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On centrally symmetric manifolds

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Abstract. We introduce some methods to construct centrally symmetric triangulated manifolds. In particular, we show the existence of some infinite series of centrally symmetric triangulated manifolds. We also enumerate centrally symmetric triangulated 2-, 3-manifolds with few vertices.

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1. Introduction

A *map* M is an embedding of a graph G on a surface S such that the closure of components of $S\backslash G$, called the *faces* of M, are homeomorphic to 2-discs. A map M is said to be a *polyhedral map* if the intersection of any two distinct faces is either empty, a common vertex, or a common edge. Here map means polyhedral map.

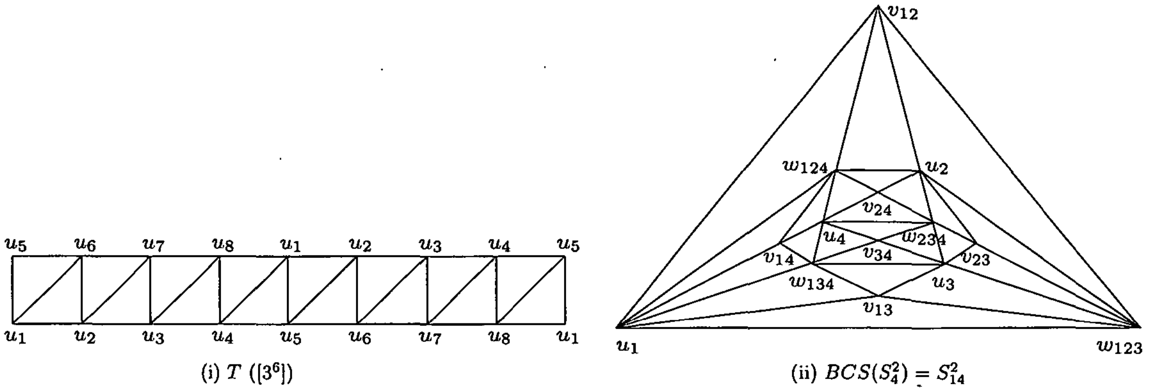
For a vertex u in a map X, the faces containing u form a cycle (called the *face-cycle* at u) C_u in the dual graph of X. So, C_u is of the form $(F_{1,1}\text{-}\cdots\text{-}F_{1,n_1})\text{-}\cdots\text{-}(F_{k,1}\text{-}\cdots\text{-}F_{k,n_k})\text{-}F_{1,1}$, where $F_{i,\ell}$ is a p_i-gon for $1 \leq \ell \leq n_i$, $1 \leq i \leq k$, $p_r \neq p_{r+1}$ for $1 \leq r \leq k-1$ and $p_n \neq p_1$. A map X is called *semiequivelar* ([2], we are including the same definition for the sake of completeness) if C_u and C_v are of same type for all $u, v \in V(X)$. More precisely, there exist integers $p_1, \ldots, p_k \geq 3$ and $n_1, \ldots, n_k \geq 1$, $p_i \neq p_{i+1}$ (addition in the suffix is modulo k) such that C_u is of the form as above for all $u \in V(X)$. In such a case, X is called a semiequivelar map of type $[p_1^{n_1}, \ldots, p_k^{n_k}]$ (or, a map of type $[p_1^{n_1}, \ldots, p_k^{n_k}]$).

All simplicial complexes considered in this paper are finite and abstract. The vertex set of a simplicial complex X will be denoted by $V(X)$. For $A \subset V(X)$, the induced subcomplex $X[A]$ of X on the vertex set A is defined by $X[A] := \{\alpha \in X : \alpha \subset A\}$. By a triangulated manifold we mean a simplicial complex whose geometric carrier is a topological manifold.

We call a simplicial complex (or a map) K *centrally symmetric* (or *CS*) if there exists an involution $I \in \mathrm{Aut}(K)$ such that $I(\alpha) \cap \alpha = \emptyset$ for each face α of K. In that case, we also say that (K, I) is centrally symmetric. See [13,15,16], for more on centrally symmetric manifolds and applications.

For $\ell \geq 1$, a simplicial complex K is called *ℓ-neighbourly* if every set with at most ℓ vertices forms a face of K. If (K, I) is centrally symmetric then for any vertex v of K, $\{v, I(v)\}$ does not form an edge. So, no face contains both v and $I(v)$. Thus, (K, I) is never ℓ-neighbourly for $\ell \geq 2$. In [7], Lutz defined centrally ℓ-neighbourly, namely, a centrally symmetric simplicial complex (K, I) is called *centrally ℓ-neighbourly* if α is a face of K for each α with at most ℓ vertices and $\alpha \cap I(\alpha) = \emptyset$. If it is centrally $\lfloor(\dim(K)+1)/2\rfloor$-neighbourly then it is called *nearly neighbourly*. Thus, a $2m$-vertex 3-dimensional centrally symmetric simplicial complex (K, I) is nearly neighbourly if and only if $f_1(K) = \binom{2m}{2} - m$.

The polytopal 3-sphere $S_2^0 * S_2^0 * S_2^0 * S_2^0$ is centrally symmetric. It is not difficult to see that it is the only centrally symmetric 3-sphere on 8 vertices. Grünbaum constructed a 10-vertex centrally symmetric polytopal 3-sphere and



(i) T ($[3^6]$)

(ii) $BCS(S_4^2) = S_{14}^2$

Figure 1. Centrally symmetric triangulations of torus and $\mathbb{S}^2$.

shown that it is unique (see [7]). Grünbaum has also shown that there does not exist such 3-sphere on 12 vertices (see [4, Page 116]). In [6], Lassmann and Sparla have shown that there are three centrally symmetric 3-neighbourly triangulations of the product $\mathbb{S}^2 \times \mathbb{S}^2$ with cyclic symmetry. In [7], Lutz has extended this result and enumerated triangulations of product of spheres using cyclic and dihedral group actions. In [5], Klee and Novik have shown that there is a centrally symmetric $(2d + 4)$-vertex triangulation of the product of spheres $\mathbb{S}^i \times \mathbb{S}^{d-i}$ for all pairs of nonnegative integers i and d with $0 \leq i \leq d$. Lutz [7] has shown existence of vertex transitive central symmetric triangulation of spheres and torus under cyclic and dihedral group action. In this article, we have relaxed the condition of vertex transitivity and constructed centrally symmetric manifolds of dimension ≥ 2 and centrally symmetric maps. In particular, we prove the following.

(i) For each $g \geq 1$, there exists a centrally symmetric $8g$-vertex triangulated 2-manifold of Euler characteristic $2 - 2g$ (see Theorem 6).
(ii) There exist series of centrally symmetric triangulated m-manifolds which are not triangulation of spheres or product of spheres for each $m \geq 2$ (see Theorem 7).
(iii) There exist series of centrally symmetric semiequivelar maps (see Corollary 9).
(iv) There exist series of centrally symmetric maps with arbitrary p-gonal faces for $p \geq 3$ (see Theorem 11).
(v) There are exactly 6303 centrally symmetric triangulated surfaces with $n \leq 12$ vertices. Out of these, 1228 are orientable and 5075 are non orientable (see Theorem 12).
(vi) There are exactly 68 centrally symmetric triangulated 3-manifolds on 12 vertices (see Theorem 13).

2. Centrally symmetric triangulated manifolds

For a triangulated manifold M, a pair of faces $\{\alpha, \beta\}$ is called a pair of *antineighbours* if there is no edge in M of the form uv, where $u \in \alpha$ & $v \in \beta$. For an m-dimensional cell complex X, let $BCS(X)$ denote the barycentric subdivision of X. Then, the vertex set of $BCS(X)$ is $\{\alpha : \alpha \text{ is a face of } X\}$ & the facets are $\{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ where $\alpha_0 < \alpha_1 < \cdots < \alpha_m$ are faces of X. We first present two examples of centrally symmetric triangulated 2-manifolds.

Example 1.

(i) *Let T be the triangulation of the torus on 8 vertices, given in Fig. 1 (i). Let $V(T) = \{u_1, \ldots, u_8\}$ be vertex set of T. Let $I_T := \prod_{t=1}^{4}(u_t, u_{t+4})$ be a map on $V(T)$. Then (T, I_T) is centrally symmetric.*
(ii) *Let S_4^2 be the 4-vertex triangulation of $\mathbb{S}^2$. Let $V(S_4^2) = \{u_1, u_2, u_3, u_4\}$. Then the faces of S_4^2 are $v_{ij} := \{u_i, u_j\}$, $w_{ijk} := \{u_i, u_j, u_k\}$, $1 \leq i, j, k \leq 4$ are distinct. Thus the vertex set of $BCS(S_4^2)$ is $\{u_i, v_{ij}, w_{ijk} : 1 \leq i, j, k \leq 4$ distinct$\}$ and facets are $\{u_i, v_{ij}, w_{ijk}\}$, $1 \leq i, j, k \leq 4$. Let $S := S_{14}^2 = BCS(S_4^2)$. Let $I_S \colon V(S) \mapsto V(S)$ be as $I_S(u_i) = w_{jkl}$, $I_S(v_{ij}) = v_{kl}$, $I_S(w_{ijk}) = u_l$, $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then I_S defines an involution on S and (S, I_S) is a CS triangulation of $\mathbb{S}^2$. Moreover, for any facet Δ of S, $\{\Delta, I_S(\Delta)\}$ is a pair of antineighbours. Note that we can generalize this construction on the $(d+2)$-vertex triangulation S_{d+2}^d of the d-sphere $\mathbb{S}^d$ for all $d \geq 2$.*

If X and Y are simplicial complex then $X \times Y$ is a cell complex whose cells are $e_\alpha \times e_\beta$ where $e_\alpha \in X$ and $e_\beta \in Y$. In general, product cell complex need not be centrally symmetric. For example, if X is the 3-vertex triangulation of

$\mathbb{S}^1$ then $X \times X$ is a 9-vertex map on $\mathbb{S}^1 \times \mathbb{S}^1$. Since $X \times X$ has odd number of vertices, $X \times X$ can not be centrally symmetric. Here we prove the following (this lemma is used in the proof of Theorem 7).

Lemma 2. *Let X and Y be two simplicial complexes. If X or Y is centrally symmetric then $X \times Y$ is a centrally symmetric cell complex.*

Proof. Without loss, assume that (X, I_X) is centrally symmetric where $V(X) = \{u_1, \dots, u_{2m}\}$ and $I_X = \prod_{i=1}^{m}(u_i, u_{i+m})$. Let $V(Y) = \{v_1, \dots, v_n\}$. Let $I := \prod_{j=1}^{m}\prod_{k=1}^{n}(u_j \times v_k, u_{j+m} \times v_k)$ be a map on the vertex set of $X \times Y$.

Claim. The $(X \times Y, I)$ is centrally symmetric.

Observe that, $I(I(u_\ell \times Y)) = u_\ell \times Y$ for all $1 \leq \ell \leq 2m$, and for $\sigma \in X$, $I(I(\sigma \times Y)) = \sigma \times Y$ since $I_X(I_X(\sigma)) = \sigma$. Again, suppose $I(\beta \times \beta') = \beta \times \beta'$ for some $\beta \in X$ and $\beta' \in Y$. Then, it implies that $I_X(\beta) = \beta$, a contradiction since (X, I_X) is centrally symmetric. Therefore, the $(X \times Y, I)$ is a centrally symmetric. This proves the result. □

Observation 3. Let (X, I) be a centrally symmetric m-dimension cell-complex. Let $\tilde{I}(\alpha) := I(\alpha)$ for $\alpha \in V(BCS(X))$. Then $(BCS(X), \tilde{I})$ is CS simplicial complex.

Definition 4. *Let K_1, K_2 be two triangulated m-manifolds. Let $\Delta_1, \Delta_1' \in K_1$ & $\Delta_2, \Delta_2' \in K_2$ be m-simplices. Let $\varphi : \Delta_1 \to \Delta_2$, $\psi : \Delta_1' \to \Delta_2'$ be bijections. If $\{\Delta_1, \Delta_1'\}$ is a pair of antineighbours in K_1 then the quotient K obtained from $(K_1 \backslash \{\Delta_1, \Delta_1'\}) \sqcup (K_2 \backslash \{\Delta_2, \Delta_2'\})$ by identify u with $\varphi(u)$ & v with $\psi(v)$ for $u \in \Delta_1$, $v \in \Delta_2$ is a simplicial complex. Clearly, $|K|$ is the space obtained from $|K_1 \#^{\varphi} K_2|$ by adding an 1-handle. We denote this K by $K_1 \text{\textcircled{\#}}^{\varphi\psi} K_2$ or simply by $K_1 \text{\textcircled{\#}} K_2$.*

We need the following lemma in the proof of Theorems 6 and 7.

Lemma 5. *Let $(K_i, I_i), i = 1, 2$, be two centrally symmetric triangulated m-manifolds. If there exists facet $\Delta_1 \in K_1$ such that $\{\Delta_1, I_1(\Delta_1)\}$ is a pair of antineighbours then for any facet $\Delta_2 \in K_2$ & any bijection $\varphi \colon \Delta_1 \mapsto \Delta_2$, there exists an involution I on $K_1 \text{\textcircled{\#}}^{\varphi\psi} K_2$, such that $(K_1 \text{\textcircled{\#}}^{\varphi\psi} K_2, I)$ is centrally symmetric, where $\psi = I_2 \circ \varphi \circ I_1$.*

Proof. Assume without loss that $I_1 = \prod_{i=1}^{n_1}(a_i, a_{2n_1-i+1})$ & $I_2 = \prod_{j=1}^{n_2}(b_j, b_{2n_2-j+1})$ and $\Delta_1 := \{a_1, a_2, \dots, a_{m+1}\}$ in K_1 and $\Delta_2 := \{b_1, b_2, \dots, b_{m+1}\}$ in K_2. We obtain $(m-1)$-spheres $\partial\Delta_1, \partial\Delta_2$ by removing interiors of Δ_1 and Δ_2. We identify $\partial\Delta_1$ with $\partial\Delta_2$ by the map $\varphi \colon a_s \to b_s$, $1 \leq s \leq m+1$. From the construction, $|K_1 \#^{\varphi} K_2|$ is the connected sum of the manifolds $|K_1|$ & $|K_2|$ and hence $K_1 \#^{\varphi} K_2$ is a triangulated m-manifold. Again, we identify $\partial I_1(\Delta_1)$ with $\partial I_2(\Delta_2)$ by the map $\psi \colon a_{2n_1-s} \to b_{2n_2-s}, 0 \leq s \leq m$. Hence, $|K_1 \text{\textcircled{\#}}^{\varphi\psi} K_2|$ can be obtain from $|K_1 \#^{\varphi} K_2|$ by an 1-handle addition. This implies $|K_1 \text{\textcircled{\#}}^{\varphi\psi} K_2|$ is a m-manifold and hence $K_1 \text{\textcircled{\#}}^{\varphi\psi} K_2$ is a triangulated m-manifold.

In $K_1 \text{\textcircled{\#}}^{\varphi\psi} K_2$, we identify a_i with b_i and the new vertices are denoted by b_i for $1 \leq i \leq m+1$. Let $K := K_1 \text{\textcircled{\#}}^{\varphi\psi} K_2$ & $I = \prod_{i=m+2}^{n_1}(a_i, a_{2n_1-i+1}) \prod_{j=1}^{n_2}(b_j, b_{2n_2-j+1})$.

Claim. The (K, I) is centrally symmetric.

For a m-simplex Δ of K, the simplex Δ or its subset, say Δ', which is a k-simplex of K belongs to K_1, K_2 or $K_1 \cap K_2$. If Δ belongs to K_1 or K_2 and $\Delta \cap I(\Delta) \neq \phi$, then by the definition of involution, $\Delta \cap I_i(\Delta) \neq \phi$ in K_i and which is a contradiction. If Δ belongs to $K_1 \cap K_2$ then both I_1 and I_2 fix the face Δ. This gives a contradiction as K_1 and K_2 are CST manifolds. We use the same argument for Δ'. Therefore, (K, I) is centrally symmetric. This proves the result. □

Theorem 6. *For each $g \geq 1$, there exists a centrally symmetric $8g$-vertex triangulated 2-manifold M_g of Euler characteristic $\chi(M_g) = 2 - 2g$.*

Proof. Let (T, I_T), (S, I_S) be the CS triangulated 2-manifolds as in Example 1. Then for any facet Δ in S, $\{\Delta, I_S(\Delta)\}$ is a pair of antineighbours. We prove the theorem by induction on g.

If $g = 1$ then T_8 serves the purpose. If $g = 2$ then we take $K_1 = S$ & $K_2 = T_8$ in Lemma 5. By Lemma 5, we get $M_2 = S \text{\textcircled{\#}} T_8$ a 16 $(= f_0(S) + f_0(T_8) - 6)$ vertex CS triangulated 2-manifold. Since $\chi(T_8) = 0$ & $\chi(S) = 2$, $\chi(M_2) = \chi(T_8) + \chi(S) - 4 = 0 + 2 - 4 = -2$. We continue with this construction and at g^{th} label, $M_g = S \text{\textcircled{\#}} M_{g-1}$. Clearly, $f_0(M_g) = f_0(S) + f_0(M_{g-1}) - 6 = 14 + 8(g-1) - 6 = 8g$, $f_1(M_g) = f_1(S) + f_1(M_{g-1}) - 6$ & $f_2(M_g) = f_2(S) + f_2(M_{g-1}) - 4$. These implies that $\chi(M_g) = 2 - 2g$. So, M_g is an $8g$-vertex CS triangulated 2-manifold and $\chi(M_g) = 2 - 2g$ for every $g \geq 1$. This proves the result. □

Let X and Y be two simplicial complexes with $V(X) \cap V(Y) = \emptyset$. The simplicial complex $X * Y := X \cup Y \cup \{\alpha \cup \beta \,:\, \alpha \in X, \beta \in Y\}$ is called the *join* of X and Y (see [1]).

It is not difficult to see that the triangulation $S_2^0 * \cdots * S_2^0$ ($d + 1$ copies) of $\mathbb{S}^d$ is a 2^{d+1}-vertex CS triangulation for all $d \geq 2$. In [5], Klee and Novik constructed $(2d + 4)$-vertex triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i}$ for $0 \leq i \leq d$ and for all d. Here we present infinitely many CS triangulated manifolds which are not triangulation of spheres or product of spheres.

Theorem 7. *There exist series of centrally symmetric triangulated m-manifolds which are not triangulation of spheres or product of spheres for each $m \geq 2$.*

Proof. Let $m \geq 2$. Let (M, I) and (M_1, I_1) be two CS triangulated m-manifolds. Assume that M has a facet Δ such that $\{\Delta, I(\Delta)\}$ is a pair of antineighbours. For example we can take $M = BCS(S^m_{m+2})$ or $BCS(S_2^0 * \cdots * S_2^0)$ or barycentric subdivision of Klee and Novik example, and $M_1 = BCS(S^m_{m+2})$ or $S_2^0 * \cdots * S_2^0$ or Klee and Novik example. Then, by Lemma 5, $M_2 := M \text{\textcircled{\#}} M_1$ is CS. Clearly, it is not a product of sphere or spheres. Inductively, let $M_k = M \text{\textcircled{\#}} M_{k-1}$ for $k \geq 2$. By Lemma 5, M_k is a CS triangulated m-manifolds for each $k \geq 2$. Thus, there are series of centrally symmetric triangulated manifolds. Moreover, if we consider these triangulated manifolds and apply Lemma 2 & Observation 3 then we get more centrally symmetric triangulated manifolds. This proves the theorem. □

3. Centrally symmetric maps on surfaces

Cycles in maps may or may not be homotopic to the generators of the fundamental group of the surface on which they lie. The cycles which are homotopic to a generator and non-genus-separating are called *non-trivial.* Those which are homotopic to a point are called *contractible cycles.* Those cycles which are homotopic to a generator and genus-separating are called *genus-separating.* See [12] for properties and results related to these topological cycles in maps on surfaces. If a map M contains a contractible cycle of even length or a non-trivial cycle then we can construct series of centrally symmetric maps from M.

Let L be a contractible cycle of even length in M, say $L = \partial D$, where D is a 2-disc in M. Take two copies of $(M \backslash D) \cup L$ and identify along L by the antipodal map on L. Let the resulting simplicial complex be $M\#^L M$. Clearly $|M\#^L M| = |M|\#|M|$.

Theorem 8. *Let M be a map.*

(a) *If M contains a non-trivial cycle then there exists a 2-fold covering $\tilde{M}$ of M, where $\tilde{M}$ is centrally symmetric.*
(b) *If M contains a contractible cycle L of even length then $M\#^L M$ is centrally symmetric.*

Proof. Let C be a non-trivial shortest cycle in M. The cycle C divides face-cycles of the vertices of C, that is, every sub-path u-v-$w \subset C$ of length two is a chord of the $face\text{-}cycle(v)$ at each vertex $v \in V(C)$. This gives that there are sequences $Y_1, Y_2, \ldots, Y_k$ and $Z_1, Z_2, \ldots, Z_\ell$ of faces incident with C on different sides of C. We cut M along the cycle C and, hence we get a map M_C which is bounded by two identical cycle C. We denote these boundary cycles by C_Y and C_Z where the faces $Y_1, Y_2, \ldots, Y_k$ are incident with C_Y and $Z_1, Z_2, \ldots, Z_\ell$ are incident with C_Z in M_C. Let $C_Y := C(u_1, \ldots, u_r)$ and $C_Z := C(w_1, \ldots, w_r)$. Then, C_Y identified with C_Z by the map $u_i \rightarrow w_i$ for all $1 \leq i \leq r$ in M, that is, $u_i = w_i$ for all i in M. So, $V(M_C) = V(M \backslash C) \cup V(C_Y) \cup V(C_Z)$ where $V(M \backslash C) = \{v_1, v_2, \ldots, v_m\}$. We consider another copy M'_C of M_C. Let $V(M'_C) = \{v'_1, v'_2, \ldots, v'_m\} \cup V(C'_Y) \cup V(C'_Z)$ and

$M'_C \cong M_C$ by $u' \to u \; \forall \, u' \in V(M'_C), u \in V(M_C)$. Then $\partial M'_C = C'_Y \; (= C(u'_1, \dots, u'_r)) \cup C'_Z \; (= C(w'_1, \dots, w'_r))$ where the faces $Y'_1, Y'_2, \dots, Y'_k$ are incident with C'_Y and $Z'_1, Z'_2, \dots, Z'_\ell$ are incident with C'_Z.

Since $f\colon M_C \cong M'_C$ by $v_j \to v'_j$, $u_i \to u'_i$ and $w_k \to w'_k$ for all i, j, k, it follows that $Y_i \mapsto Y'_i$ and $Z_j \mapsto Z'_j$ by the map f. We identify C_Y with C'_Z by the map $h_1\colon u_i \to w'_i$ and C_Z with C'_Y by the map $h_2\colon w_i \to u'_i$ for all $1 \leq i \leq r$. Hence, we get a map, namely, $\tilde{M} = M_C \textcircled{\#}^{h_1 h_2} M'_C$ of genus $g(\tilde{M}) = g(M_C) + g(M'_C) + 1$. Without loss, assume that the vertices w'_i are replaced by u_i and w_i are replaced by u'_i in $\tilde{M}$. Let $V(\tilde{M}) = \{v_1, \dots, v_m\} \cup \{v'_1, \dots, v'_m\} \cup \{u_1, \dots, u_r\} \cup \{u'_1, \dots, u'_r\}$ and $I_{\tilde{M}} := \prod_{i=1}^m (v_i, v'_i) \prod_{j=1}^r (u_j, u'_j)$.

Claim. The $(\tilde{M}, I_{\tilde{M}})$ is centrally symmetric.

Let $F = [x_{i_1}, \dots, x_{i_t}]$ be a facet in $\tilde{M}$. Then $I_{\tilde{M}}(F) = I_{\tilde{M}}([x_{i_1}, \dots, x_{i_t}]) = [x'_{i_1}, \dots, x'_{i_t}] = f([x_{i_1}, \dots, x_{i_t}])$ if $F \in F(\tilde{M}) \setminus \{\{Y_t, Y'_t\colon 1 \leq t \leq k\} \cup \{Z_s, Z'_s\colon 1 \leq s \leq \ell\}\}$, $I_{\tilde{M}}(Y_t) = Y'_t$ for $1 \leq t \leq k$, $I_{\tilde{M}}(Z_s) = Z'_s$ for $1 \leq s \leq \ell$, $I_{\tilde{M}}(Y'_t) = Y_t$ for $1 \leq t \leq k$ & $I_{\tilde{M}}(Z'_s) = Z_s$ for $1 \leq s \leq \ell$. Hence $I_{\tilde{M}}(I_{\tilde{M}}(F)) = F$ for all $F \in F(\tilde{M})$. So, $(\tilde{M}, I_{\tilde{M}})$ is centrally symmetric. This proves the claim, and hence Part (a).

The proof of Part (b) is similar to Lemma 5. In construction, consider boundary cycle $L = \partial D$ in place of boundary of simplex. The idea is as follows. Take two copies of $(M \setminus D) \cup L$, namely, $(M_1 \setminus D_1) \cup L_1$ & $(M_2 \setminus D_2) \cup L_2$. Let $V(M_1 \setminus D_1) = \{a_1, \dots, a_\ell\}$, $V(M_2 \setminus D_2) = \{b_1, \dots, b_\ell\}$, $L_1 = C(x_1, x_2, \dots, x_{2m})$ and $L_2 = (y_1, y_2, \dots, y_{2m})$ where $((M_1 \setminus D_1) \cup L_1) \cong ((M_2 \setminus D_2) \cup L_2)$ by $a_i \to b_i$ for all i and $x_j \to y_j$ for all j. Let $I_{L_1, L_2} := \prod_{i=1}^m (x_i, y_{m+i}) \prod_{i=m+1}^{2m} (x_i, y_{i-m})$. Then, $((M_1 \setminus D_1) \cup L_1) \#^{I_{L_1, L_2}} ((M_2 \setminus D_2) \cup L_2)$ is centrally symmetric by the involution $\prod_{t=1}^{\ell} (a_t, b_t) I_{L_1, L_2}$. This completes the proof. □

Corollary 9. *There are series of centrally symmetric semiequivelar maps of different genera.*

Proof. Let M be a semiequivelar map of type X. By the construction as in the proof of Theorem 8(a), the map $\tilde{M}$ is a semiequivelar map of type X. Again, consider $\tilde{M}$ and we get a 2-fold cover $\tilde{\tilde{M}}$ of $\tilde{M}$ of type X by Theorem 8(a). By repeating the construction as in the proof of Theorem 8(a), there is a series of centrally symmetric semiequivelar maps of type X from M of different genera. Here, we present an application of the construction on one example.

Let $K := \{a_1a_2a_3,\ a_1a_2a_{12},\ a_1a_{11}a_{12},\ a_1a_7a_9,\ a_1a_5a_9,\ a_1a_5a_{11},\ a_1a_3a_7,\ a_3a_4a_5,\ a_3a_5a_9,\ a_7a_8a_9,\ a_{10}a_{11}a_{12},\ a_4a_{10}a_{12},\ a_2a_3a_4,\ a_2a_4a_8,\ a_4a_8a_{12},\ a_6a_8a_{12},\ a_2a_6a_{12},\ a_2a_6a_{10},\ a_2a_8a_{10},\ a_8a_9a_{10},\ a_4a_6a_{10},\ a_4a_5a_6,\ a_5a_6a_7,\ a_6a_7a_8,\ a_5a_7a_{11},\ a_3a_7a_{11},\ a_3a_9a_{11},\ a_9a_{10}a_{11}\}$ (in [3, (Section 2, N_1)]) which is a semiequivelar map of type $[3^7]$ on the 2-torus. The cycle $L = C_3(a_2, a_{12}, a_4)$ in K is non-trivial. We cut K along L. Hence, we get a map Y with two boundary cycles C_1, C_2. We represent $(Y, C_{Y,1}, C_{Y,2})$ to be a map Y with two boundary cycles $C_{Y,1}, C_{Y,2}$. Let $(K_i, C_{K_i,1}, C_{K_i,2})$ for $i = 1, 2$ be two isomorphic copies of $(Y, C_{Y,1}, C_{Y,2})$, i.e., $K_i \cong Y, C_{K_i,1} \cong C_{Y,1}$, $C_{K_i,2} \cong C_{Y,2}$. Consider the map $Z := K_1 \textcircled{\#}^{g_1 g_2} K_2$ where $C_{K_1,1}$ identified with $C_{K_2,2}$ by $g_1\colon C_{K_1,1} \to C_{K_2,2}$ & $C_{K_2,1}$ identified with $C_{K_1,2}$ by $g_2\colon C_{K_2,1} \to C_{K_1,2}$ in Z. Let I_Z be an involution on $V(Z)$ by $K_1 \cong K_2, g_1\colon C_{K_1,1} \to C_{K_2,2}, g_2\colon C_{K_2,1} \to C_{K_1,2}$ as in the proof of Theorem 8(a). Clearly, (Z, I_Z) is a centrally symmetric map of type $[3^7]$ of genus 3. This Z is 2-fold cover of K. Again, consider two copies of Z and repeat the same construction as above with the same cycle. We repeat this process and each step, we get a centrally symmetric semiequivelar map of type $[3^7]$ with different genus.

From [2], we know that there are semiequivelar maps of type $[p^q]$ for each $[p^q]$ in $\{[3^7], [4^5], [4^6], [3^{3\ell-1}], [3^{3\ell}], [k^k]\colon \ell \geq 3, k \geq 5\}$. We consider these maps and apply above construction. Hence we get series of centrally symmetric semiequivelar maps. These prove the result. □

Lemma 10. *The dual of a centrally symmetric map is centrally symmetric.*

Proof. Let (M, I) be a centrally symmetric map. Let K denote the dual map of M. By the definition of duality, the map K has for its vertices the set of facets of M and two vertices of M are ends of an edge of M if the corresponding facets in M have an edge in common.

Let F be a facet of M. Then, $Orbit(F)$ contains exactly two disjoint facets under I. Let k denote the number of orbits of facets in M. Then, the map M contains $2k$ number of facets which is even. Let F_i for $i = 1, 2, \dots, 2k$ denote facets in M such that $I(F_i) = F_{2k+1-i}$ for all $1 \leq i \leq 2k$. Let u_i be the dual vertex of F_i in K. Let $I' := \prod_{i=1}^k (u_i, u_{2k+1-i})$ since $Orbit(F_i) = \{F_i, F_{2k-(i-1)}\} = Orbit(F_{2k-(i-1)})$.

Claim. The (K, I') is centrally symmetric.

Suppose there is a face F which is fix under the involution I', that is, $I'(F) = F$. Let the dual faces of $I'(F)$ and F be $I(X)$ and X in M respectively. By the definition of duality, $I(X) = X$. Hence, X is a fixed face in M under I. This shows that M is not centrally symmetric, a contradiction. Therefore, the map K is centrally symmetric under the involution I'. This proves the result. □

Theorem 11. *For any $p \geq 3$, there exist a series of centrally symmetric maps whose faces are p-gons.*

Proof. In [9], Lutz listed an enumerated results on vertex-transitive triangulations up to 15 vertices. This list contains d-regular triangulations for $3 \leq d \leq 12$. We take dual of these maps. Hence, we get a list of semiequivelar maps of all whose faces are p-gons for $3 \leq p \leq 12$. From these maps as in the Corollary 9, there are series of CS maps of all whose faces are p-gons for each $3 \leq p \leq 12$. Also, by Theorem 8(b), there are series of CS maps of all whose faces are p-gons for each $3 \leq p \leq 12$. Consider these series of maps and apply Lemma 10, hence, there are series of centrally symmetric maps with arbitrary faces.

One can also use the MANIFOLD_VT [8] to construct higher degree d (≥ 13) vertex-transitive triangulated maps and apply above arguments. So, there are series of CS maps of all whose faces are p-gons for many surfaces. Hence by Lemma 10, there are series of centrally symmetric maps with arbitrary faces. This proves the result. □

4. Enumeration of CST manifolds using computer

Theorem 12. *There are exactly* 6303 *centrally symmetric triangulated surfaces with at most* 12 *vertices. Out of these,* 1228 *are orientable and* 5075 *are non orientable.*

Proof. We present an enumeration of CST surfaces by a program which is modified version of MANIFOLD_VT [8] as follows. Lutz (in MANIFOLD_VT [8]) has used the cyclic group $Z_{2m} = \langle(1, 2, 3, \ldots, 2m)\rangle$ and the dihedral group $D_{2m} = \langle(1, 2, 3, \ldots, 2m), (1, 2m)(2, 2m-1)\cdots(m, m+1)\rangle$, and generated CS vertex transitive triangulated maps. We have replaced the groups Z_{2m}, D_{2m} by $\mathbb{Z}_2 = \langle I \colon I = (1, 2m)(2, 2m-1)\cdots(m, m+1)\rangle$ on the set $\{1, 2, \ldots, 2m\}$ and relaxed the criteria of vertex transitivity. It generates all the possible 1- and 2-orbits, that is, 1 and 2 dimensional orbits. We neglect those 2-orbits containing F and $I(F)$ for which $F \cap I(F) \neq \phi$. We also ignore those 1-orbits for which $e = I(e)$. The remaining orbits are called admissible orbits. Thus, for fixed $n = 2m$, we obtained all admissible 1- and 2-orbits under the group action $\mathbb{Z}_2$. In the process, we check link of m vertices namely $1, 2, \ldots, m$ which are use to define I. We also compute reduced homology groups to check orientability of the objects using [14]. Hence we get all possible non isomorphic CST maps.

As a result for $m = 3, 4$ and 5, we have listed the objects in *Table 1* [11]. For $m = 3$ the object 6_{tight} obtained in *Table 1* [11] is isomorphic to Lutz's object [7]. For $m = 4$ we get 4 objects out of which the object 8_{tight} in *Table 1* [11] is isomorphic to that of Lutz's object [7]. For $m = 6$, we give the number of non isomorphic objects for different genus in *Table 2* [11]. In this case and for $\chi = -8$, we give the list of all the objects in *Table 3* [11]. *Table 1* [11] gives the list of centrally symmetric triangulated surfaces for n $\leq$ 10 vertices. *Table 2* [11] gives number of different objects on 12 vertices. The total number of objects is 6303. It is clear from the tables by looking at homology groups that 1228 are orientable and 5075 are non orientable. □

Theorem 13. *There are exactly* 68 *centrally symmetric triangulated* 3*-manifolds on* 12 *vertices.*

Proof. Similarly as above, we have modified the program MANIFOLD_VT [8]. In this case, the modified program generates all possible 2- and 3-orbits. Let F_d be a d-orbit for $d \in \{2, 3\}$. We ignore those 3-orbit for which $F_3 \cap I(F_3) \neq \phi$. Also, we ignore those 2-orbit for which $F_2 \cap I(F_2) \neq \phi$. Hence, we get all possible admissible 2- and 3-orbits. We check link of m vertices namely $1, 2, \ldots, m$ which are used to define I. We also compute reduced homology groups of the objects using [14]. Hence we get all possible non isomorphic CST 3-manifolds.

As a result for $m = 6$, we have listed all possible 3-manifolds in *Table 4* [11]. *Table 4* [11] gives the list of centrally symmetric 3-manifolds on 12 vertices. The total number of objects is 68. By looking at homology groups we deduce that the objects are orientable and triangulation of homological $\mathbb{S}^2 \times \mathbb{S}^1$ (i.e. objects and $\mathbb{S}^2 \times \mathbb{S}^1$ have same reduced homology groups). □

Remark 14. We know from Tables 1, 2, 3 in [11] that the number of CS triangulated 2- & 3-manifolds is very large. Thus, we are not listing those here. These list of triangulated manifolds are available with the first author.

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