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On $G(\lambda)$-strictly pseudocontractive mapping in Hilbert spaces

F. U. Ogbuisi[1,2,3] and O. T. Mewomo[4]

[1] *School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa*
e-mail: 215082189@stu.ukzn.ac.za

[2] *DST-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), South Africa*

[3] *Department of Mathematics, University of Nigeria Nsukka, Nigeria*
e-mail: ferdinard.ogbuisi@unn.edu.ng, fudochukwu@yahoo.com

[4] *School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa*
e-mail: mewomoo@ukzn.ac.za

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Abstract. The purpose of this paper is to study $G(\lambda)$-strictly pseudocontractive mapping in a Hilbert space endowed with a directed graph. Moreover, we extend the results of Tiammee *et al.* [Tiammee *et al.* On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs, Fixed point theory and applications (2015) 2015:187 DOI 10.1186/s13663-015-0436-9] obtained for G-nonexpansive mappings to $G(\lambda)$-strictly pseudocontractive mapping in Hilbert spaces.

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1. Introduction

Let H be a real Hilbert space. A mapping $T : H \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \ \ \forall x, y \in H, \tag{1.1}$$

and $T : H \to H$ is said to be λ*-strictly pseudocontractive* if there exists $\lambda > 0$ such that,

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2, \ \ \forall x, y \in H, \tag{1.2}$$

or equivalently

$$\langle (I - T)x - (I - T)y, x - y\rangle \geq \lambda\|(I - T)x - (I - T)y\|^2, \ \ \forall x, y \in H. \tag{1.3}$$

It is obvious that (1.2) (and consequently (1.3)) for $\lambda \in \left(0, \frac{1}{2}\right)$, is identical to the inequality

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \ \ \forall x, y \in H, \tag{1.4}$$

where $k = 1 - 2\lambda < 1$. Let C be a nonempty, closed and convex subset of a Hilbert space H and let $T : C \to C$ be a mapping, then a point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T)$.

Let C be a convex subset of a Hilbert space H and $T : C \to C$, the sequence $\{x_n\}$ defined iteratively by $x_1 \in C$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \ n \geq 1, \tag{1.5}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0, 1] satisfying the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, is generally referred to as the Mann sequence in the light of [21]. Many authors have studied the approximation of fixed points of λ-strictly pseudocontractive mappings using the Mann iteration (1.5) (see, for example, [1,13–17,20,22,23,25,30–33] and the references therein). The Mann iteration (1.5) in general yield only weak convergence for approximating fixed points of nonexpansive mappings and to obtain strong convergence, the Mann iteration has to be modified.

In 2007, Marino and Xu [22] obtained weak convergence results using Mann iteration (1.5) for λ-strictly pseudocontractive mappings in Hilbert spaces and used the "CQ" algorithm to obtain the strong convergence for a finite family of λ-strictly pseudocontractive mappings.

Let $G = (V(G), E(G))$ be a directed graph where $V(G)$ is the set of vertices of the graph and $E(G)$ be a set of its edges. Assume that G has no parallel edges. We denote by G^{-1} the directed graph obtained from G by reversing the direction of edges. That is

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

If x and y are vertices in G, then a path in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^{n}$ of $n + 1$ vertices such that $x_0 = x, x_n = y, (x_{i-1}, x_i) \in G$ for $i = 1, 2, \ldots, n$. A graph G is connected if there is a (directed) path between any two vertices of G. The power behind using graphs instead of partial orders was first recognized by Jachymski in [19]. Jachymski [19] studied fixed point theory in a metric space endowed with directed graph, introduced the idea of G-contractions and extend the Banach contraction principle to metric space endowed with a directed graph.

Definition 1.1 ([19]). *Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains loops, i.e., $\Delta = \{(x, x) : x \in X\} \subseteq E(G)$. Then a mapping $f : X \to X$ is said to be a G-contraction if f preserves edges of G, i.e.,*

$$x, y \in X, \quad (x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G), \tag{1.6}$$

and there exists $\alpha \in (0, 1)$ such that for any $x, y \in X$,

$$(x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leq \alpha d(x, y).$$

Jachymski [19] further obtained the following theorem.

Theorem 1.2 ([19]). *Let (X, d) be a complete metric space and let the triple (X, d, G) be such that for any $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, there exists a subsequence $\{x_{n_k}\}$ with $(x_{n_k}, x) \in E(G)$ for $n \in \mathbb{N}$. Let f be a G-contraction, and $X_f = \{x \in X : (x, f(x)) \in E(G)\}$. Then $F(T) \neq \emptyset$ if and only if $X_f \neq \emptyset$.*

Many authors (for example see, [3–9,11,18,27]) have obtained results that improved and extended in various ways Theorem 1.2.

Definition 1.3 ([28]). *Let C be a nonempty convex subset of a Banach space X, $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $T : C \to C$. Then T is said to be G-nonexpansive if the following conditions hold:*

(1) *T is edge-preserving, i.e., for any $x, y \in C$ such that $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$;*
(2) *$\|Tx - Ty\| \leq \|x - y\|$, whenever $(x, y) \in E(G)$ for any $x, y \in C$.*

Tiammee *et al.* [28], proved Browder's convergence theorem and a strong convergence theorem with the Halpern iteration process for a G-nonexpansive mapping in a Hilbert space endowed with a directed graph. Precisely, they obtained the following two convergence theorems:

Theorem 1.4 ([28], Theorem 3.4). *Let C be a bounded closed convex subset of a Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. suppose C has property G and $T : C \to C$ is a G-nonexpansive mapping. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Define $T_n : C \to C$ by*

$$T_n x = (1 - \alpha_n)Tx + \alpha_n x_0,$$

for each $x \in C$ and $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\alpha_n \to 0$. Then the following holds:

(i) *T_n has a fixed point $u_n \in C$,*
(ii) *$F(T) \neq \emptyset$,*
(iii) *if $F(T) \times F(T) \subseteq E(G)$ and $P_{F(T)}x_0$ is dominated by $\{u_n\}$, then the sequence $\{u_n\}$ converges strongly to $w_0 = P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection onto $F(T)$.*

Theorem 1.5 ([28], Theorem 4.5). *Let C be a nonempty closed convex subset of a Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G)$ is convex and G is transitive. Suppose C has property G. Let $T : C \to C$ be a G-nonexpansive mapping. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Suppose that $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq E(G)$. Let $\{x_n\}$ be a sequence defined iteratively by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad n \geq 0, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ satisfies; (i) $\alpha_n \in [0, 1]$, (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (iii) $\lim_{n\to\infty} \alpha_n = 0$ and (iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. If $\{x_n\}$ is dominated by $P_{F(T)}x_0$ and $\{x_n\}$ dominates x_0, then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection on $F(T)$.

In this paper, we study $G(\lambda)$-strictly pseudocontractive mapping which is an important generalization of the G-nonexpansive mappings that have been recently considered by many authors. We further extended the results of Tiammee *et al.* [28] obtained for G-nonexpansive mappings to $G(\lambda)$-strictly pseudocontractive mappings.

2. Preliminaries

In this section, we present some important definitions and notable results that will be needed in the sequel.

Lemma 2.1 ([29]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\Sigma_{n=0}^{\infty}\gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\Sigma_{n=0}^{\infty}|\delta_n\gamma_n| < \infty$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. *Let H be a real Hilbert space. Then the following result holds*

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2,$$

and

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall\, x, y \in H.$$

Lemma 2.3. *Let H be a Hilbert space, then $\forall\, x, y \in H$ and $\alpha \in (0, 1)$, we have*

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$$

Lemma 2.4 ([2]). *Let X be a Hilbert space. For any $x, y \in X$, if $\|x + y\| = \|x\| + \|y\|$ then there exists $t \geq 0$ such that $y = tx$ or $x = ty$.*

Lemma 2.5. *Let X be a Banach space. Then X is reflexive if and only if every bounded sequence $\{x_n\}$ in X has a weakly convergent subsequence $\{x_{n_k}\}$.*

Definition 2.6. *Let C be a nonempty closed and convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that*

$$\|x - P_C x\| \leq \|x - y\| \quad \forall\, y \in C.$$

P_C is called the metric projection of H onto C and is characterized by the variational inequality:

$$\langle x - P_C x, z - P_C x\rangle \leq 0, \quad \forall\, z \in C.$$

Lemma 2.7. *Let H be a Hilbert space and let $\{x_n\}$ be a sequence of H with $x_n \rightharpoonup x$. If $x \neq y$, then*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Definition 2.8. *Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$, where $V(G) = C$ is a directed graph. C is said to have property G if every sequence $\{x_n\}$ in C converging weakly to $x \in C$ has a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

Definition 2.9. *Let C be a nonempty closed convex subset of a Hilbert space H and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then T is said to be G-monotone if $\langle Tx - Ty, x - y\rangle \geq 0$ whenever $(x, y) \in E(G)$ for any $x, y \in C$.*

Definition 2.10 ([10,26]). *Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a dominating set if for every $v \in V(G)\backslash X$ there exists $x \in X$ such that $(x, v) \in E(G)$ and we say that x dominates v or v is dominated by x. Let $v \in V$, then a set $X \subseteq V$ is dominated by v if $(v, x) \in E(G)$ for any $x \in X$ and we say that X dominates v if $(x, v) \in E(G)$ for all $x \in X$.*

In this paper, we will always assume $E(G)$ contains all loops.

3. Main results

In this section, we define $G(\lambda)$-strictly pseudocontractive mapping and prove a fixed point theorem for $G(\lambda)-$strictly pseudocontractive mapping in a Hilbert space. We will start by given the definition of $G(\lambda)-$strictly pseudocontractive mapping, an example of $G(\lambda)$-strictly pseudocontractive mapping, some properties of $G(\lambda)$-strictly pseudocontractive mapping and the structure of the fixed point set of $G(\lambda)$-strictly pseudocontractive mapping in Hilbert spaces.

Definition 3.1. *Let C be a nonempty closed and convex subset of a Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then a mapping $T : C \to C$ is $G(\lambda)$-strictly pseudocontractive if the following conditions hold:*

(1) *T is edge-preserving, i.e., for any $x, y \in C$ such that $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$;*
(2) *There exists $\lambda > 0$ such that $\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2$, whenever $(x, y) \in E(G)$ for any $x, y \in C$.*

Clearly, $I - T$ is G-monotone if T is $G(\lambda)$-strictly pseudocontractive.

Example 3.2. Let $X = C$ be the Banach space l_2 and let $G = (l_2, E(G))$ and $E(G) = \{(\{x_n\}, \{y_n\})\}$: for all $n \in \mathbb{N}, x_n, y_n \in \mathbb{Z}$ and $y_n = 2x_n, n \geq 1\}$. Define a mapping $T : l_2 \to l_2$ by

$$T(x_1, x_2, \ldots, x_i, \ldots) = \begin{cases} (-5x_1, -5x_2, -5x_3, \ldots) \text{ if } x_n \in \mathbb{Z} \text{ for all } n \in \mathbb{N}, \\ (5x_1, 5x_2, 5x_3, \ldots) \text{ if } x_n \notin \mathbb{Z} \text{ for some } n \in \mathbb{N}. \end{cases} \tag{3.1}$$

Clearly, T is edge preserving. Now let $\bar{x} = (x_1, x_2, x_3, \ldots)$ and $\bar{y} = (x_1, x_2, x_3, \ldots)$ be such that $\bar{x}, \bar{y} \in l_2$ and $(\bar{x}, \bar{y}) \in E(G)$.

Then

$$\begin{aligned} \|\bar{x} - T\bar{x} - (\bar{y} - T\bar{y})\|^2 &= \sum_{i=1}^{\infty} |x_i - Tx_i - (y_i - Ty_i)|^2 \\ &= \sum_{i=1}^{\infty} |6x_i - 6y_i|^2 \\ &= 36\sum_{i=1}^{\infty} |x_i - y_i|^2 \\ &= 36\|\bar{x} - \bar{y}\|^2, \end{aligned}$$

which implies

$$\|\bar{x}-\bar{y}\|^2 = \frac{1}{36}\|\bar{x}-T\bar{x}-(\bar{y}-T\bar{y})\|^2.$$

But

$$\begin{aligned}\|T\bar{x}-T\bar{y}\|^2 &= \sum_{i=1}^{\infty}|5x_i-5y_i|^2\\ &= 25\|\bar{x}-\bar{y}\|^2\\ &= \|\bar{x}-\bar{y}\|^2+24\|\bar{x}-\bar{y}\|^2\\ &= \|\bar{x}-\bar{y}\|^2+\frac{2}{3}\|\bar{x}-T\bar{x}-(\bar{y}-T\bar{y})\|^2. \end{aligned} \tag{3.2}$$

Thus, T is $G(\lambda)$-strictly pseudocontractive. Choose $x^* = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,\dots\right)$ and $y^* = \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},0,0,0,\dots\right)$, then

$$\begin{aligned}\|Tx^*-Ty^*\|^2 = 25 > 17 &= \|x^*-y^*\|^2+\|x^*-Tx^*-(y^*-Ty^*)\|^2\\ &> \|x^*-y^*\|^2+k\|x^*-Tx^*-(y^*-Ty^*)\|^2, \quad \forall\, k\in(0,1),\end{aligned}$$

which implies T is not λ-strictly pseudocontractive. Furthermore, for $(\bar{x},\bar{y})\in E(G)$,

$$\|T\bar{x}-T\bar{y}\| = 5\|\bar{x}-\bar{y}\| > \|\bar{x}-\bar{y}\|.$$

Hence, T is not G-nonexpansive.

Remark 3.3. Example 3.2 shows that there exists $G(\lambda)$-strictly pseudocontractive mappings which are neither λ-strictly pseudocontractive nor G-nonexpansive.

Proposition 3.4. *Let H be a Hilbert space and $G=(V(G),E(G))$ a directed graph with $V(G)=X$. Suppose $T: H\to H$ is a $G(\lambda)$-strictly pseudocontractive mapping. If H has Property G, then T is continuous.*

Proof. Let $\{x_n\}$ be a sequence in H such that $x_n\to x$, we show that $Tx_n\to Tx$.

Let $\{Tx_{n_j}\}$ be a subsequence of $\{Tx_n\}$, since $x_{n_j}\to x$, by property G, there is a subsequence $\{x_{m_j}\}$ such that $(x_{m_j},x)\in E(G)$ for $j\in\mathbb{N}$. Since T is $G(\lambda)$-strictly pseudocontractive and $(x_{m_j},x)\in E(G)$, we obtain

$$\begin{aligned}\|Tx_{m_j}-Tx\|^2 &\le \|x_{m_j}-x\|^2+k\|x_{m_j}-Tx_{m_j}-(x-Tx)\|^2\\ &\le \|x_{m_j}-x\|^2+k\|x_{m_j}-x\|^2+k\|Tx_{m_j}-Tx\|^2+2k\|x_{m_j}-x\|\|Tx_{m_j}-Tx\|,\end{aligned}$$

which implies

$$(1-k)\|Tx_{m_j}-Tx\|^2 \le (1+k)\|x_{m_j}-x\|^2+2k\|x_{m_j}-x\|\|Tx_{m_j}-Tx\|\to 0,\ j\to\infty.$$

By the double extract subsequence principle, we conclude that $Tx_n\to Tx$. Thus T is continuous. □

Theorem 3.5. *Let H be a Hilbert space and C be a subset of H having Property G. Let $G=(V(G),E(G))$ be a directed graph such that $V(G)=C$ and $E(G)$ is convex. Suppose $T: C\to C$ is a $G(\lambda)$-strictly pseudocontractive mapping, $F(T)\neq\emptyset$ and $F(T)\times F(T)\subseteq E(G)$. Then $F(T)$ is closed and convex.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x$. Since C has Property G, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $(x_{n_j}, x) \in E(G)$ for all $k \in \mathbb{N}$. Moreover, by T being $G(\lambda)$-strictly pseudocontractive, we obtain

$$\begin{aligned}
\|x - Tx\|^2 &= \|x - x_{n_j} + x_{n_j} - Tx\|^2 \\
&\leq \|x - x_{n_j}\|^2 + \|x_{n_j} - Tx\|^2 + 2\|x - x_{n_j}\|\|x_{n_j} - Tx\| \\
&= \|x - x_{n_j}\|^2 + \|Tx_{n_j} - Tx\|^2 + 2\|x - x_{n_j}\|\|x_{n_j} - Tx\| \\
&\leq \|x - x_{n_j}\|^2 + \|x_{n_j} - x\|^2 + k\|(I - T)x_{n_j} - (I - T)x\|^2 + 2\|x - x_{n_j}\|\|x_{n_j} - Tx\| \\
&= \|x - x_{n_j}\|^2 + \|x_{n_j} - x\|^2 + k\|x - Tx\|^2 + 2\|x - x_{n_j}\|\|x_{n_j} - Tx\|.
\end{aligned}$$

Therefore,

$$(1 - k)\|x - Tx\|^2 \leq 2\|x - x_{n_j}\|^2 + 2\|x - x_{n_j}\|\|x_{n_j} - Tx\| \to 0, \quad j \to \infty.$$

Thus, $x \in F(T)$ and so $F(T)$ is closed.

Next, we show that $F(T)$ is convex.

Let $x, y \in F(T)$ and $\alpha \in [0, 1]$, then $(x, x), (x, y) \in E(G)$. Set $z_\alpha = \alpha x + (1 - \alpha)y$, then since $E(G)$ is convex we have $(x, z_\alpha) = (\alpha x + (1 - \alpha)x, \alpha x + (1 - \alpha)y) \in E(G)$. Similarly, we have $(y, z_\alpha) \in E(G)$. Since T is $G(\lambda)$-strictly pseudocontractive, we get

$$\begin{aligned}
\|x - Tz_\alpha\|^2 &= \|Tx - Tz_\alpha\|^2 \\
&\leq \|x - z_\alpha\|^2 + k\|(I - T)x - (I - T)z_\alpha\|^2 \\
&= \|x - z_\alpha\|^2 + k\|z_\alpha - Tz_\alpha\|^2.
\end{aligned} \tag{3.3}$$

Similarly,

$$\|y - Tz_\alpha\|^2 \leq \|y - z_\alpha\|^2 + k\|z_\alpha - Tz_\alpha\|^2. \tag{3.4}$$

But,

$$\begin{aligned}
\|z_\alpha - Tz_\alpha\|^2 &= \|\alpha x + (1 - \alpha)y - Tz_\alpha\|^2 \\
&= \|\alpha(x - Tz_\alpha) + (1 - \alpha)(y - Tz_\alpha)\|^2 \\
&= \alpha\|x - Tz_\alpha\|^2 + (1 - \alpha)\|y - Tz_\alpha\|^2 - \alpha(1 - \alpha)\|x - y\|^2.
\end{aligned} \tag{3.5}$$

Substituting (3.3) and (3.4) into (3.5), we have

$$\begin{aligned}
\|z_\alpha - Tz_\alpha\|^2 &\leq \alpha\|x - z_\alpha\|^2 + \alpha k\|z_\alpha - Tz_\alpha\|^2 + (1 - \alpha)\|y - z_\alpha\|^2 \\
&\quad + (1 - \alpha)k\|z_\alpha - Tz_\alpha\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\
&= \alpha\|(1 - \alpha)(x - y)\| + (1 - \alpha)\|\alpha(y - x)\|^2 \\
&\quad + k\|z_\alpha - Tz_\alpha\|^2 - \alpha(1 - \alpha)\|x - y\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1 - k)\|z_\alpha - Tz_\alpha\|^2 &\leq \alpha(1 - \alpha)^2\|x - y\|^2 + (1 - \alpha)\alpha^2\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\
&= \alpha(1 - \alpha)[1 - \alpha + \alpha - 1]\|x - y\|^2 = 0.
\end{aligned}$$

Thus, $F(T)$ is convex. $\square$

Theorem 3.6. *Let C be a bounded closed and convex subset of a Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Suppose C has Property G and let $T : C \to C$ be a $G(\lambda)$-strictly pseudocontractive mapping. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Define $T_{n,\beta} : C \to C$ by*

$$T_{n,\beta}x = (1-\alpha_n)[(1-\beta)x + \beta Tx] + \alpha_n x_0$$

for each $x \in C$ and $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\alpha_n \to 0$ and $\beta \in (0, \min\{1, 2\lambda\})$. Then the following hold:

(i) *$T_{n,\beta}$ has a fixed point $u_n \in C$,*
(ii) *$F(T) \neq \emptyset$,*
(iii) *if $F(T) \times F(T) \subseteq E(G)$ and $P_{F(T)}x_0$ is dominated by $\{u_n\}$, then the sequence $\{u_n\}$ converges strongly to $w_0 = P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection onto $F(T)$.*

Proof.

(i) Let x_0 be such that $(x_0, Tx_0) \in E(G)$, we show that $T_{n,\beta}$ is a G-contraction for all $n \in \mathbb{N}$ and $\beta \in (0, \min\{1, 2\lambda\}) \subset (0, 1)$.

$$\begin{aligned}\|T_{n,\beta}x - T_{n,\beta}y\|^2 &= \|(1-\alpha_n)[(1-\beta)x + \beta Tx] - (1-\alpha_n)[(1-\alpha)y + \beta Ty]\|^2 \\ &= (1-\alpha_n)^2\|x - y - \beta[x - Tx - (y - Ty)]\|^2 \\ &= (1-\alpha_n)^2[\|x-y\|^2 - 2\beta\langle x - Tx - (y - Ty), x - y\rangle + \beta^2\|x - Tx - (y - Ty)\|^2] \\ &\leq (1-\alpha_n)^2[\|x-y\|^2 - 2\beta\lambda\|x - Tx - (y - Ty)\|^2 + \beta^2\|x - Tx - (y - Ty)\|^2] \\ &= (1-\alpha_n)^2[\|x-y\|^2 - \beta(2\lambda - \beta)\|x - Tx - (y - Ty)\|^2] \\ &\leq (1-\alpha_n)^2\|x-y\|^2. \end{aligned} \tag{3.6}$$

Therefore,

$$\|T_{n,\beta}x - T_{n,\beta}y\| \leq (1-\alpha_n)\|x-y\|.$$

Again, since T is edge preserving, $(Tx, Ty) \in E(G)$, for $x, y \in E(G)$, and we have

$$(T_{n,\beta}x, T_{n,\beta}y) = ((1-\alpha_n)[(1-\beta)x + \beta Tx] + \alpha_n x_0, (1-\alpha_n)[(1-\beta)y + \beta Ty] + \alpha_n x_0) \in E(G).$$

Therefore, $T_{n,\beta}$ is G-contraction. For any sequence $\{x_n\}$ in C such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, by Property G of C, there is a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$. Since $E(G)$ is convex and $(x_0, x_0) \in E(G)$, we obtain

$$(x_0, T_{n,\beta}x_0) = ((1-\alpha_n)[(1-\beta)x_0 + \beta x_0] + \alpha_n x_0, (1-\alpha_n)[(1-\beta)x_0 + \beta Tx_0] + \alpha_n x_0) \in E(G).$$

Therefore all conditions of Theorem 1.2 are satisfied, so $T_{n,\beta}$ has a fixed point u_n, i.e., $u_n = T_{n,\beta}u_n$.

(ii) We now show that $F(T) \neq \emptyset$. Since $\{u_n\}$ is bounded , there is a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup v$ for some $v \in C$. Suppose $Tv \neq v$. By Property G, without loss of generality, we may assume that $(u_{n_i}, v) \in E(G)$ for all $i \in \mathbb{N}$. First, we show that $u_n - Tu_n \to 0, n \to \infty$.

$$\begin{aligned}\|u_n - Tu_n\| &= \|(1-\alpha_n)[(1-\beta)u_n + \beta Tu_n] + \alpha_n x_0 - Tu_n\| \\ &\leq \|(1-\alpha_n)[(1-\beta)u_n + \beta Tu_n - (1-\alpha_n)[(1-\beta)Tu_n + \beta Tu_n]\| \\ &\quad + \|\alpha_n[(1-\beta)Tu_n + \beta Tu_n] - \alpha_n x_0\| \\ &= (1-\alpha_n)(1-\beta)\|u_n - Tu_n\| + \alpha_n\|[(1-\beta)Tu_n + \beta Tu_n] - x_0\|. \end{aligned} \tag{3.7}$$

Hence,

$$(1 - (1-\alpha_n)(1-\beta))\|u_n - Tu_n\| \leq \alpha_n\|[(1-\beta)Tu_n + \beta Tu_n] - x_0\|,$$

and taking lim sup, we have $\|u_n - Tu_n\| \to 0, n \to \infty$. Observe that

$$\begin{aligned}
\|u_{n_i} - Tv\|^2 &= \|u_{n_i} - Tu_{n_i} + Tu_{n_i} - Tv\|^2 \\
&\leq \|u_{n_i} - Tu_{n_i}\|^2 + \|Tu_{n_i} - Tv\|^2 + 2\|u_{n_i} - Tu_{n_i}\|Tu_{n_i} - Tv\| \\
&\leq \|u_{n_i} - Tu_{n_i}\|^2 + 2\|u_{n_i} - Tu_{n_i}\|Tu_{n_i} - Tv\| \\
&\quad + \|u_{n_i} - v\|^2 + k\|(I-T)u_{n_i} - (I-T)v\|^2 \\
&\leq \|u_{n_i} - Tu_{n_i}\|^2 + 2\|u_{n_i} - Tu_{n_i}\|Tu_{n_i} - Tv\| \\
&\quad + \|u_{n_i} - v\|^2 + \frac{k}{\lambda}\langle (I-T)u_{n_i} - (I-T)v, u_{n_i} - v\rangle \\
&\leq \|u_{n_i} - Tu_{n_i}\|^2 + 2\|u_{n_i} - Tu_{n_i}\|Tu_{n_i} - Tv\| \\
&\quad + \|u_{n_i} - v\|^2 + \frac{k}{\lambda}\|(I-T)u_{n_i}\|\|u_{n_i} - v\| - \frac{k}{\lambda}\langle (I-T)v, u_{n_i} - v\rangle.
\end{aligned}$$

Then from Lemma 2.7, we have

$$\begin{aligned}
\liminf_{n\to\infty} \|u_{n_i} - v\|^2 &< \liminf_{n\to\infty} \|u_{n_i} - Tv\|^2 \\
&\leq \liminf_{n\to\infty} [\|u_{n_i} - Tu_{n_i}\|^2 + 2\|u_{n_i} - Tu_{n_i}\|Tu_{n_i} - Tv\| \\
&\quad + \|u_{n_i} - v\|^2 + \frac{k}{\lambda}\|(I-T)u_{n_i}\|\|u_{n_i} - v\| - \frac{k}{\lambda}\langle (I-T)v, u_{n_i} - v\rangle] \\
&= \liminf_{n\to\infty} \|u_{n_i} - v\|^2,
\end{aligned}$$

which is a contradiction. Hence $Tv = v$.

(iii) Suppose that $F(T) \times F(T) \subseteq E(G)$ and $\{P_{F(T)}x_0\}$ is dominated by $\{u_n\}$, we show that $u_n \to w_0 = P_{F(T)}x_0$. Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$, we denote $v_i = u_{n_i}$. For each i, v_i is a fixed point of $T_{n_i,\beta}$. Hence, we have

$$\begin{aligned}
v_i &= (1-\alpha_{n_i})[(1-\beta)v_i + \beta Tv_i] + \alpha_{n_i}x_0 \\
&\Rightarrow (1-\alpha_{n_i})[(1-\beta)v_i + \beta v_i] + \alpha_{n_i}v_i = (1-\alpha_{n_i})[(1-\beta)v_i + \beta Tv_i] + \alpha_{n_i}x_0 \\
&\Rightarrow \alpha_{n_i}v_i + (1-\alpha_{n_i})\beta(v_i - Tv_i) = \alpha_{n_i}x_0.
\end{aligned} \tag{3.8}$$

Since w_0 is a fixed point of T, we have

$$\alpha_{n_i}w_0 + (1-\alpha_{n_i})\beta(w_0 - Tw_0) = \alpha_{n_i}w_0. \tag{3.9}$$

By subtracting (3.9) from (3.8) and taking the inner product of the difference with $v_i - w_0$, we obtain

$$\alpha_{n_i}\langle v_i - w_0, v_i - w_0\rangle + (1-\alpha_{n_i})\beta\langle (I-T)v_i - (I-T)w_0, v_i - w_0\rangle = \alpha_{n_i}\langle x_0 - w_0, v_i - w_0\rangle. \tag{3.10}$$

Again, since $P_{F(T)}x_0$ is dominated by $\{u_n\}$, we have $(v_i, w_0) \in E(G)$ for all $i \in \mathbb{N}$. Moreover, since T is $G(\lambda)$-strictly pseudocontractive, then $\langle (I-T)v_i - (I-T)w_0, v_i - w_0\rangle \geq 0$ for all $i \in \mathbb{N}$. Thus from (3.10), we have

$$\alpha_{n_i}\|v_i - w_0\|^2 \leq \alpha_{n_i}\langle x_0 - w_0, v_i - w_0\rangle.$$

Hence,

$$\begin{aligned}
\|v_i - w_0\|^2 &\leq \langle x_0 - w_0, v_i - w_0\rangle \\
&= \langle x_0 - w_0, v - w_0\rangle + \langle x_0 - w_0, v_i - v\rangle.
\end{aligned}$$

By the variational characterization of the metric projection, we know that $\langle x_0 - w_0, v - w_0\rangle \leq 0$ and thus we have

$$\|v_i - w_0\|^2 \leq \langle x_0 - w_0, v_i - v\rangle \to 0, \ i \to \infty.$$

Therefore, $v_i \to w_0 = P_{F(T)}x_0$ and by the double extract subsequence principle, we can conclude that $u_n \to w_0 = P_{F(T)}x_0$.

□

Proposition 3.7. *Let C be a convex subset of a vector space X and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$ and $E(G)$ convex. Let G be transitive and $T : C \to C$ be edge-preserving. Let $\{x_n\}$ be a sequence defined for initial $x_0 \in C$,*

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)[(1 - \beta)x_n + \beta T x_n], \tag{3.11}$$

where $\alpha_n \in [0, 1]$ and $\beta \in (0, 1)$. If $\{x_n\}$ dominates x_0 and $(x_0, Tx_0) \in E(G)$ then (x_n, x_{n+1}), (x_0, x_0) and (x_n, Tx_n) are in $E(G)$ for any $n \in \mathbb{N}$.

Proof. The proof is similar to the proof of Proposition 4.4 of Tiammee *et al.*, [28] and is omitted. □

Theorem 3.8. *Let C be a nonempty closed and convex subset of a Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G)$ is convex and G is transitive. Suppose C has property G. Let $T : C \to C$ be a $G(\lambda)-$strictly pseudocontractive. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Suppose that $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq E(G)$. Let $\{x_n\}$ be as in (3.11), where $\beta \in (0, \min\{1, 2\lambda\})$ and $\alpha_n \in [0, 1]$ satisfies the following condition:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iii) $\lim_{n\to\infty} \alpha_n = 0$.

If $\{x_n\}$ dominates $P_{F(T)}x_0$ and x_0, then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection on $F(T)$.

Proof. Let $z_0 = P_{F(T)}x_0$. From Proposition 3.7, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Since $z_0 \in F(T)$ and $z_0 = P_{F(T)}x_0$ is dominated by $\{x_n\}$, we have $(x_n, z_0) \in E(G)$ and

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n x_0 + (1 - \alpha_n)[(1 - \beta)x_n + \beta T x_n] - z_0\|^2 \\
&= \|\alpha_n(x_0 - z_0) + (1 - \alpha_n)[(1 - \beta)x_n + \beta T x_n - z_0]\|^2 \\
&= \alpha_n\|x_0 - z_0\|^2 + (1 - \alpha_n)\|(1 - \beta)x_n + \beta T x_n - z_0\|^2 \\
&\quad - \alpha_n(1 - \alpha_n)\|(1 - \beta)x_n + \beta T x_n - x_0\|^2 \\
&\leq \alpha_n\|x_0 - z_0\|^2 + (1 - \alpha_n)\|(1 - \beta)x_n + \beta T x_n - z_0\|^2.
\end{aligned} \tag{3.12}$$

But,

$$\begin{aligned}
\|(1 - \beta)x_n + \beta T x_n - z_0\|^2 &= \|(1 - \beta)(x_n - z_0) + \beta(T x_n - z_0)\|^2 \\
&= (1 - \beta)^2\|x_n - z_0\|^2 + \beta^2\|T x_n - z_0\|^2 + 2\beta(1 - \beta)\langle x_n - z_0, T x_n - z_0\rangle \\
&\leq (1 - \beta)^2\|x_n - z_0\|^2 + \beta^2[\|x_n - z_0\|^2 + k\|x_n - T x_n\|^2] \\
&\quad + 2\beta(1 - \beta)\left[\|x_n - z_0\|^2 - \frac{1 - k}{2}\|x_n - T x_n\|^2\right]
\end{aligned}$$

$$\begin{aligned}
&= (1-2\beta+\beta^2)\|x_n - z_0\|^2 + \beta^2[\|x_n - z_0\|^2 + k\|x_n - Tx_n\|^2] \\
&\quad + 2\beta\|x_n - z_0\|^2 - 2\beta^2\|x_n - z_0\|^2 - \beta(1-\beta)(1-k)\|x_n - Tx_n\|^2] \\
&= \|x_n - z_0\|^2 + \beta[k+\beta-1]\|x_n - Tx_n\|^2 \\
&\leq \|x_n - z_0\|^2. \qquad (3.13)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \alpha_n\|x_0 - z_0\|^2 + (1-\alpha_n)\|x_n - z_0\|^2 \\
&\leq \max\{\|x_0 - z_0\|^2, \|x_n - z_0\|^2\}.
\end{aligned}$$

Thus, $\{x_n\}$ is bounded and consequently, $\{Tx_n\}$ is bounded for all $n \in \mathbb{N}$. From (3.12) and (3.13), we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \alpha_n\|x_0 - z_0\|^2 + (1-\alpha_n)[\|x_n - z_0\|^2 + \beta(\beta-(1-k))\|x_n - Tx_n\|^2] \\
&\leq \alpha_n\|x_0 - z_0\|^2 + \|x_n - z_0\|^2 + \beta(\beta-(1-k))\|x_n - Tx_n\|^2. \qquad (3.14)
\end{aligned}$$

Now,

$$\begin{aligned}
\|(1-\beta)(x_n - x_{n-1}) + \beta(Tx_n - Tx_{n-1})\|^2 &= \|x_n - x_{n-1} - \beta(x_n - Tx_n - (x_{n-1} - Tx_{n-1}))\|^2 \\
&= \|x_n - x_{n-1}\|^2 + \beta^2\|x_n - Tx_n - (x_{n-1} - Tx_{n-1})\|^2 \\
&\quad - 2\beta\langle x_n - x_{n-1}, x_n - Tx_n - (x_{n-1} - Tx_{n-1})\rangle \\
&\leq \|x_n - x_{n-1}\|^2 + \beta^2\|x_n - Tx_n - (x_{n-1} - Tx_{n-1})\|^2 \\
&\quad - 2\beta\lambda\|x_n - Tx_n - (x_{n-1} - Tx_{n-1})\|^2 \\
&= \|x_n - x_{n-1}\|^2 - \beta(2\lambda-\beta)\|x_n - Tx_n - (x_{n-1} - Tx_{n-1})\|^2 \\
&\leq \|x_n - x_{n-1}\|^2, \qquad (3.15)
\end{aligned}$$

which implies

$$\|(1-\beta)(x_n - x_{n-1}) + \beta(Tx_n - Tx_{n-1})\| \leq \|x_n - x_{n-1}\|. \qquad (3.16)$$

Thus

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n x_0 + (1-\alpha_n)[(1-\beta)x_n + \beta Tx_n] - \alpha_n x_0 - (1-\alpha_{n-1})[(1-\beta)x_{n-1} + \beta Tx_{n-1}]\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|x_0 - (1-\beta)x_{n-1} - \beta Tx_{n-1}\| \\
&\quad + (1-\alpha_n)\|(1-\beta)(x_n - x_{n-1}) + \beta(Tx_n - Tx_{n-1})\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|x_0 - (1-\beta)x_{n-1} - \beta Tx_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}|D + (1-\alpha_n)\|x_n - x_{n-1}\|, \qquad (3.17)
\end{aligned}$$

where $D = \sup\{\|x_0 - (1-\beta)x_{n-1} - \beta Tx_{n-1}\| : n \in \mathbb{N}\}$.

But for $m, n \in \mathbb{N}$ and from (3.17), we have

$$\begin{aligned}
\|x_{n+m+1} - x_{n+m}\| &\leq \left(\sum_{i=m}^{n+m-1} |\alpha_{i+1} - \alpha_i|\right)D + \left(\prod_{i=m}^{n+m-1} |1 - \alpha_{i+1}|\right)\|x_{m+1} - x_m\| \\
&\leq \left(\sum_{i=m}^{n+m-1} |\alpha_{i+1} - \alpha_i|\right)D + \exp\left(-\sum_{i=m}^{n+m-1} \alpha_{i+1}\right)\|x_{m+1} - x_m\|. \qquad (3.18)
\end{aligned}$$

Since $\{x_n\}$ is bounded and $\sum_{i=0}^{\infty} \alpha_i = \infty$, we obtain

$$\limsup_{n\to\infty} \|x_{n+1} - x_n\| = \limsup_{n\to\infty} \|x_{n+m+1} - x_{n+m}\| \leq \left(\sum_{i=m}^{\infty} |\alpha_{i+1} - \alpha_i|\right) D, \quad \forall\, m \in \mathbb{N}.$$

Hence by $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Moreover,

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n x_0 + (1-\alpha_n)[(1-\beta)x_n + \beta Tx_n] - Tx_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \|x_0 - Tx_n\| + (1-\alpha_n)\|(1-\beta)x_n + \beta Tx_n - Tx_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \|x_0 - Tx_n\| + (1-\alpha_n)\|(1-\beta)(x_n - Tx_n)\|.
\end{aligned} \tag{3.19}$$

Therefore

$$(1 - [(1-\alpha_n)(1-\beta)])\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| + \alpha_n \|x_0 - Tx_n\| \to 0,\ n \to \infty, \tag{3.20}$$

Hence,

$$\|x_n - Tx_n\| \to 0, n \to \infty.$$

Since $\{x_n\}$ is bounded and H is a Hilbert space, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to $y \in C$.

We next show that $\limsup\langle x_n - z_0, x_0 - z_0\rangle \leq 0$. Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty}\langle x_n - z_0, x_0 - z_0\rangle = \lim_{n\to\infty}\langle x_{n_j} - z_0, x_0 - z_0\rangle$. Since C has Property G, then $(\{x_{n_j}\}, y) \in E(G)$. Thus, as in the proof of Theorem 3.6 (ii), we conclude that $y = Ty$.

Hence, by the variational characterization of the projection mapping, we get

$$\lim_{k\to\infty} \langle x_{n_j} - z_0, x_0 - z_0\rangle = \langle y - z_0, x_0 - z_0\rangle \leq 0.$$

Therefore, $\langle x_n - z_0, x_0 - z_0\rangle \leq 0$.

Furthermore,

$$(1-\alpha_n)[(1-\beta)x_n + \beta Tx_n - z_0] = (x_{n+1} - z_0) - \alpha_n(x_0 - z_0),$$

and

$$\begin{aligned}
\|(1-\alpha_n)[(1-\beta)x_n + \beta Tx_n - z_0]\|^2 &= \|x_{n+1} - z_0\|^2 + \alpha_n^2\|x_0 - z_0\|^2 - 2\alpha_n\langle x_{n+1} - z_0, x_0 - z_0\rangle \\
&\geq \|x_{n+1} - z_0\|^2 - 2\alpha_n\langle x_{n+1} - z_0, x_0 - z_0\rangle.
\end{aligned} \tag{3.21}$$

But,

$$\begin{aligned}
\|(1-\alpha_n)[(1-\beta)x_n + \beta Tx_n - z_0]\|^2 &= (1-\alpha_n)^2\|x_n - z_0 - \beta(x_n - Tx_n)\|^2 \\
&= (1-\alpha_n)^2[\|x_n - z_0\|^2 - 2\beta\langle x_n - Tx_n, x_n - z_0\rangle + \beta^2\|x_n - Tx_n\|^2] \\
&\leq (1-\alpha_n)^2[\|x_n - z_0\|^2 - 2\beta\lambda\|x_n - Tx_n\|^2 + \beta^2\|x_n - Tx_n\|^2] \\
&= (1-\alpha_n)^2[\|x_n - z_0\|^2 - \beta(\lambda - \beta)\|x_n - Tx_n\|^2 \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2.
\end{aligned} \tag{3.22}$$

Thus,

$$\begin{aligned}(1-\alpha_n)\|x_n - z_0\|^2 &\geq \|(1-\alpha_n)[(1-\beta)x_n + \beta T x_n - z_0\|^2 \\ &\geq \|x_{n+1} - z_0\|^2 - 2\alpha_n\langle x_{n+1} - z_0, x_0 - z_0\rangle. \end{aligned} \tag{3.23}$$

Hence,

$$\|x_{n+1} - z_0\|^2 \leq (1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle x_{n+1} - z_0, x_0 - z_0\rangle.$$

It then follows from Lemma 2.1, that $\lim_{n\to\infty}\|x_n - z_0\|^2 = 0$, i.e., $x_n \to z_0$. □

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