$

It is customary to put $\sigma(0) = -\frac{1}{24}$. For $n \ge 0$, define $\sigma^*(n)$ by

$$\sigma^*(n) = \begin{cases} \sigma(n) & \text{if } 4 \nmid n, \\ \sigma(n) - 4\sigma\left(\frac{n}{4}\right) & \text{if } 4|n. \end{cases}$$

Then, Jacobi's formula for $r_4(n)$ is given by

$$r_4(n) = 8\sigma^*(n). \tag{1.6}$$

Using (1.3) and (1.6), one can write $r_5(n)$ as

$$\frac{r_5(n)}{8} = \sum_{|j| \le \sqrt{n}} \sigma^*(n - j^2) = \sum_{|j| \le \sqrt{n}} \sigma(n - j^2) - 4 \sum_{\substack{|j| \le \sqrt{n} \\ j^2 \equiv n(4)}} \sigma\left(\frac{n - j^2}{4}\right) \tag{1.7}$$

Bateman derived asymptotic formulas for the sums on the right hand side of the equation (1.7) and improved the error term in (1.5). More precisely, he showed

$$\sum_{|j| \le \sqrt{n}} \sigma^*(n - j^2) = \frac{\pi^2}{6} n^{3/2} \chi_2(n) \sum_{\substack{k>0 \\ k \text{ odd}}} A_k(n) + O(n(1 + \log n)^3) \tag{1.8}$$

where for prime p,

$$\chi_p(n) = \sum_{j=0}^{\infty} A_{p^j}(n).$$

It is worth noting that the main term in (1.8) is the same as Hardy's exact formula for $\frac{r_5(n)}{8} = \sum_{|j|\le\sqrt{n}} \sigma^*(n-j^2)$. With this in mind, Bateman conjectured in the same paper in 1995 that perhaps there is a similar exact formula for the sum

$$\sum_{|j|\le\sqrt{n}} \sigma(n-j^2).$$

His conjecture can be stated as follows:

Conjecture. [Bateman's Conjecture]

$$\frac{\pi^2}{6} n^{3/2}\left(\frac{5}{3}-\frac{\chi_2(n)}{3}\right)\sum_{\substack{k>0\\ k\text{ odd}}} A_k(n) = \begin{cases} \sum_{|j|\le\sqrt{n}} \sigma(n-j^2), & \text{if } n \text{ is not a perfect square,}\\ \sum_{|j|\le\sqrt{n}} \sigma(n-j^2)+2n, & \text{if } n \text{ is a perfect square.}\end{cases} \tag{1.9}$$

Denote the sum in the left hand side of (1.9) by $S(n)$. One can see that the conjecture is trivially true in the case when $n \equiv 2, 3 \pmod 4$, because in that case $\sigma(n-j^2) = \sigma^*(n-j^2)$ and $\chi_2(n) = 5/3 - \chi_2(n)/3$. Our goal in this paper is to investigate $S(n)$ in the case when $n \equiv 1 \pmod 4$ and n is square-free.

Bateman's conjecture was proved by Knopp and Bateman in [2] in 1998 using the theory of half-integral weight modular forms. In this paper, we give an "elementary" proof of Bateman's conjecture in the case $n \equiv 1 (\mathrm{mod}\ 4)$ and n square-free. We do not use the theory of half integral weight modular forms. The essential non-trivial ingredient is Siegel's formula (as refined by Zagier) for $\zeta_K(-1)$ where K is a real quadratic field and $\zeta_K(s)$ is the Dedekind zeta function of K. It is likely the method extends to deal with the case when n is not square-free, however, we have not pursued this here for the sake of brevity.

2. Preliminaries

In the first half of this section, we recall a few results about congruences which will be an important tool in the proof of our main theorem. In the other half, we recall some definitions from algebraic number theory and state the Siegel-Zagier formula for special values of certain Dedekind zeta functions. These will be used in the proof of our main result (Theorem 3.5).

Definition 1. *Suppose that $(a, m) = 1$. Then a is called a quadratic residue of m if the congruence $x^2 \equiv a (\mathrm{mod}\ m)$ has a solution. If there is no solution, then a is called a quadratic non-residue of m.*

Hensel's lemma provides a criterion for "lifting" solutions of a polynomial modulo successive powers of a prime p. We recall this below.

Lemma 2.1 (Hensel's lemma). *Let p be a prime and k an arbitrary positive integer. Let $f \in \mathbb{Z}[x]$. If a is a solution of $f(x) \equiv 0 (\mathrm{mod}\ p)$ and $p \nmid f'(a)$, then, for every $k \ge 2$, there exists precisely one solution b of $f(x) \equiv 0 (\mathrm{mod}\ p^k)$ such that $b \equiv a (\mathrm{mod}\ p)$.*

For a proof of Hensel's lemma, see p. 157 of [5].

Lemma 2.2. *If p is an odd prime and $(a, p) = 1$, then $x^2 \equiv a (\mathrm{mod}\ p^k)$ has exactly two solutions if a is a quadratic residue of p, and no solutions if a is a quadratic non-residue of p.*

Proof. First, it is clear that if a is a quadratic residue of p, then the congruence $x^2 \equiv a(\text{mod } p)$ has exactly two roots. Now setting $f(x) = x^2 - a$, $f'(a) = 2a$, so that using lemma (2.1), we deduce that the equation $x^2 \equiv a(\text{mod } p^k)$ has exactly two roots for each k if a is a quadratic residue. Since every solution of this congruence also solves the congruence $x^2 \equiv a(\text{mod } p)$, there can be no solution if a is a quadratic non-residue of p. □

Lemma 2.3. *If p is an odd prime and $p|n$, say, $n = p^\beta n_0$, $p \nmid n_0$, then the number of solutions of $x^2 \equiv a(\text{mod } p^\alpha)$ is given by,*

$$\begin{cases} 1 + \left(\frac{n_0}{p}\right), & \text{if } \beta < \alpha, \beta \text{ is even.} \\ 0, & \text{if } \beta < \alpha, \beta \text{ is odd.} \\ \sum_{\gamma \geq \alpha/2}^{\alpha} \phi(p^{\alpha-\gamma}), & \text{if } \beta \geq \alpha. \end{cases}$$

where $\phi(n)$ denotes the Euler totient function.

Proof. Given the congruence $x^2 \equiv p^\beta n_0 \ (\text{mod } p^\alpha)$, it is clear that $p|x^2$ which implies that $p|x$. Let $x = p^\gamma t$ where $(t, p) = 1$. So we have $p^{2\gamma} \equiv p^\beta n_0 \ (\text{mod } p^\alpha)$.

(i) When $\beta < \alpha$, $p^\beta | p^{2\gamma}$, implying that $2\gamma \geq \beta$. When we divide this congruence by p^β, we get

$$p^{2\gamma-\beta} t^2 \equiv n_0(\text{mod } p^{\alpha-\beta}),$$

where $2\gamma \geq \beta$. But $p \nmid n_0$ and since $(t, p) = 1$, $2\gamma = \beta$. If β is even, then there is exactly one solution for γ and then $t^2 \equiv n_0(\text{mod } p^{\alpha-\beta})$ and the number of choices for such t is $1 + \left(\frac{n_0}{p}\right)$. If β is odd, there is no solution.

(ii) When $\beta \geq \alpha$, we get

$$p^{2\gamma} t^2 \equiv 0(\text{mod } p^\alpha)$$

where $\alpha/2 \leq \gamma \leq \alpha$. Hence, the number of solutions in this case is $\sum_{\gamma \geq \alpha/2}^{\alpha} \phi(p^{\alpha-\gamma})$.

□

For $p = 2$, the story is completely different and is given by the following lemma.

Lemma 2.4. *Suppose a is odd. Then*

(i) *the congruence $x^2 \equiv a(\text{mod } 2)$ is always solvable and has exactly one solution;*
(ii) *the congruence $x^2 \equiv a(\text{mod } 4)$ is solvable if and only if $a \equiv 1(\text{mod } 4)$, in which case there are precisely two solutions;*
(iii) *the congruence $x^2 \equiv a(\text{mod } 2^k)$, with $k \geq 3$, is solvable if and only if $a \equiv 1(\text{mod } 8)$, in which case there are exactly four solutions. If x_0 is a solution, then all solutions are given by $\pm x_0$ and $\pm x_0 + 2^{k-1}$.*

Proof. The first two cases are clear. For case (iii), suppose $x^2 \equiv a(\text{mod } 2^k)$ has a solution x_0. Then obviously $x_0^2 \equiv a(\text{mod } 8)$, and x_0 is odd since a is odd. But the square of an odd number is congruent to 1 modulo 8, and hence $a \equiv 1(\text{mod } 8)$. This proves the necessity of the condition $a \equiv 1(\text{mod } 8)$ for the existence of a solution. Moreover, $(-x_0)^2 = x_0^2 \equiv a(\text{mod } 2^k)$ and $(\pm x_0 + 2^{k-1})^2 = x_0^2 \pm 2^k x_0 + 2^{2k-2} \equiv x_0^2 \equiv a(\text{mod } 2^k)$, since $2k - 2 \geq k$. It is easily verified that the four numbers $\pm x_0$ and $\pm x_0 + 2^{k-1}$ are incongruent modulo 2^k. Hence, the congruence has at least four solutions if there are any.

It remains to verify that the condition on a is sufficient and that there are at most four solutions. We show sufficiency by induction on k. For the base case $k = 3$, it is clear as $x^2 \equiv 1(\text{mod } 8)$ has the solution $x \equiv 1$. Now assume that $x^2 \equiv a(\text{mod } 2^k)$ is solvable with a solution x_0. Then we know that $\pm x_0$ and $\pm x_0 + 2^{k-1}$ solve the congruence and we will prove that one of them also solves the congruence

$$x^2 \equiv a(\text{mod } 2^{k+1}). \tag{2.1}$$

We know that $x_0^2 = a + 2^k n$ for some integer n. If n is even, then x_0 is a solution of (2.1). If n is odd, then

$$(x_0 + 2^{k-1})^2 = x_0^2 + 2^k x_0 + 2^{2k-2} = a + 2^k(n + x_0) + 2^{2k-2} \equiv a(\text{mod } 2^{k+1}),$$

because $n + x_0$ is even (as both n and x_0 are odd) and $(2k - 2) \geq (k + 1)$. This completes the induction step.

Finally, in the interval $[1, 2^k]$, there are 2^{k-3} integers a that are congruent to 1 modulo 8. For each such number a we have already found 4 different solutions of the congruence $x^2 \equiv a(\text{mod } 2^k)$ in the same interval, all of them odd. Taking all these solutions together we get $4.2^{k-3} = 2^{k-1}$ solutions. But there are exactly 2^{k-1} odd numbers in the interval, so there is no room for any more solutions. Hence, each equation has exactly four solutions. □

In the rest of the section we will assume the reader is familiar with algebraic number theory. If not, we refer the reader to [3] for an introduction to the subject. However, to keep the article self-contained, we review some basic facts.

We begin by discussing the main properties of the zeta function of a number field. Let K be an algebraic number field of degree n and let $\mathcal{O}_K$ denote the ring of integers of K. For any non-zero ideal $\mathfrak{A}$ of $\mathcal{O}_K$, the norm $N(\mathfrak{A})$ is defined as the number of elements in the quotient $\mathcal{O}_K/\mathfrak{A}$ which is finite by a theorem of Dedekind. The Dedekind zeta function $\zeta_K(s)$ is defined for Re $(s) > 1$ as the infinite series

$$\zeta_K(s) = \sum_{\mathfrak{A}} \frac{1}{(N\mathfrak{A})^s},$$

where the sum is over all non-zero ideals in $\mathcal{O}_K$. Also, for Re $(s) > 1$,

$$\zeta_K(s) = \prod_{\mathfrak{P}} \left(1 - \frac{1}{(N\mathfrak{P})^s}\right)^{-1},$$

where the product is over all prime ideals in $\mathcal{O}_K$. $\zeta_K(s)$ can be analytically continued to all of $\mathbb{C}$, except for a simple pole at $s = 1$. Moreover, it has a functional equation relating $\zeta_K(s)$ to $\zeta_K(1-s)$. In the case of a totally real field K, this functional equation takes the form

$$F(s) = F(1-s), \tag{2.2}$$

where

$$F(s) = D^{s/2}\pi^{-ns/2}\Gamma\left(\frac{s}{2}\right)^n \zeta_K(s), \tag{2.3}$$

where D is the absolute discriminant of K and $n = [K : \mathbb{Q}]$.

In particular, we have

$$\zeta_K(-2m) = 0$$

and

$$\zeta_K(1-2m) = \{(-1)^m(2m-1)!/2^{2m-1}\pi^{2m}\}^n D^{2m-1/2}\zeta_K(2m),$$

for $m = 1, 2, \ldots$.

Thus, it is equivalent to give values of $\zeta_K(s)$ at $s = 2, 4, 6, \ldots$ or at $s = -1, -3, -5, \ldots$. Hecke conjectured that these numbers namely $\zeta_K(-1), \zeta_K(-3), \ldots$ are rational; in fact, they generalise Bernoulli numbers (corresponding to the special case $K = \mathbb{Q}$). Siegel proved this and gave an explicit formula for these numbers in [8] which is also stated in p. 59 of [10]. For the sake of simplicity, we will content ourselves by stating the specialization of this formula to a real quadratic number field.

In [10], Zagier exploited the simple arithmetic of quadratic fields to evaluate in elementary form the terms appearing in Siegel's formula , thus giving a more simplified expression for $\zeta_K(1-2m)$ (K a quadratic field), involving only rational numbers. Define $s_l^K(2m)$ as

$$s_l^K(2m) = \sum_{j \mid l} \chi(j) j^{2m-1} e_{2m-1}((l/j)^2 D), \tag{2.4}$$

where D is the discriminant of K, the arithmetic function $e_r(n)$ is defined as

$$e_r(n) = \sum_{\substack{|x|\le\sqrt{n} \\ x^2\equiv n \pmod 4}} \sigma_r\left(\frac{n-x^2}{4}\right), \tag{2.5}$$

where r is a positive integer and not a perfect square and $\sigma_r(n) = \sum_{d|n} d^r$.

Theorem 2.5 (Siegel-Zagier formula). *Let $m = 1, 2, \ldots$ be a natural number and K a quadratic field. Then,*

$$\zeta_K(1-2m) = 4\sum_{l=1}^{r} b_l(4m)s_l^K(2m), \tag{2.6}$$

where $r = \left[\frac{m}{3}\right] + 1$ and the coefficients $b_l(4m)$ are computable rational numbers which are tabulated on p.60 of [10] for $1 \le l \le 10$.

For a proof of the Siegel-Zagier formula, see section 3 of [10]. For $m = 1$, the formula reduces to

$$\zeta_K(-1) = \frac{1}{60}e_1(D). \tag{2.7}$$

For further reading on the Siegel's formula, we refer the interested reader to [8] and [10].

3. Bateman's conjecture in the case $n \equiv 1 \pmod 4$ and square-free

In this section, we first reformulate Bateman's conjecture and then show that in the case when n is square-free and congruent to 1 modulo 4, the conjecture is true.

Using (1.7) and (1.9), the conjecture can be reformulated in terms of the sum

$$S^*(n) = \frac{\pi^2}{6}n^{3/2}\left(\frac{5}{3} - \frac{4}{3}\chi_2(n)\right)\sum_{\substack{k>0 \\ k \text{ odd}}} A_k(n)$$

as follows:

$$S^*(n) = \begin{cases} 4\sum\limits_{\substack{|j|\le\sqrt{n} \\ j^2\equiv n(4)}} \sigma\left(\frac{n-j^2}{4}\right), & \text{if } n \text{ is not a perfect square} \\ 4\sum\limits_{\substack{|j|\le\sqrt{n} \\ j^2\equiv n(4)}} \sigma\left(\frac{n-j^2}{4}\right) + 2n, & \text{if } n \text{ is a perfect square.} \end{cases} \tag{3.1}$$

For $n \in \mathbb{N}$, denote by $f_d(n)$ the number of integers m such that

$$m^2 \equiv n(\text{mod}), \quad 0 \le m \le d-1.$$

Clearly, for an odd prime p, $f_p(n) = 1 + \left(\frac{n}{p}\right)$ where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Jacobi symbol.

Lemma 3.1. *For p odd, $n \equiv 1 \pmod 4$ and square-free, and $\alpha > 1$*

$$f_{p^\alpha}(n) = \begin{cases} 1 + \left(\frac{n}{p}\right), & p \nmid n \\ 0, & p | n. \end{cases}$$

Proof. If $p \nmid n$, then this follows directly using Lemma 2.2. If $p|n$, then the result follows from Lemma 2.3. □

Lemma 3.2. *With $f_d(n)$ defined as before, we have*

$$f_{2^\beta}(n) = \begin{cases} 2, & \beta = 2, \quad n \equiv 1 (\text{mod } 4) \\ 4, & \beta > 2, \quad n \equiv 1 (\text{mod } 8) \\ 0, & \beta > 2, \quad n \equiv 5 (\text{mod } 8). \end{cases}$$

Proof. This follows directly from Lemma (2.4). □

Lemma 3.3.

$$\sum_{\substack{d>0 \\ d \text{ odd}}} \frac{f_d(n)}{d^2} = \frac{\pi^2}{8} \sum_{\substack{c>0 \\ c \text{ odd}}} A_c(n)$$

where $f_d(n)$ and $A_c(n)$ are defined as before.

Proof. See pg. 134 of [1] or Lemma 2.12 of [9]. □

Lemma 3.4. *For* Re $(s) > 0$, $n \equiv 1(\text{mod } 4)$ *and square-free,*

$$\sum_{\substack{d>0 \\ d \text{ odd}}} \frac{f_d(n)}{d^s} = \left(\frac{2^s}{2^s+1}\right)\left(\frac{2^s - \left(\frac{n}{2}\right)}{2^s}\right)\left(\frac{\zeta_K(s)}{\zeta(2s)}\right)$$

where $\zeta_K(s)$ is the Dedekind zeta function associated to the field $K = \mathbb{Q}(\sqrt{n})$.

Proof. Since $f_d(n)$ is multiplicative function of d, we have,

$$\sum_{\substack{d>0 \\ d \text{ odd}}} \frac{f_d(n)}{d^s} = \prod_{p \text{ odd}} (1 + f_p(n)p^{-s} + f_{p^2}(n)p^{-2s} + \cdots)$$

$$= \left[\prod_{p \nmid n, p \text{ odd}} (1 + f_p(n)p^{-s} + f_{p^2}(n)p^{-2s} + \cdots)\right]\left[\prod_{p \mid n, p \text{ odd}} (1 + f_p(n)p^{-s} + f_{p^2}(n)p^{-2s} + \cdots)\right]$$

For d odd, using Lemma 3.1, the above is same as

$$= \left[\prod_{p \nmid n, p \text{ odd}} \left(1 + \frac{1+\left(\frac{n}{p}\right)}{p^s} + \frac{1+\left(\frac{n}{p}\right)}{p^{2s}} + \cdots\right)\right]\left[\prod_{p \mid n, p \text{ odd}} \left(1 + \frac{1}{p^s}\right)\right]$$

$$= \left[\prod_{p \nmid n, p \text{ odd}} \left(1 + \left(1 + \left(\frac{n}{p}\right)\right)\frac{1}{p^s - 1}\right)\right]\left[\prod_{p \mid n, p \text{ odd}} \left(1 + \frac{1}{p^s}\right)\right]$$

$$= \left[\prod_{p \nmid n, p \text{ odd}} \left(1 + \left(\frac{n}{p}\right)p^{-s}\right)\right]\left[\prod_{p \nmid n, p \text{ odd}} \left(\frac{p^s}{p^s - 1}\right)\right]\left[\prod_{p \mid n, p \text{ odd}} \left(1 + \frac{1}{p^s}\right)\right]$$

$$= \left[\prod_{p \nmid n, p \neq 2} \frac{\left(1 - \left(\frac{n}{p}\right)^2 p^{-2s}\right)}{\left(1 - \left(\frac{n}{p}\right)p^{-s}\right)}\right]\left[\zeta(s)\prod_{p \mid 2n}\left(\frac{p^s - 1}{p^s}\right)\right]\left[\prod_{p \mid n, p \neq 2}\left(1 + \frac{1}{p^s}\right)\right]$$

$$= \left[\frac{L(s,\chi_n)\prod_{p|2n}\left(1-\left(\frac{n}{p}\right)p^{-s}\right)}{\prod_p\left(1-p^{-2s}\right)^{-1}\prod_{p|2n}\left(1-p^{-2s}\right)} \right] \left[\zeta(s)\prod_{p|2n}\left(1-\frac{1}{p^s}\right) \right] \left[\prod_{p|n,p\neq 2}\left(1+\frac{1}{p^s}\right) \right]$$

$$= \frac{\zeta(s)L(s,\chi_n)}{\zeta_{2n}(2s)}\left(\frac{2^s-\left(\frac{n}{2}\right)}{2^s}\right)\prod_{p|2n}\left(1-\frac{1}{p^s}\right)\left[\prod_{p|2n}\left(1+\frac{1}{p^s}\right)\left(1+\frac{1}{2^s}\right)^{-1}\right].$$

where $\zeta_{2n}(s) = \zeta(s)\prod_{p|2n}(1-p^{-s})$ and $\chi_n(p) = \left(\frac{n}{p}\right)$ is the primitive quadratic character modulo n and $L(s,\chi_n)$ is the $L-$function attached to χ_n. Since $\zeta(s)L(s,\chi_n) = \zeta_K(s)$, where $\zeta_K(s)$ is the Dedekind zeta function associated to the field $K = \mathbb{Q}(\sqrt{n})$, we get

$$\sum_{\substack{d>0 \\ d \text{ odd}}} \frac{f_d(n)}{d^s} = \left(\frac{\zeta_K(s)}{\zeta(2s)\prod_{p|2n}\left(1-p^{-2s}\right)}\right)\left(\frac{2^s-\left(\frac{n}{2}\right)}{2^s}\right)\left(\frac{2^s}{2^s+1}\right)\prod_{p|2n}\left(1-p^{-2s}\right)$$

$$= \left(\frac{2^s}{2^s+1}\right)\left(\frac{2^s-\left(\frac{n}{2}\right)}{2^s}\right)\left(\frac{\zeta_K(s)}{\zeta(2s)}\right),$$

which is what we wanted to show. □

Theorem 3.5. *For n square-free and congruent to* 1 *modulo* 4,

$$S^*(n) = 4\left[\sum_{\substack{|j|\leq\sqrt{n} \\ j^2\equiv n(4)}} \sigma\left(\frac{n-j^2}{4}\right)\right].$$

Proof. By definition,

$$S^*(n) = \frac{\pi^2}{6}n^{3/2}\left(\frac{5}{3}-\frac{4}{3}\chi_2(n)\right)\sum_{\substack{k>0 \\ k \text{ odd}}} A_k(n).$$

Using Lemma (3.3), we get

$$S^*(n) = \frac{4}{3}n^{3/2}\left(\frac{5}{3}-\frac{4\chi_2(n)}{3}\right)\sum_{\substack{d>0 \\ d \text{ odd}}} \frac{f_d(n)}{d^2}.$$

Now, applying Lemma (3.4) for $s = 2$, we get

$$S^*(n) = \frac{4}{3}n^{3/2}\left(\frac{5}{3}-\frac{4\chi_2(n)}{3}\right)\frac{4}{5}\left(1-\frac{\left(\frac{n}{2}\right)}{4}\right)\frac{\zeta_K(2)}{\zeta(4)}$$

Using (2.2), we get $\zeta_K(2) = n^{-3/2}4\pi^4\zeta_K(-1)$. We also know that $\zeta(4) = \frac{\pi^4}{90}$. Using these, we get,

$$S^*(n) = \frac{4.4}{3.5}\left(1-\frac{\left(\frac{n}{2}\right)}{4}\right)\left(\frac{5}{3}-\frac{4\chi_2(n)}{3}\right)4\pi^4\frac{\zeta_K(-1)}{\zeta(4)}$$

$$= (4^3\times 6)\left(1-\frac{\left(\frac{n}{2}\right)}{4}\right)\left(\frac{5}{3}-\frac{4\chi_2(n)}{3}\right)\zeta_K(-1).$$

Now, applying (2.7), we get

$$\zeta_K(-1) = \frac{1}{60}\left[\sum_{\substack{|j|\le\sqrt{n} \\ j^2\equiv n(4)}} \sigma\left(\frac{n-j^2}{4}\right)\right]$$

Also,

$$\left(\frac{5}{3} - \frac{4\chi_2(n)}{3}\right) = 4\sum_{\beta=2}^{\infty}\frac{f_{2^\beta}(n)}{2^{2\beta}}.$$

(See p. 135 of [1]). Using these results, we get

$$S^*(n) = E^*(n)\left[\sum_{\substack{|j|\le\sqrt{n} \\ j^2\equiv n(4)}} \sigma\left(\frac{n-j^2}{4}\right)\right]$$

where

$$E^*(n) = \frac{4^4\times 6}{60}\left(1 - \frac{\left(\frac{n}{2}\right)}{4}\right)\left(\sum_{\beta=2}^{\infty}\frac{f_{2^\beta}(n)}{2^{2\beta}}\right).$$

Now, using Lemma (3.2), for $n \equiv 1 (\text{mod } 8)$,

$$\sum_{\beta=2}^{\infty}\frac{f_{2^\beta}(n)}{2^{2\beta}} = \frac{2}{2^4} + \sum_{\beta=3}^{\infty}\frac{4}{2^{2\beta}} = \frac{5}{24}$$

and for $n \equiv 5 (\text{mod } 8)$,

$$\sum_{\beta=2}^{\infty}\frac{f_{2^\beta}(n)}{2^{2\beta}} = \frac{2}{2^4} + \sum_{\beta=3}^{\infty}\frac{0}{2^{2\beta}} = \frac{1}{8}.$$

Using the fact that

$$\left(\frac{n}{2}\right) = \begin{cases} 1, & n \equiv 1 (\text{mod } 8) \\ -1, & n \equiv 5 (\text{mod } 8), \end{cases}$$

we get that for $n \equiv 1 (\text{mod } 4)$, $E^*(n) = 4$. This completes the proof. □

4. Concluding remarks

It should be possible to extend our results for all moduli. Part of the difficulty in doing this is first to extend Zagier's results to allow for this case and we hope to return to this at a later date.

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