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On a complex sequence of vanishing moments

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Abstract. This paper shows that vanishing of all moments of the complex sequence $\{z_j\}$ implies that $\{z_j\}$ is identically zero, provided $\{z_j\}$ is in $l^p, 1 \leq p < \infty$. This proof is different from one given by Priestley [Proc. Amer. Math. Soc. 116 (1992) 437–444] and shows an interesting connection of this problem with heat kernel.

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1. Introduction

Moment of a complex sequence $\{z_j\}$ of order n is defined to be $\sum_{j=1}^{\infty} z_j^n$. In response to a question of W. M. Priestley, Lenard [1] constructed a non-vanishing infinite sequence of complex numbers $\{z_j\}$ such that all the power-sums (moments) $\sum_{j=1}^{\infty} z_j^n$ $(n = 1, 2, 3, \dots)$ vanish. If a finite number of complex numbers $z_1, \dots, z_k$ has the property that all the power-sums vanish, then it follows that all of them must be zero. The question raised by W. M. Priestley is that whether the above statement is true for a complex sequence having infinitely many terms. He conjectured that this is not the case. This conjecture is confirmed by generating an explicit example of an infinite sequence, see for the detail [1]. Note that this sequence constructed by Lenard is a bounded sequence. Later on Priestly proved that the conjecture is not true when the sequence lies in l^p, see [2]. In this paper we provide a different proof of the statement that

> "*If $\{z_j\}$ is a sequence of complex numbers whose all moments vanish and $\{z_j\}_{j=1}^{\infty} \in l^p, 1 \leq p < \infty$, then the sequence is identically zero.*"

The idea of the proof is to consider one parametric family (time parameter) of expressions which is zero for all time $t > 0$, because of the vanishing power sums of the given complex sequence. For the localizing effect as t tends to zero, the sequence gets separated. Thus we argue that the sequence vanishes identically.

2. Vanishing moment problem

Lemma 2.1. *Let $\{z_j\}_{j=1}^{\infty}$ be an absolutly summable sequence. Then for any bounded measurable function ϕ having compact support in $\mathbb{R}$, the following holds:*

$$\sum_{j=1}^{\infty} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \left[e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}} \right] \phi(x) dx = \frac{1}{\sqrt{4\pi t}} \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} (z_j/\sqrt{4t})^n \int_{\mathbb{R}} \frac{H_n(x/\sqrt{4t})}{n!} e^{\frac{-x^2}{4t}} \phi(x) dx \right). \tag{2.1}$$

Here $H_n(x)$ is real Hermite polynomial of order n.

Proof. Since $\{z_j\}_{j=1}^{\infty} \in l^1$, there exists a $m > 0$ such that $|z_j| < m$ for all j. Now

$$\sum_{j=1}^{\infty} |z_j|^n \leq m^{n-1} \|\{z_j\}\|_{l^1}, \tag{2.2}$$

for $n \geq 1$. Hermite polynomials satisfy the following identity.

$$\int_{\mathbb{R}} H_m(x) H_n(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{mn}.$$

Employing this identity,

$$\begin{aligned}
&\sum_{j=1}^{\infty} \int_{\mathbb{R}} \left| \frac{H_n(x/\sqrt{4t})}{n!} (z_j/\sqrt{4t})^n e^{\frac{-x^2}{4t}} \phi(x) \right| dx \\
&\leq \sum_{j=1}^{\infty} \int_{\mathbb{R}} \left| \frac{H_n(x/\sqrt{4t})}{n!} (z_j/\sqrt{4t})^n \phi(x) \right| dx \\
&\leq \frac{m^{n-1}}{(4t)^{\frac{n}{2}} n!} \|\{z_j\}\|_{l^1} \left[\int_{\mathbb{R}} \left| H_n(x/\sqrt{4t}) \right|^2 e^{\frac{-x^2}{4t}} dx \right]^{1/2} \left[\int_{\mathbb{R}} |\phi(x)|^2 e^{\frac{x^2}{4t}} dx \right]^{1/2} \\
&\leq C_1 \frac{m^{n-1} 2^{\frac{n}{2}} \pi^{\frac{1}{4}}}{(4t)^{\frac{n-1}{2}} \sqrt{n!}} \|\{z_j\}\|_{l^1}, \quad \text{for some positive constant } C_1.
\end{aligned} \tag{2.3}$$

Thus we obtain

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \left| \int_{\mathbb{R}} \frac{H_n(x/\sqrt{4t})}{n!} (z_j/\sqrt{4t})^n \phi(x) e^{\frac{-x^2}{4t}} dx \right| \leq C_1 \sum_{n=1}^{\infty} \frac{m^{n-1} 2^{\frac{n}{2}} \pi^{\frac{1}{4}}}{(4t)^{\frac{n-1}{2}} \sqrt{n!}} \|\{z_j\}\|_{l^1} < \infty. \tag{2.4}$$

Let us denote $a_{nj} := \int_{\mathbb{R}} \frac{H_n(x/\sqrt{4t})}{n!} (z_j/\sqrt{4t})^n \phi(x) e^{\frac{-x^2}{4t}} dx.$

Then (2.4) leads to $\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |a_{nj}| < \infty$. Thus, we obtain

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} a_{nj} = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} a_{nj}. \tag{2.5}$$

Applying term by term integration, one gets

$$\sum_{n=1}^{\infty} a_{nj} = \sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{H_n(x/\sqrt{4t})}{n!} (z_j/\sqrt{4t})^n \phi(x) e^{\frac{-x^2}{4t}} dx = \int_{\mathbb{R}} \sum_{n=1}^{\infty} \frac{H_n(x/\sqrt{4t})}{n!} (z_j/\sqrt{4t})^n \phi(x) e^{\frac{-x^2}{4t}} dx.$$

Now using the following fact in the above equation

$$e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}} = \sum_{n=1}^{\infty} \frac{H_n(x/\sqrt{4t})}{n!} e^{\frac{-x^2}{4t}} \left(\frac{z_j}{\sqrt{4t}} \right)^n,$$

we get

$$\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} a_{nj} = \sum_{j=1}^{\infty} \int_{\mathbb{R}} \left[e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}} \right] \phi(x) dx. \tag{2.6}$$

On the other hand

$$\sum_{j=1}^{\infty} a_{nj} = \sum_{j=1}^{\infty} \int_{\mathbb{R}} \frac{H_n(x/\sqrt{4t})}{n!}(z_j/\sqrt{4t})^n \phi(x) e^{\frac{-x^2}{4t}} dx$$
$$= \sum_{j=1}^{\infty} (z_j/\sqrt{4t})^n \int_{\mathbb{R}} \frac{H_n(x/\sqrt{4t})}{n!} \phi(x) e^{\frac{-x^2}{4t}} dx.$$

Therefore, one has

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} a_{nj} = \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} (z_j/\sqrt{4t})^n \int_{\mathbb{R}} \frac{H_n(x/\sqrt{4t})}{n!} \phi(x) e^{\frac{-x^2}{4t}} dx \right). \tag{2.7}$$

In view of the equations (2.5), (2.6) and (2.7), the proof is completed. □

Theorem 2.2. *If all the moments of the complex sequence* $\{z_j\}_{j=1}^{\infty}$ *vanish and* $\{z_j\}_{j=1}^{\infty} \in l^1$, *then* $z_j = 0$ *for all* $j \in \mathbb{N}$.

Proof. Let $z_j = x_j + iy_j$. Our aim is to show that imaginary part of the complex sequence is identically zero. Suppose this is not the case. Clearly $y_j^2 \to 0$ as $j \to \infty$. Hence there exists a y_k such that $y_k^2 = \max_j y_j^2$.

Pick up a neighborhood B of x_k. Now take a continuous function ψ supported in B and define $\phi(x) = \psi(x)\theta(x)$, where

$$\theta(x) = \left| e^{-\frac{(x-z_k)^2}{4t}} \right| e^{\frac{(x-z_k)^2}{4t}}.$$

Now

$$\left| e^{-\frac{(x-z_j)^2}{4t}} \right| = e^{-\frac{(x-x_j)^2}{4t}} e^{\frac{y_j^2}{4t}}.$$

Again

$$\left| e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}} \right| = \left| \int_0^1 \frac{d}{ds}\left[e^{-\frac{(x-sz_j)^2}{4t}} \right] ds \right|$$
$$\leq |z_j| \int_0^1 \left| \left[e^{-\frac{(x-sz_j)^2}{4t}} \right] 2 \left(\frac{x - sz_j}{4t} \right) \right| ds$$
$$= 2|z_j| \int_0^1 e^{-\frac{(x-sx_j)^2}{4t}} e^{\frac{s^2y_j^2}{4t}} \sqrt{\frac{(x - sx_j)^2}{16t^2} + s^2 \frac{y_j^2}{16t^2}} ds$$
$$\leq 2|z_j| e^{\frac{y_j^2}{4t}} \int_0^1 e^{-\frac{(x-sx_j)^2}{4t}} \sqrt{\frac{(x - sx_j)^2}{16t^2} + s^2 \frac{y_j^2}{16t^2}} ds.$$

Therefore

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\left[e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}}\right]\phi(x)dx\right| \\
&\leq 2|z_j|e^{\frac{y_j^2}{4t}} \int_0^1 \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-sx_j)^2}{4t}} \sqrt{\frac{(x-sx_j)^2}{16t^2} + s^2\frac{y_j^2}{16t^2}}|\phi(x)|dxds \\
&\leq 2|z_j|e^{\frac{y_j^2}{4t}} \int_0^1 \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-sx_j)^2}{4t}} \left[1 + \frac{(x-sx_j)^2}{16t^2} + s^2\frac{y_j^2}{16t^2}\right]|\phi(x)|dxds \\
&= \frac{2}{\sqrt{\pi}}|z_j|e^{\frac{y_j^2}{4t}} \int_0^1 \int_{\mathbb{R}} e^{-u^2}\left[1 + \frac{u^2}{4t} + s^2\frac{y_j^2}{16t^2}\right]|\phi(\sqrt{4t}u + sx_j)|duds \\
&\leq C_2(\phi)|z_j|e^{\frac{y_j^2}{4t}}\left[1 + \frac{1}{t} + \frac{1}{t^2}\right],
\end{aligned}
$$

for some positive constant $C_2(\phi)$, which only depends on ϕ.

Since all the moments of the sequence $\{z_j\}_{j=1}^{\infty}$ vanish, Lemma 2.1 implies

$$
\begin{aligned}
&e^{-\frac{y_k^2}{4t}} \sum_{j=1}^{N} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\left[e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}}\right]\phi(x)dx \\
&\quad = -e^{-\frac{y_k^2}{4t}} \sum_{j=N+1}^{\infty} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\left[e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}}\right]\phi(x)dx.
\end{aligned}
$$

Choose sufficiently large N such that $y_j \neq y_k$ for $j \geq N$. This implies

$$
\begin{aligned}
\left|e^{-\frac{y_k^2}{4t}} \sum_{j=1}^{N} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\left[e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}}\right]\phi(x)dx\right| &= \left|e^{-\frac{y_k^2}{4t}} \sum_{j=N+1}^{\infty} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\left[e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}}\right]\phi(x)dx\right| \\
&\leq C_2(\phi) \sum_{j=N+1}^{\infty} |z_j|e^{\frac{y_j^2-y_k^2}{4t}}\left[1 + \frac{1}{t} + \frac{1}{t^2}\right] \\
&\leq C_2(\phi)e^{\frac{-\alpha}{4t}}\left[1 + \frac{1}{t} + \frac{1}{t^2}\right]\|\{z_j\}\|_1,
\end{aligned}
$$

where $-\alpha = \max_{j \geq N+1}(y_j^2 - y_k^2) < 0$.

Passing to the limit as t tends to zero, we have

$$
\lim_{t\to 0}\left|e^{-\frac{y_k^2}{4t}} \sum_{j=1}^{N} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\left[e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}}\right]\phi(x)dx\right| = 0. \tag{2.8}
$$

Now left hand side of the expression (2.8) can be written as:

$$\left| e^{-\frac{y_k^2}{4t}} \sum_{j=1}^{N} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \left[e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}} \right] \phi(x) dx \right|$$

$$= \left| e^{-\frac{y_k^2}{4t}} \sum_{\substack{j=1 \\ y_j \neq y_k}}^{N} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \left[e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}} \right] \phi(x) dx \right.$$

$$\left. + e^{-\frac{y_k^2}{4t}} \sum_{\substack{j=1 \\ y_j = y_k}}^{N} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \left[e^{-\frac{(x-z_j)^2}{4t}} - e^{-\frac{x^2}{4t}} \right] \phi(x) dx \right| := |I_1 + I_2|. \tag{2.9}$$

Now let us consider

$$\left| e^{-\frac{y_k^2}{4t}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-z_j)^2}{4t}} \phi(x) dx \right| \leq e^{-\frac{y_k^2}{4t}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-x_j)^2 + y_j^2}{4t}} |\phi(x)| dx$$

$$= e^{\frac{y_j^2 - y_k^2}{4t}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-x_j)^2}{4t}} |\phi(x)| dx$$

$$\to 0 \quad \text{as} \quad t \to 0 \quad \text{for} \quad y_j \neq y_k. \tag{2.10}$$

Similarly we can show that

$$e^{-\frac{y_k^2}{4t}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \phi(x) dx \to 0 \quad \text{as} \quad t \to 0. \tag{2.11}$$

In light of (2.10) and (2.11), I_1 of the equation (2.9) approaches zero as t approaches zero. In order to understand the limit of I_2, we consider two subcases for the case $y_j = y_k$, i.e., $x_j = x_k$ and $x_j \neq x_k$.

If $x_j = x_k$, consider

$$e^{-\frac{y_k^2}{4t}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-z_k)^2}{4t}} \phi(x) dx = e^{-\frac{y_k^2}{4t}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-z_k)^2}{4t}} \psi(x)\theta(x) dx$$

$$= e^{-\frac{y_k^2}{4t}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-x_k)^2 + y_k^2}{4t}} \psi(x) dx$$

$$\to \psi(x_k) \quad \text{as} \quad t \to 0, \tag{2.12}$$

by the property of heat kernel.

If $x_j \neq x_k$, consider

$$\left| e^{-\frac{y_k^2}{4t}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-z_j)^2}{4t}} \phi(x) dx \right| \leq e^{-\frac{y_k^2}{4t}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-x_j)^2 + y_k^2}{4t}} |\phi(x)|\, dx$$

$$\to |\phi(x_j)| \quad \text{as} \quad t \to 0. \tag{2.13}$$

Now squeezing the ball B, we can assume $x_j \notin B$, for $x_j \neq x_k$, $1 \leq j \leq N$. This ensures that $|\phi(x_j)| = 0$.

In view of (2.12) and (2.13), the I_2 part of the equation (2.9) approaches $N_1 |\psi(x_k)|$ as t approaches zero, where $N_1 = \sum_{\substack{j=1 \\ z_j = z_k}}^{N} 1$.

Hence $N_1|\psi(x_k)| = 0$. This is true for all ψ supported in the ball B, which is a contradiction. So the only possibility is $y_k^2 = 0$. So all the complex numbers are real numbers. Therefore all the complex numbers are identically zero. □

Remark 2.3. Same conclusion of the Theorem 2.2 holds if we take $\{z_j\}_{j=1}^{\infty} \in l^p$, $1 < p < \infty$, and all the moments of the sequence vanish. In this case, we will use the following expression to prove it:

$$\sum_{j=1}^{\infty} \int_{\mathbb{R}} \frac{1}{4\pi t} \left[e^{-\frac{(x-z_j{}^l)^2}{4t}} - e^{-\frac{x^2}{4t}} \right] \phi(x) dx,$$

where l is an integer greater than p. In this way, one can also prove that, if $\{z_j\} \in l^p$ and $\sum_{j=1}^{\infty} z_i{}^{pn} = 0$ for all $n \in \mathbb{N}$ imply $z_j = 0$ for all $n \in \mathbb{N}$.

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