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On congruences involving special numbers

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Abstract. In this paper, using some special numbers and combinatorial identities, we show some interesting congruences: for a prime $p > 3$,

$$\sum_{k=0}^{(p-1)/2} \frac{C_k}{9^k(k+1)} \equiv \frac{2^{p-1}}{p}\left(\frac{L_{2p}}{3^{p-2}} - 9\right) - 5\left(\frac{5}{p}\right) + 9 \pmod{p},$$

$$\sum_{k=1}^{(p-1)/2} \binom{2k}{k}\frac{H_k^2}{3^k} \equiv -\frac{2}{3}\left(-\frac{1}{3}\right)^{(p-1)/2}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p},$$

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}\frac{F_k}{4^k(2k+1)} \equiv \frac{1}{2p}(F_{(1-3p)/2} - F_{(1+3p)/2}) \pmod{p^2},$$

where $B_n(x)$ is the Bernoulli polynomial, C_n, H_n, F_n and L_n are the nth Catalan number, the nth harmonic number, the nth Fibonacci number and the nth Lucas number, respectively. $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

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1. Introduction

The Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ are defined by the following recursions:

$$F_{n+1} = F_n + F_{n-1} \quad \text{and} \quad L_{n+1} = L_n + L_{n-1}, \quad n > 0,$$

where $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively. If α and β are the roots of equation $x^2 - x - 1 = 0$, the Binet formulas of the sequences $\{F_n\}$ and $\{L_n\}$ have the forms

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

respectively.

The Pell sequence $\{P_n\}$ and the Pell-Lucas sequence $\{Q_n\}$ are defined recursively by

$$P_{n+1} = 2P_n + P_{n-1} \quad \text{and} \quad Q_{n+1} = 2Q_n + Q_{n-1}, \quad n > 0,$$

in which $P_0 = 0$, $P_1 = 1$ and $Q_0 = Q_1 = 2$, respectively. If γ and δ are the roots of equation $x^2 - 2x - 1 = 0$, the Binet formulas of the sequences $\{P_n\}$ and $\{Q_n\}$ have the forms

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad Q_n = \gamma^n + \delta^n,$$

respectively.

The harmonic numbers have interesting applications in many fields of mathematics, such as number theory, combinatorics, analysis and computer science. Harmonic numbers H_n are defined as for $n > 0$

$$H_n = \sum_{k=1}^{n} \frac{1}{k},$$

where $H_0 = 0$. The first few harmonic numbers are $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \ldots$ and for $m \in \mathbb{N}$, harmonic numbers of order m are those rational numbers

$$H_{0,m} = 0, \quad H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}, \quad n > 0.$$

Some elementary combinatorial properties of the Catalan numbers are given in [3,4,9,15]. The Catalan numbers are given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}, \quad n \in \mathbb{N}.$$

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ are defined by

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n = 2, 3, 4, \ldots),$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad (n = 0, 1, 2, \ldots),$$

respectively [5].

For a prime p and an integer a with $p \nmid a$, we write the Fermat quotient $q_p(a) = (a^{p-1} - 1)/p$. For an odd prime p and an integer a, $\left(\frac{a}{p}\right)$ denotes the Legendre symbol defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

In [16], Z. W. Sun obtained that

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m(m-4)}{p}\right) \pmod{p},$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k(k+1)} \equiv \frac{m}{2} - \frac{m-4}{2}\left(\frac{m(m-4)}{p}\right) \pmod{p},$$

where p is an odd prime and m is any integer not divisible by p.

Let p be a fixed prime > 3. Define

$$q(x) = \frac{x^p - (x-1)^p - 1}{p} \quad \text{and} \quad G_n(x) = \sum_{k=1}^{p-1} \frac{x^k}{k^n},$$

where x is a variable.

In [2], A. Granville showed that

$$q(x) \equiv -G_1(x) \pmod{p},$$
$$G_2(x) \equiv G_2(1-x) + x^p G_2(1-1/x) \pmod{p},$$
$$q(x)^2 \equiv -2x^p G_2(x) - 2(1-x^p)G_2(1-x) \pmod{p}.$$

The author gave that for any integer $n > 1$

$$\sum_{k=1}^{n-1} \binom{n-1}{k-1} \frac{(-x)^k}{k^2} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{(1-x)^k - 1}{k} + \frac{(1-x)^n - (-x)^n - 1}{n^2}. \tag{1.1}$$

In [12], Z. H. Sun obtained the following congruences: for an odd prime p and $G_n(x) \in \mathbb{Z}_p[x]$,

$$G_2(x) \equiv \frac{1}{p}\left(\frac{1+(x-1)^p - x^p}{p} - \sum_{i=1}^{p-1} \frac{(1-x)^i - 1}{i}\right) + p\sum_{r=2}^{p-1} \frac{x^r}{r^2} \sum_{s=1}^{r-1} \frac{1}{s} \pmod{p^2},$$

and for a prime $p > 3$ and $n \in \mathbb{N}$,

$$npG_{n+1}(x) \equiv (-1)^n x^p G_n(1/x) - G_n(x) \pmod{p^2}.$$

In [18], Z. W. Sun showed that for a prime $p > 3$,

$$\sum_{k=1}^{p-1} \frac{H_k L_k}{k} \equiv 0 \pmod{p} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{H_k F_k}{k} \equiv \frac{2}{p} \sum_{k=1}^{p-1} \frac{F_k}{k} \pmod{p}. \tag{1.2}$$

In [7], H. Pan and Z. W. Sun obtained that for a prime $p > 5$,

$$\sum_{k=1}^{p-1} \frac{L_k}{k^2} \equiv 0 \pmod{p}, \tag{1.3}$$

and for a prime $p \neq 2, 5$,

$$\sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k} \equiv \left(\frac{p^a}{5}\right)\left(1 - 2F_{p^a - \left(\frac{p^a}{5}\right)}\right) \pmod{p^3},$$

where a is a positive integer.

In [1], J. Choi and H. M. Srivastava showed that for $m, n \in \mathbb{N}$,

$$\sum_{k=m}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right). \tag{1.4}$$

In [4], S. Koparal and N. Ömür gave that for $n > 1$ and $x \in \mathbb{R}$,

$$\sum_{k=1}^{n-1} \binom{n-1}{k-1} \frac{(-x)^k}{k} H_{k-1} = \frac{(1-x)^n - (-x)^n - 1}{n} H_{n-1} - \sum_{k=1}^{n-1} \frac{(1-x)^k - 1}{k(n-k)}. \tag{1.5}$$

In [10], J. Spieß obtained that

$$\sum_{k=0}^{n} \binom{k}{p} H_{n-k} = \binom{n+1}{p+1}(H_{n+1} - H_{p+1}), \tag{1.6}$$

$$\sum_{k=j}^{n-1} \binom{k-1}{j-1} \frac{H_{k-1}}{n-k} = \binom{n-1}{j-1}(H_{n-1}(H_{n-1} - H_{j-1}) - H_{n-1,2} + H_{j-1,2}), \tag{1.7}$$

and

$$\sum_{k=j}^{n-1}\frac{2}{n-k}\binom{k}{j}H_{n-k-1}=\binom{n}{j}((H_n-H_j)^2-H_{n,2}+H_{j,2}). \tag{1.8}$$

In [8], K. H. Pilehrood et all gave that for a prime $p\neq 2,5$,

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}F_{2k+1}}{(2k+1)16^k}\equiv(-1)^{(p+1)/2}\frac{F_p-(\frac{p}{5})}{p}\pmod{p^2},$$

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}L_{2k+1}}{(2k+1)16^k}\equiv(-1)^{(p+1)/2}\frac{L_p-1}{p}\pmod{p^2}.$$

In [6], S. Mattarei and R. Tauraso showed that for a prime $p>3$,

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k^2}\equiv 2(G_2(\lambda)+G_2(\mu))\pmod{p}, \tag{1.9}$$

where $\lambda=\frac{1}{2}(1+\sqrt{1-4x})$ and $\mu=\frac{1}{2}(1-\sqrt{1-4x})$.

In [19], R. Tauraso obtained that for a prime $p>3$,

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k}H_k\equiv 2(\lambda-\mu)^p(G_2(\lambda/(\lambda-\mu))-G_2(\mu/(\mu-\lambda)))\pmod{p}, \tag{1.10}$$

where λ and μ are defined as before.

2. Some congruences involving special numbers

In this section, we will give the congruences involving some special numbers. Firstly, we will give some auxilary lemmas involving harmonic numbers:

Lemma 1. *For $n>1$ and $x\in\mathbb{R}$, we have*

$$\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}H_{k-1}=H_n\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}+\frac{(1-x)^n-(-x)^n-1}{n^2}-\sum_{k=1}^{n-1}\binom{n}{k}\frac{(-x)^k}{k^2}.$$

Proof. From binomial theorem, we have

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}H_{k-1}&=\sum_{k=1}^{n-1}\frac{H_{k-1}}{k}\sum_{j=1}^{k}\binom{k}{j}(-x)^j\\&=\sum_{k=1}^{n-1}H_{k-1}\sum_{j=1}^{k}\binom{k-1}{j-1}\frac{(-x)^j}{j}\\&=\sum_{j=1}^{n-1}\frac{(-x)^j}{j}\sum_{k=j}^{n-1}\binom{k-1}{j-1}H_{k-1}.\end{aligned}$$

By taking $n-2$, $j-1$ instead of n, m in (1.4), respectively, we write

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}H_{k-1} &= \sum_{j=1}^{n-1}\frac{(-x)^j}{j}\binom{n-1}{j}\left(H_{n-1}-\frac{1}{j}\right)\\ &= \sum_{j=1}^{n-1}\frac{(-x)^j}{j}\left(\binom{n}{j}-\binom{n-1}{j-1}\right)\left(H_{n-1}-\frac{1}{j}\right)\\ &= \sum_{j=1}^{n-1}\frac{(-x)^j}{j}\left(\frac{n}{j}\binom{n-1}{j-1}-\frac{j}{n}\binom{n}{j}\right)\left(H_{n-1}-\frac{1}{j}\right)\\ &= n\sum_{j=1}^{n-1}\frac{(-x)^j}{j^2}\binom{n-1}{j-1}\left(H_{n-1}-\frac{1}{j}\right)-\frac{1}{n}\sum_{j=1}^{n-1}(-x)^j\binom{n}{j}\left(H_{n-1}-\frac{1}{j}\right)\\ &= nH_{n-1}\sum_{j=1}^{n-1}\frac{(-x)^j}{j^2}\binom{n-1}{j-1}-n\sum_{j=1}^{n-1}\frac{(-x)^j}{j^3}\binom{n-1}{j-1}\\ &\quad -\frac{H_{n-1}}{n}\sum_{j=1}^{n-1}\binom{n}{j}(-x)^j+\frac{1}{n}\sum_{j=1}^{n-1}\binom{n}{j}\frac{(-x)^j}{j},\end{aligned}$$

and from binomial theorem, we get

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}H_{k-1} &= (nH_{n-1}+1)\sum_{k=1}^{n-1}\frac{(-x)^k}{k^2}\binom{n-1}{k-1}-n\sum_{k=1}^{n-1}\frac{(-x)^k}{k^3}\binom{n-1}{k-1}\\ &\quad -H_{n-1}\frac{(1-x)^n-(-x)^n-1}{n}.\end{aligned}$$

With help of the sum in (1.1), we write

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}H_{k-1} &= \frac{nH_{n-1}+1}{n}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}+\frac{nH_{n-1}+1}{n^2}((1-x)^n-(-x)^n-1)\\ &\quad -n\sum_{k=1}^{n-1}\frac{(-x)^k}{k^3}\binom{n-1}{k-1}-H_{n-1}\frac{(1-x)^n-(-x)^n-1}{n}\\ &= H_n\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}+\frac{(1-x)^n-(-x)^n-1}{n^2}-\sum_{k=1}^{n-1}\frac{(-x)^k}{k^2}\binom{n}{k}.\end{aligned}$$

Thus, this ends the proof. □

Lemma 2. *For $n > 1$ and $x \in \mathbb{R}\backslash\{1\}$, we have*

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{n-k}H_{k-1} &= H_{n,2}(1-(1-x)^n)+H_{n-1}\sum_{k=1}^{n-1}\frac{(1-x)^{n-k}-1}{k}-\frac{1}{n}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}\\ &\quad -(1-x)^n\sum_{k=1}^{n}\binom{n}{k}\frac{H_k}{k}\left(\frac{x}{1-x}\right)^k.\end{aligned} \tag{2.1}$$

Proof. By (1.7), it is clearly that $\sum_{k=1}^{n-1}\frac{H_{n-k-1}}{k}=H_{n-1}^2-H_{n-1,2}$. Then

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{n-k}H_{k-1}&=\sum_{k=1}^{n-1}\frac{(1-x)^{n-k}-1}{k}H_{n-k-1}\\&=(1-x)^n\sum_{k=1}^{n-1}\frac{(1-x)^{-k}-1}{k}H_{n-k-1}+((1-x)^n-1)\sum_{k=1}^{n-1}\frac{H_{n-k-1}}{k}\\&=(1-x)^n\sum_{k=1}^{n-1}\frac{(1-x)^{-k}-1}{k}H_{n-k-1}+((1-x)^n-1)(H_{n-1}^2-H_{n-1,2}).\end{aligned}$$

So it suffices to show $\sum_{k=1}^{n-1}\frac{(1-x)^{-k}-1}{k}H_{n-k-1}$ which implies (2.1). From binomial theorem, we have

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^{-k}-1}{k}H_{n-k-1}&=\sum_{k=1}^{n-1}\left(\left(1-\frac{x}{x-1}\right)^k-1\right)\frac{H_{n-k-1}}{k}\\&=\sum_{k=1}^{n-1}\frac{H_{n-k-1}}{k}\sum_{j=1}^{k}\binom{k}{j}\left(\frac{x}{1-x}\right)^j\\&=\sum_{k=1}^{n-1}H_{n-k-1}\sum_{j=1}^{k}\binom{k-1}{j-1}\frac{1}{j}\left(\frac{x}{1-x}\right)^j\\&=\sum_{j=1}^{n-1}\frac{1}{j}\left(\frac{x}{1-x}\right)^j\sum_{k=j}^{n-1}\binom{k-1}{j-1}H_{n-k-1}.\end{aligned}$$

By binomial properties and replacing $n-2$ by n and $j-1$ by p in (1.6), respectively, we show

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^{-k}-1}{k}H_{n-k-1}&=\sum_{j=1}^{n-1}\frac{1}{j}\left(\frac{x}{1-x}\right)^j\binom{n-1}{j}(H_{n-1}-H_j)\\&=\sum_{j=1}^{n-1}\frac{1}{j}\left(\frac{x}{1-x}\right)^j\left(\binom{n}{j}-\binom{n-1}{j-1}\right)(H_{n-1}-H_j)\\&=\sum_{j=1}^{n-1}\frac{1}{j}\left(\frac{x}{1-x}\right)^j\left(\frac{n}{j}\binom{n-1}{j-1}-\frac{j}{n}\binom{n}{j}\right)(H_{n-1}-H_j)\\&=n\sum_{j=1}^{n-1}\frac{1}{j^2}\binom{n-1}{j-1}(H_{n-1}-H_j)\left(\frac{x}{1-x}\right)^j-\frac{1}{n}\sum_{j=1}^{n-1}\binom{n}{j}(H_{n-1}-H_j)\left(\frac{x}{1-x}\right)^j\\&=nH_{n-1}\sum_{j=1}^{n-1}\frac{1}{j^2}\binom{n-1}{j-1}\left(\frac{x}{1-x}\right)^j-n\sum_{j=1}^{n-1}\binom{n-1}{j-1}\frac{H_j}{j^2}\left(\frac{x}{1-x}\right)^j\\&\quad-\frac{H_{n-1}}{n}\sum_{j=1}^{n-1}\binom{n}{j}\left(\frac{x}{1-x}\right)^j+\frac{1}{n}\sum_{j=1}^{n-1}\binom{n}{j}H_j\left(\frac{x}{1-x}\right)^j.\end{aligned}$$

Substituting $\frac{x}{x-1}$ replace of x in (1.1) and (1.5), we have

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^{-k}-1}{k}H_{n-k-1} &= H_{n-1}\sum_{k=1}^{n-1}\frac{(1-x)^{-k}-1}{k}+\frac{H_{n-1}}{n}((1-x)^{-n}-x^n(1-x)^{-n}-1)\\ &\quad -\frac{H_{n-1}}{n}\sum_{k=1}^{n-1}\binom{n}{k}\left(\frac{x}{1-x}\right)^k-n\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{H_k}{k^2}\left(\frac{x}{1-x}\right)^k\\ &\quad +\frac{H_n}{n}((1-x)^{-n}-x^n(1-x)^{-n}-1)-\frac{1}{n}\sum_{k=1}^{n-1}\frac{(1-x)^{-k}-1}{n-k}.\end{aligned}$$

From binomial theorem, the desired result is given. □

Lemma 3. *For $n>1$ and $x\in\mathbb{R}\backslash\{1\}$, we have*

$$\begin{aligned}\sum_{k=1}^{n-1}\binom{n}{k}(-x)^kH_{k-1,2} &= -H_{n-1,2}(-x)^n-\frac{(1-x)^n-1}{n^2}-H_n\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}\\ &\quad -(1-x)^n\sum_{k=1}^{n}\binom{n}{k}\frac{H_k}{k}\left(\frac{x}{1-x}\right)^k+\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}H_{k-1}.\end{aligned}$$

Proof. From binomial theorem, we have

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k(n-k)}H_{k-1} &= \sum_{k=1}^{n-1}\frac{H_{k-1}}{k(n-k)}\sum_{j=1}^{k}\binom{k}{j}(-x)^j\\ &= \sum_{k=1}^{n-1}\frac{H_{k-1}}{n-k}\sum_{j=1}^{k}\binom{k-1}{j-1}\frac{(-x)^j}{j}\\ &= \sum_{j=1}^{n-1}\frac{(-x)^j}{j}\sum_{k=j}^{n-1}\binom{k-1}{j-1}\frac{H_{k-1}}{n-k}.\end{aligned}$$

By (1.7), we rewrite

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k(n-k)}H_{k-1} &= \sum_{k=1}^{n-1}\frac{(-x)^k}{k}\binom{n-1}{k-1}(H_{n-1}(H_{n-1}-H_{k-1})-H_{n-1,2}+H_{k-1,2})\\ &= \sum_{k=1}^{n-1}\frac{(-x)^k}{n}\binom{n}{k}(H_{n-1}(H_{n-1}-H_{k-1})-H_{n-1,2}+H_{k-1,2})\\ &= \frac{H_{n-1}^2-H_{n-1,2}}{n}\sum_{k=1}^{n-1}\binom{n}{k}(-x)^k-\frac{H_{n-1}}{n}\sum_{k=1}^{n-1}\binom{n}{k}(-x)^kH_{k-1}\\ &\quad +\frac{1}{n}\sum_{k=1}^{n-1}\binom{n}{k}(-x)^kH_{k-1,2}.\end{aligned}$$

Then, by binomial theorem and (1.5), we get

$$\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k(n-k)}H_{k-1}=(H_{n-1}^2-H_{n-1,2})\frac{(1-x)^n-(-x)^n-1}{n}$$
$$-H_{n-1}\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{(-x)^k}{k}H_{k-1}+\frac{1}{n}\sum_{k=1}^{n-1}\binom{n}{k}(-x)^kH_{k-1,2}$$
$$=(H_{n-1}^2-H_{n-1,2})\frac{(1-x)^n-(-x)^n-1}{n}-H_{n-1}^2\frac{(1-x)^n-(-x)^n-1}{n}$$
$$+H_{n-1}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k(n-k)}+\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{(-x)^k}{k}H_{k-1,2}.$$

So, we have

$$\sum_{k=1}^{n-1}\binom{n}{k}(-x)^kH_{k-1,2}=H_{n-1,2}((1-x)^n-(-x)^n-1)$$
$$+n\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k(n-k)}H_{k-1}-nH_{n-1}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k(n-k)}$$
$$=H_{n-1,2}((1-x)^n-(-x)^n-1)+\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}H_{k-1}+\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{n-k}H_{k-1}$$
$$-H_{n-1}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}-H_{n-1}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{n-k}.$$

With help of the sum in Lemma 2, the desired result is obtained. □

Theorem 1. *For $n \in$ and $x \in \mathbb{R}\backslash\{1\}$, we have*

$$\sum_{k=1}^{n}\binom{n}{k}H_k^2(-x)^k=(1-x)^n\sum_{k=1}^{n}\binom{n}{k}\left(\frac{H_{k-1}}{k}-\frac{1}{k^2}\right)\left(\frac{x}{1-x}\right)^k.$$

Proof. From binomial theorem, we have

$$2\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{n-k}H_{n-k-1}=\sum_{k=1}^{n-1}\frac{2}{n-k}H_{n-k-1}\sum_{j=1}^{k}\binom{k}{j}(-x)^j$$
$$=\sum_{j=1}^{n-1}(-x)^j\sum_{k=j}^{n-1}\frac{2}{n-k}\binom{k}{j}H_{n-k-1}.$$

By (1.8), we write

$$2\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{n-k}H_{n-k-1}=\sum_{k=1}^{n-1}\binom{n}{k}((H_n-H_k)^2-H_{n,2}+H_{k,2})(-x)^k$$
$$=(H_n^2-H_{n,2})\sum_{k=1}^{n-1}\binom{n}{k}(-x)^k-2H_n\sum_{k=1}^{n-1}\binom{n}{k}H_k(-x)^k$$
$$+\sum_{k=1}^{n-1}\binom{n}{k}H_k^2(-x)^k+\sum_{k=1}^{n-1}\binom{n}{k}H_{k,2}(-x)^k$$
$$=\sum_{k=1}^{n-1}\binom{n}{k}H_k^2(-x)^k+\sum_{k=1}^{n-1}\binom{n}{k}H_{k,2}(-x)^k$$
$$+(H_n^2-H_{n,2})((1-x)^n-(-x)^n-1)-2H_n\sum_{k=1}^{n-1}\binom{n}{k}H_k(-x)^k,$$

and from some arrangements,

$$\sum_{k=1}^{n-1}\binom{n}{k}H_k^2(-x)^k=2\sum_{k=1}^{n-1}\frac{(1-x)^{n-k}-1}{k}H_{k-1}-\sum_{k=1}^{n-1}\binom{n}{k}H_{k,2}(-x)^k$$
$$-(H_n^2-H_{n,2})((1-x)^n-(-x)^n-1)+2H_n\sum_{k=1}^{n-1}\binom{n}{k}H_k(-x)^k.$$

By (1.1) and (1.5), we write

$$\sum_{k=1}^{n-1}\binom{n}{k}H_k^2(-x)^k=2\sum_{k=1}^{n-1}\frac{(1-x)^{n-k}-1}{k}H_{k-1}-\sum_{k=1}^{n-1}\binom{n}{k}H_{k,2}(-x)^k$$
$$-(H_n^2-H_{n,2})((1-x)^n-(-x)^n-1)$$
$$+2H_n^2((1-x)^n-(-x)^n-1)-2H_n\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{n-k}.$$

With help of Lemmas 1 and 3, we rewrite

$$\sum_{k=1}^{n-1}\binom{n}{k}H_k^2(-x)^k=2\sum_{k=1}^{n-1}\frac{(1-x)^{n-k}-1}{k}H_{k-1}+(-x)^nH_{n,2}+(1-x)^n\sum_{k=1}^{n}\binom{n}{k}\frac{H_k}{k}\left(\frac{x}{1-x}\right)^k$$
$$+(H_n^2+H_{n,2})((1-x)^n-(-x)^n-1)-2H_n\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{n-k}.$$

If we replace x by $\frac{x}{x-1}$ in Lemma 1, we have

$$\sum_{k=1}^{n-1}\binom{n}{k}H_k^2(-x)^k=2(1-x)^n\sum_{k=1}^{n-1}\frac{(1-x)^{-k}-1}{k}H_{k-1}+2((1-x)^n-1)\sum_{k=1}^{n-1}\frac{H_{k-1}}{k}$$
$$+(1-x)^n\sum_{k=1}^{n}\binom{n}{k}\frac{H_k}{k}\left(\frac{x}{1-x}\right)^k-2H_n\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{n-k}$$
$$+(-x)^nH_{n,2}+(H_n^2+H_{n,2})((1-x)^n-(-x)^n-1)$$

$$= 2H_n(1-x)^n \sum_{k=1}^{n-1} \frac{(1-x)^{-k}-1}{k} + 2\frac{1-x^n-(1-x)^n}{n^2}$$
$$- 2(1-x)^n \sum_{k=1}^{n} \binom{n}{k}\frac{1}{k^2}\left(\frac{x}{1-x}\right)^k + 2((1-x)^n-1)\sum_{k=1}^{n-1}\frac{H_{k-1}}{k}$$
$$+ (1-x)^n \sum_{k=1}^{n} \binom{n}{k}\frac{H_k}{k}\left(\frac{x}{1-x}\right)^k - 2H_n\sum_{k=1}^{n-1}\frac{(1-x)^k}{n-k}$$
$$+ (-x)^n H_{n,2} + (H_n^2 + H_{n,2})((1-x)^n - (-x)^n - 1).$$

From $\sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2}(H_n^2 + H_{n,2})$ in [1], we have the conclusion. □

Now, we will give applications of Theorem 1.

Corollary 1. *Let $p > 3$ be a prime. For any rational number x such that $x \neq 1$ and $x^{-1} \equiv 0 \pmod p$,*

$$\sum_{k=1}^{p-1} H_k^2 x^k \equiv -(1-x)^{p-1}(G_2(1/(1-x)) + G_2(x/(x-1))) \pmod p.$$

Proof. If we take $p-1$ replace of n in Theorem 1, we have

$$\sum_{k=1}^{p-1}\binom{p-1}{k} H_k^2(-x)^k = (1-x)^{p-1}\sum_{k=1}^{p-1}\binom{p-1}{k}\left(\frac{H_{k-1}}{k}-\frac{1}{k^2}\right)\left(\frac{x}{1-x}\right)^k.$$

For $k = 0, 1, \dots p-1$, $\binom{p-1}{k} \equiv (-1)^k \pmod p$, we get

$$\sum_{k=1}^{p-1} H_k^2 x^k \equiv (1-x)^{p-1}\sum_{k=1}^{p-1}\left(\frac{H_{k-1}}{k}-\frac{1}{k^2}\right)\left(\frac{x}{x-1}\right)^k \pmod p.$$

By Lemma 1, it is shown that

$$\sum_{k=1}^{p-2}\frac{(1-x)^k}{k}H_{k-1} \equiv (1-x)^{p-1} - \sum_{k=1}^{p-1}\frac{x^k}{k^2} \pmod p.$$

Then

$$\sum_{k=1}^{p-1} H_k^2 x^k \equiv -(1-x)^{p-1}\sum_{k=1}^{p-1}\frac{1+(-x)^k}{k^2(1-x)^k}$$
$$= -(1-x)^{p-1}(G_2(1/(1-x)) + G_2(x/(x-1))) \pmod p.$$

The proof is completed. □

When $x = 2$ in Corollary 1, we obtain the following congruence: for $p > 3$,

$$\sum_{k=1}^{p-1} 2^k H_k^2 \equiv q_p(2)^2 \pmod p.$$

Corollary 2. *Let $p > 3$ be a prime. For any rational number x such that $x \neq 4^{-1}$ and $x^{-1} \equiv 0 \pmod{p}$,*

$$\sum_{k=1}^{(p-1)/2} \binom{2k}{k} H_k^2 x^k \equiv 2(\lambda-\mu)^{-1}(G_2(\lambda) - G_2(\mu))$$
$$- 4(\lambda-\mu)^{p-1}(G_2(\lambda/(\lambda-\mu)) + G_2(\mu/(\mu-\lambda))) \pmod{p},$$

where λ and μ are defined as before.

Proof. If we replace $\frac{p-1}{2}$ by n and $4x$ by x in Theorem 1, we obtain

$$\sum_{k=1}^{(p-1)/2} \binom{(p-1)/2}{k} H_k^2 (-4x)^k = (1-4x)^{(p-1)/2} \sum_{k=1}^{(p-1)/2} \binom{(p-1)/2}{k} \left(\frac{H_{k-1}}{k} - \frac{1}{k^2}\right)\left(\frac{4x}{1-4x}\right)^k$$
$$= (1-4x)^{(p-1)/2} \sum_{k=1}^{p-1} \binom{(p-1)/2}{k} \left(\frac{H_{k-1}}{k} - \frac{1}{k^2}\right)\left(\frac{4x}{1-4x}\right)^k.$$

For $k = 0, 1, \ldots, (p-1)/2$, $\binom{(p-1)/2}{k} \equiv \frac{1}{(-4)^k}\binom{2k}{k} \pmod{p}$, we write

$$\sum_{k=1}^{(p-1)/2} \binom{2k}{k} H_k^2 x^k \equiv (1-4x)^{(p-1)/2} \left(\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_k}{k} \left(\frac{x}{4x-1}\right)^k - 2\sum_{k=1}^{p-1} \binom{2k}{k} \frac{1}{k^2} \left(\frac{x}{4x-1}\right)^k\right) \pmod{p}.$$

From (1.9) and (1.10), we have the desired result. □

For example, when $x = 3^{-1}$ in Corollary 2, for a prime $p > 3$, we have

$$\sum_{k=1}^{(p-1)/2} \binom{2k}{k} \frac{H_k^2}{3^k} \equiv -\frac{2}{3}\left(-\frac{1}{3}\right)^{(p-1)/2} \left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p}.$$

Corollary 3. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1} H_k^2 F_{2k+1-p} \equiv 0 \pmod{p},$$

and for $p > 5$,

$$\sum_{k=1}^{p-1} H_k^2 L_{2k+1-p} \equiv 0 \pmod{p}. \tag{2.2}$$

Proof. We will give the proof of (2.2). From Theorem 1 and Binet formula of the Lucas sequence $\{L_n\}$, we have

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k+n} H_k^2 L_{2k-n}$$
$$= \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-n} H_k^2 \alpha^{2k-n} + \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-n} H_k^2 \beta^{2k-n}$$
$$= \sum_{k=1}^{n} \binom{n}{k} \frac{(-\alpha)^k}{k} H_k - 2\sum_{k=1}^{n} \binom{n}{k} \frac{(-\alpha)^k}{k^2} + \sum_{k=1}^{n} \binom{n}{k} \frac{(-\beta)^k}{k} H_k - 2\sum_{k=1}^{n} \binom{n}{k} \frac{(-\beta)^k}{k^2}$$
$$= \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k H_k L_k}{k} - 2\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k L_k}{k^2}.$$

Also for $n = p-1$, we write

$$\sum_{k=1}^{p-1}\binom{p-1}{k}(-1)^k H_k^2 L_{2k+1-p} = \sum_{k=1}^{p-1}\binom{p-1}{k}\frac{(-1)^k H_k L_k}{k} - 2\sum_{k=1}^{p-1}\binom{p-1}{k}\frac{(-1)^k L_k}{k^2}.$$

For $k = 0, \ldots, p-1$, $\binom{p-1}{k} \equiv (-1)^k \pmod p$, (1.2) and (1.3), the desired result is obtained. Similarly the other congruence is given. □

Now, we derive the Theorem 2 by evaluating the sum $S(n) = \sum_{k=0}^{n}\binom{n+k}{2k}\frac{(4x)^k}{2k+1}$ using technical manipulations on sums. By Zeilberger's algorithm, we obtain the recurrence relation

$$(2n+1)S(n) - (4x+2)(2n+3)S(n+1) + (2n+5)S(n+2) = 0.$$

Theorem 2. *For $n \in \mathbb{N}$ and $x \in \mathbb{R}\backslash\{0\}$, we have*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(4x)^k}{2k+1} = \frac{1}{2n+1}\frac{(\sqrt{x+1}+\sqrt{x})^{2n+1} - (\sqrt{x+1}-\sqrt{x})^{2n+1}}{2\sqrt{x}}.$$

Proof. For $x = -1$, Z.W. Sun[17] have the identity

$$\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(-4)^k}{2k+1} = \frac{(-1)^n}{2n+1}. \tag{2.3}$$

For $x \neq -1$, we can show the proof. By [11], it is known that

$$\binom{n+k}{2k}\frac{2n+1}{2k+1}4^k(-1)^n = \sum_{j=k}^{n}\binom{n+j}{2j}\binom{j}{k}(-4)^j.$$

Therefore

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(4x)^k}{2k+1} &= \frac{(-1)^n}{2n+1}\sum_{k=0}^{n}x^k\sum_{j=k}^{n}\binom{n+j}{2j}\binom{j}{k}(-4)^j \\
&= \sum_{j=0}^{n}\frac{(-1)^{n-j}4^j}{2n+1}\binom{n+j}{2j}\sum_{k=0}^{j}\binom{j}{k}x^k \\
&= \sum_{j=0}^{n}\frac{(-1)^{n-j}4^j}{2n+1}\binom{n+j}{2j}(1+x)^j \\
&= \sum_{j=0}^{n}\frac{(-1)^{n-j}4^j(1+x)^j}{2n+1}\binom{n+j}{n-j} \\
&= \sum_{j=0}^{n}\frac{(-1)^j(4(1+x))^{n-j}}{2n+1}\binom{2n-j}{j} \\
&= \frac{(1+x)^n}{2n+1}\sum_{j=0}^{n}\left(-\frac{1}{1+x}\right)^j 2^{2n-2j}\binom{2n-j}{j}.
\end{aligned}$$

By applying the equality $\sum_{k=j}^{n/2}\binom{n+1}{2k+1}\binom{k}{j}=2^{n-2j}\binom{n-j}{j}$ in [11], we obtain from the above that

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(4x)^k}{2k+1}&=\frac{(1+x)^n}{2n+1}\sum_{j=0}^{n}\left(-\frac{1}{1+x}\right)^j\sum_{i=j}^{n}\binom{2n+1}{2i+1}\binom{i}{j}\\&=\frac{(1+x)^n}{2n+1}\sum_{i=0}^{n}\binom{2n+1}{2i+1}\sum_{j=0}^{i}\binom{i}{j}\left(-\frac{1}{1+x}\right)^j\\&=\frac{(1+x)^n}{2n+1}\sum_{i=0}^{n}\binom{2n+1}{2i+1}\left(1-\frac{1}{1+x}\right)^i\\&=\frac{1}{2n+1}\sum_{i=0}^{n}\binom{2n+1}{2i+1}x^i(1+x)^{n-i}\\&=\sum_{i=0}^{n}\binom{2n}{2i}\frac{x^i(1+x)^{n-i}}{2i+1}.\end{aligned}$$

Taking $\frac{x}{1+x}$ and $2n+1$ instead of x and n in $\sum_{i=0}^{(n-1)/2}\binom{n}{2i+1}x^i=\frac{(1+\sqrt{x})^n-(1-\sqrt{x})^n}{2\sqrt{x}}$ [11], respectively, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(4x)^k}{2k+1}&=\frac{(1+x)^n}{2n+1}\sum_{i=0}^{n}\binom{2n+1}{2i+1}(1+x)^{-i}x^i\\&=\frac{(1+x)^n}{2n+1}\frac{\left(1+\sqrt{\frac{x}{x+1}}\right)^{2n+1}-(1-\sqrt{\frac{x}{x+1}})^{2n+1}}{2\sqrt{\frac{x}{x+1}}}\\&=\frac{1}{2n+1}\frac{(\sqrt{x+1}+\sqrt{x})^{2n+1}-(\sqrt{x+1}-\sqrt{x})^{2n+1}}{2\sqrt{x}}.\end{aligned}\tag{2.4}$$

Combining (2.3) and (2.4), we have the proof of desired result. □

Corollary 4. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{F_k}{4^k(2k+1)}\equiv\frac{1}{2p}(F_{(1-3p)/2}-F_{(1+3p)/2})\pmod{p^2},\tag{2.5}$$

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{L_k}{4^k(2k+1)}\equiv\frac{1}{2p}(L_{(1+3p)/2}-L_{(1-3p)/2})\pmod{p^2},$$

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{P_k}{8^k(2k+1)}\equiv\frac{1}{4p}((-1)^{(p-1)/2}Q_{(p-1)/2}-Q_{(p+1)/2})\pmod{p^2},$$

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{Q_k}{8^k(2k+1)}\equiv\frac{2}{p}((-1)^{(p-1)/2}P_{(p-1)/2}+P_{(p+1)/2})\pmod{p^2}.$$

Proof. For the proof (2.5), considering $x = -\alpha$ and $x = -\beta$ in Theorem 2, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(-4)^k}{2k+1}F_k &= \frac{1}{\alpha-\beta}\left(\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(-4\alpha)^k}{2k+1}-\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(-4\beta)^k}{2k+1}\right)\\ &= \frac{1}{2n+1}\frac{1}{\alpha-\beta}\left(\frac{(\sqrt{\beta}+1/\sqrt{\beta})^{2n+1}-(\sqrt{\beta}-1/\sqrt{\beta})^{2n+1}}{2/\sqrt{\beta}}\right.\\ &\qquad\left.-\frac{(\sqrt{\alpha}+1/\sqrt{\alpha})^{2n+1}-(\sqrt{\alpha}-1/\sqrt{\alpha})^{2n+1}}{2/\sqrt{\alpha}}\right)\\ &= \frac{1}{2(2n+1)}\left(\frac{\beta^{3n+2}-(-\alpha)^{3n+1}-\beta^{3n+2}+(-\alpha)^{3n+1}}{\alpha-\beta}\right)\\ &= \frac{1}{2(2n+1)}((-1)^nF_{3n+1}-F_{3n+2}).\end{aligned}$$

For $k = 0, \ldots, (p-1)/2$, $\binom{(p-1)/2+k}{2k}(-16)^k \equiv \binom{2k}{k} \pmod{p^2}$, the desired result is obtained. Similarly the other congruences are given. □

Now, we have the following results of Theorem 2:

Corollary 5. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{\lfloor\frac{p}{3}\rfloor} C_k^{(2)}F_k\left(\frac{4}{27}\right)^k \equiv \frac{3}{4(p-r)+6}(F_{-p+r-1}-F_{p-r+2}) \pmod{p},$$

$$\sum_{k=0}^{\lfloor\frac{p}{3}\rfloor} C_k^{(2)}L_k\left(\frac{4}{27}\right)^k \equiv \frac{3}{4(p-r)+6}(L_{p-r+2}-L_{-p+r-1}) \pmod{p},$$

$$\sum_{k=0}^{\lfloor\frac{p}{3}\rfloor} C_k^{(2)}P_k\left(\frac{2}{27}\right)^k \equiv \frac{3}{4(2p-2r+3)}((-1)^{(p-r)/3}Q_{(p-r)/3}-Q_{(p-r)/3+1}) \pmod{p},$$

$$\sum_{k=0}^{\lfloor\frac{p}{3}\rfloor} C_k^{(2)}Q_k\left(\frac{2}{27}\right)^k \equiv \frac{6}{2(p-r)+3}((-1)^{(p-r)/3}P_{(p-r)/3}+P_{(p-r)/3+1}) \pmod{p},$$

where $C_n^{(2)} = \frac{1}{2n+1}\binom{3n}{n}$ and $r = 1$ or 2 according as $3|p-r$.

Proof. For $k = 0, \ldots, \frac{p-r}{3}$,

$$\binom{(p-r)/3+k}{2k}(-27)^k \equiv \binom{3k}{k} \pmod{p}$$

where $r = 1$ or 2 according as $3|p-1$ or $3|p-2$ in [13], the proof is completed. □

Corollary 6. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{\lfloor\frac{p}{4}\rfloor} \frac{C_{2k}F_k}{16^k} \equiv \frac{1}{p-s+2}(F_{-3(p-s)/4-1}-F_{3(p-s)/4+2}) \pmod{p},$$

$$\sum_{k=0}^{\lfloor\frac{p}{4}\rfloor}\frac{C_{2k}L_k}{16^k}\equiv\frac{1}{p-s+2}(L_{3(p-s)/4+2}-L_{-3(p-s)/4-1})\pmod p,$$

$$\sum_{k=0}^{\lfloor\frac{p}{4}\rfloor}\frac{C_{2k}P_k}{32^k}\equiv\frac{1}{2(p-s+2)}((-1)^{(p-s)/4}Q_{(p-s)/4}-Q_{(p-s)/4+1})\pmod p,$$

$$\sum_{k=0}^{\lfloor\frac{p}{4}\rfloor}\frac{C_{2k}Q_k}{32^k}\equiv\frac{4}{p-s+2}((-1)^{(p-s)/4}P_{(p-s)/4}+P_{(p-s)/4+1})\pmod p,$$

where C_{2n} is the Catalan number and $s=1$ or 3 according as $4|p-s$.

Proof. For $k=0,\dots,\frac{p-s}{4}$,

$$\binom{(p-s)/4+k}{2k}(-64)^k\equiv\binom{4k}{2k}\pmod p,$$

where $s=1$ or 3 according as $4|p-1$ or $4|p-3$ in [14], we have the proof. □

Corollary 7. *Let $p>5$ be a prixme. Then*

$$\sum_{k=0}^{\lfloor\frac{p}{6}\rfloor}\binom{6k}{3k}\binom{2k}{k}^{-1}\frac{C_k^{(2)}F_k}{108^k}\equiv\frac{3}{2(p-t+3)}(F_{-(p-t)/2-1}-F_{(p-t)/2+2})\pmod p,$$

$$\sum_{k=0}^{\lfloor\frac{p}{6}\rfloor}\binom{6k}{3k}\binom{2k}{k}^{-1}\frac{C_k^{(2)}L_k}{108^k}\equiv\frac{3}{2(p-t+3)}(L_{(p-t)/2+2}-L_{-(p-t)/2-1})\pmod p,$$

$$\sum_{k=0}^{\lfloor\frac{p}{6}\rfloor}\binom{6k}{3k}\binom{2k}{k}^{-1}\frac{C_k^{(2)}P_k}{6^{3k}}\equiv\frac{3}{4(p-t+3)}((-1)^{(p-t)/6}Q_{(p-t)/6}-Q_{(p-t)/6+1})\pmod p,$$

$$\sum_{k=0}^{\lfloor\frac{p}{6}\rfloor}\binom{6k}{3k}\binom{2k}{k}^{-1}\frac{C_k^{(2)}Q_k}{6^{3k}}\equiv\frac{6}{p-t+3}((-1)^{(p-t)/6}P_{(p-t)/6}+P_{(p-t)/6+1})\pmod p,$$

where $C_n^{(2)}$ is defined as before and $t=1$ or 5 according as $6|p-t$.

Proof. For $k=0,\dots,\frac{p-t}{6}$,

$$\binom{(p-t)/6+k}{2k}(-432)^k\equiv\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}^{-1}\pmod p,$$

where $t=1$ or 5 according as $6|p-1$ or $6|p-5$, the proof is completed. □

Theorem 3. *Let $p>3$ be a prime. For any rational number x such that $x^{-1}\equiv 0\pmod p$,*

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{(k+1)}x^{k+1}\equiv 1-(\lambda-\mu)^{p+1}+\frac{2^{p-1}}{p}(\lambda^p+\mu^p-1)\pmod p, \tag{2.6}$$

where λ and μ are defined as before.

Proof. Replacing n and x by $\frac{p+1}{2}$ and $4x$ in (1.1), respectively, we have

$$\frac{p+1}{2}\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}\frac{(-4x)^{k+1}}{(k+1)^2}$$
$$=\sum_{k=1}^{(p-1)/2}\frac{(1-4x)^k}{k}+2\frac{(1-4x)^{(p+1)/2}-1}{p+1}-H_{(p-1)/2}.$$

For $k=1,\ldots,(p-1)/2$, $\binom{(p-1)/2}{k}(-4)^k\equiv\binom{2k}{k} \pmod p$, $H_{(p-1)/2}\equiv -2q_p(2) \pmod p$ and for $k=1,\ldots,p-1$, $\frac{1}{p+k}\equiv\frac{1}{k} \pmod p$, we write

$$-2\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{x^{k+1}}{(k+1)^2}\equiv 2((1-4x)^{(p+1)/2}-1)+\sum_{k=1}^{(p-1)/2}\frac{(1-4x)^k}{k}+2q_p(2) \pmod p. \tag{2.7}$$

For $k=0,1,\ldots,p-1$, $\binom{p-1}{k}\equiv(-1)^k \pmod p$, we get

$$\sum_{k=1}^{(p-1)/2}\frac{(1-4x)^k}{k}-\sum_{k=1}^{(p-1)/2}\binom{p-1}{2k-1}\frac{(1-4x)^k}{k}=-\frac{2}{p}\sum_{k=1}^{(p-1)/2}\binom{p}{2k}(1-4x)^k$$
$$=\frac{2-(1+\sqrt{1-4x})^p-(1-\sqrt{1-4x})^p}{p} \pmod p. \tag{2.8}$$

Substituting (2.8) into (2.7), we have

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{(k+1)}x^{k+1}=\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{x^{k+1}}{(k+1)^2}$$
$$\equiv 1-(1-4x)^{(p+1)/2}-q_p(2)-\frac{(\sqrt{1-4x}-1)^p-(\sqrt{1-4x}+1)^p+2}{2p}$$
$$=1-(\lambda-\mu)^{p+1}+\frac{2^{p-1}}{p}(\lambda^p+\mu^p-1) \pmod p.$$

Thus, this concludes the proof. □

From Theorem 3, we immediately deduce the following results.

Corollary 8. *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{4^k(k+1)}\equiv 4(1-q_p(2)) \pmod p,$$

$$\sum_{k=0}^{(p-1)/2}\frac{(-1)^kC_k}{k+1}\equiv 2^{p-1}\frac{1-L_p}{p}+5\left(\frac{5}{p}\right)-1 \pmod p,$$

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{8^k(k+1)}\equiv 2^{(7-p)/2}\frac{P_p}{p}-4\left(\frac{2}{p}\right)+8-\frac{2^{p+2}}{p} \pmod p,$$

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{(-4)^k(k+1)}\equiv -\frac{2Q_p}{p}+\frac{2^{p+1}}{p}+8\left(\frac{2}{p}\right)-4 \pmod p,$$

$$\sum_{k=0}^{(p-1)/2} \frac{C_k}{(-16)^k(k+1)} \equiv -\frac{L_{3p}}{2^{p-3}p} + \frac{2^{p+3}}{p} + 20\left(\frac{5}{p}\right) - 16 \pmod p$$

and for $p > 3$

$$\sum_{k=0}^{(p-1)/2} \frac{C_k}{9^k(k+1)} \equiv \frac{2^{p-1}}{p}\left(\frac{L_{2p}}{3^{p-2}} - 9\right) - 5\left(\frac{5}{p}\right) + 9 \pmod p.$$

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