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On simultaneous non-vanishing of twisted L-functions associated to newforms on $\Gamma_0(N)$

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Abstract. Given two normalized Hecke eigenforms f_1, f_2 of weight k_1 and k_2 respectively of squarefree level N, we consider the product of twisted L-functions associated with f_1 and f_2 by a quadratic charater χ, and show that if this product does not vanish at the center of the critical strip $s = 1/2$, then it does not vanish for infinitely many such twists. This is a generalisation of the work due to R. Munshi (J. Number Theory, 132, 666-674 [2012]) for the full modular group to the case of any congruence subgroup of squarfree level N.

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1. Introduction

Let f_1 and f_2 be two normalized Hecke eigenforms of weight k_1 and k_2, respectively on $\Gamma_0(N)$. Let $\mathcal{D}$ denote the set of fundamental discriminants. Define the set

$$\Delta(f_1, f_2) := \{d \in \mathcal{D} : L(f_1 \otimes \chi_d, 1/2)L(f_2 \otimes \chi_d, 1/2) \neq 0\},$$

where $L(f_i \otimes \chi_d, s)$ denotes the twisted L-function associated with f_i $(i = 1, 2)$ by quadratic character χ_d for $d \in \mathcal{D}$, and χ_d is defined by $\chi_d(n) = \left(\frac{d}{n}\right)$, where $\left(\frac{d}{n}\right)$ is the generalised Jacobi symbol.

R. Munshi [6] proved that for given two normalized Hecke eigenforms f_1 and f_2 on $SL_2(\mathbb{Z})$, if the set $\Delta(f_1, f_2)$ is nonempty then the cardinality of the set $\Delta(f_1, f_2)$ is infinite. We genaralise the work of Munshi [6] for the congruence subgroup $\Gamma_0(N)$, when N is squarefree. We follow the same exposition as given in [6]. The proof uses the connection between half-integral weight modular form and integral weight modular form developed by Shimura [7] and Waldspurger formula in the context that vanishing of the twisted L-function can be determined by vanishing of the coefficients of the associated half-integral weight modular form. We now state the main theorem of the paper.

Theorem 1.1. *Let f_1 and f_2 be two normalized Hecke eigenforms of weight k_1 and k_2, respectively on $\Gamma_0(N)$, where N is squarefree. Suppose that there exists a fundamental discriminant d such that $(d, N) = 1$ and*

$$L(f_1 \otimes \chi_d, 1/2)L(f_2 \otimes \chi_d, 1/2) \neq 0,$$

then there are infinitely many such fundamental discriminants d with the above property. In other words, the cardinality of the set $\Delta(f_1, f_2)$ is either zero or infinite.

2. Preliminaries and basic results

In this section, we briefly recall some basic definitions and properties of modular forms.

The full modular group $SL_2(\mathbb{Z})$ and congruence subgroup $\Gamma_0(N)$ of level $N \in \mathbb{N}$ are defined as follows;

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\}.$$

The group $SL_2(\mathbb{Z})$ acts on the complex upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$ via fractional linear transformation as follows;

$$\gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \quad \text{and} \quad \tau \in \mathcal{H}.$$

Let $k \in \mathbb{Z}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, then for a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ define the weight k slash operator as follows;

$$f \mid_k \gamma(\tau) := (c\tau + d)^{-k} f(\gamma \cdot \tau). \tag{1}$$

Definition 2.1. *A modular form of weight $k \in \mathbb{Z}$ for $\Gamma_0(N)$ is a complex-valued holomorphic function f on $\mathcal{H}$ satisfying;*

$$f \mid_k \gamma(\tau) = f(\tau), \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

and holomorphic at the cusps of $\Gamma_0(N)$. Further if f also vanishes at the cusps of $\Gamma_0(N)$, then f is called a cusp form.

We denote $M_k(N)$ (respectively $S_k(N)$) the space of modular forms (respectively cusp forms) of weight k for $\Gamma_0(N)$.

For $N \in 4\mathbb{N}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and a holomorphic function $f : \mathcal{H} \to \mathbb{C}$, define the weight $k + \frac{1}{2}$ slash operator as follows;

$$f\tilde{|}_{k+\frac{1}{2}}\gamma(\tau) := \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{k+\frac{1}{2}} (c\tau + d)^{-k-\frac{1}{2}} f(\gamma \cdot \tau), \tag{2}$$

where $\left(\frac{c}{d}\right)$ is the extended quadratic residue symbol as in [7].

For a fixed positive integer N such that $4|N$, we denote by $M_{k+\frac{1}{2}}(N)$, the space of modular forms of weight $k + \frac{1}{2}$ for $\Gamma_0(N)$, that is the space of all complex-valued holomorphic functions f satisfying;

$$f\tilde{|}_{k+\frac{1}{2}}\gamma(\tau) = f(\tau), \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

and holomorphic at the cusps of $\Gamma_0(N)$. Further, if f vanishes at cusps of $\Gamma_0(N)$, then f is called a cusp form and we denote the space of all cusp forms by $S_{k+\frac{1}{2}}(N)$.

For every positive integer n, one can define certain linear operator T_n, called n-th Hecke operator on $M_k(N)$. The family of Hecke operators acting on $S_k(N)$ are commuting and self-adjoint with respect to the Petersson scalar product, therefore there exist an orthonormal basis consisting of cusp forms (called eigenforms) which are eigenfunction for all the Hecke operators. We call an eigenform f with Fourier coefficients $a_f(n)$ normalized Hecke eigenform if $a_f(1) = 1$.

For more details on the theory of modular forms, we refer to [3] and [7].

Let $f(\tau) = \sum_{n=1}^{\infty} a_f(n)e(n\tau) \in S_{2k}(N)$ be a normalized Hecke eigenform of weight $2k$ on $\Gamma_0(N)$. The L-function associated with f is defined by

$$L(f, s) := \sum_{n=1}^{\infty} \frac{a_f(n)}{n^{s+k-1/2}}.$$

This series converges absolutely for $\sigma = Re(s) > 1$ and satisfies a functional equation with $s \to 1 - s$.

For $d \in \mathcal{D}$, let χ_d denotes the associated quadratic character as defined before, we define the twisted L-function $L(f \otimes \chi_d, s)$ as

$$L(f \otimes \chi_d, s) := \sum_{n=1}^{\infty} \frac{\chi_d(n) a_f(n)}{n^{s+k-1/2}}.$$

The sign of the functional equation of the twisted L-function $L(f \otimes \chi_d, s)$ is given by $\epsilon_d = (-1)^k \operatorname{sgn}(d)$. As a consequence of this we get $L(f \otimes \chi_d, 1/2) = 0$ if $(-1)^k d < 0$.

The Kohnen plus space in $S_{k+\frac{1}{2}}(4N)$ is defined by

$$S^+_{k+\frac{1}{2}}(4N) := \left\{ f \in S_{k+\frac{1}{2}}(4N) : (-1)^k n \equiv 2, 3 \pmod 4 \Rightarrow a_f(n) = 0 \right\}.$$

If $f(\tau) = \sum_{n=1}^{\infty} a_f(n) e(n\tau) \in S_{2k}(N)$ is a normalized Hecke eigenform (i.e., $a_f(1) = 1$) and $g(\tau) = \sum_{n=1}^{\infty} a_g(n) e(n\tau) \in S^+_{k+\frac{1}{2}}(4N)$ is the corresponding form of half-integral weight (in the sense of [4]), then the Fourier coefficients of f and g are related by the formula

$$a_g(n^2|d|) = a_g(|d|) \sum_{\substack{\delta | n \\ (\delta, N)=1}} \mu(\delta) \left(\frac{d}{\delta}\right) \delta^{k-1} a_f(n/\delta), \tag{3}$$

for every fundamental discriminant $d \in \mathcal{D}$ with $(-1)^k d > 0$.

For a prime p dividing N, define $\omega_p \in \{\pm 1\}$ by $f \mid W_p = \omega_p f$, where W_p is the Atkin-Lehner involution on $S_{2k}(N)$ defined by

$$f \mid W_p = f \mid_{2k} \frac{1}{\sqrt{p}} \begin{pmatrix} p & a \\ N & pb \end{pmatrix} \quad (a, b \in \mathbb{Z},\ p^2 b - Na = p).$$

Proposition 2.2 ([5] Corollary 1, p. 242). *Let d be fundamental discriminant with $(-1)^k d > 0$ and suppose that for all prime divisors p of N we have $\left(\frac{d}{p}\right) = \omega_p$. Then*

$$\frac{|a_g(|d|)|^2}{\langle g, g \rangle} = 2^{\sigma(N)} \frac{(k-1)!}{\pi^k} |d|^{k-1} \frac{L(f \otimes \chi_d, 1/2)}{\langle f, f \rangle}, \tag{4}$$

where $\sigma(N)$ denotes the number of different prime divisors of N.

Given a positive integer k and $n \in \mathbb{N}$ such that $n \equiv 0, (-1)^k \pmod 4$, n can be uniquely written as

$$n = |d| m^2, \quad \text{where } d \in \mathcal{D}, m \in \mathbb{Z} \quad \text{and} \quad (-1)^k d > 0. \tag{5}$$

It follows from equations (3) and (4) that

$$a_g(n) \neq 0 \Rightarrow L(f \otimes \chi_d, 1/2) \neq 0.$$

3. Proof of Theorem 1.1

In this section we give a proof of the Theorem 1.1.

Let N be a fixed positive squarefree integer. Suppose that

$$f_i(\tau) = \sum_{n=1}^{\infty} a_i(n) e(n\tau) \in S_{2k_i}(N), \quad i = 1, 2$$

be two normalized Hecke eigenforms of weight $2k_i$ on $\Gamma_0(N)$. Let

$$F_i(\tau) = \sum_{n=1}^{\infty} A_i(n)e(n\tau) \in S^+_{k+\frac{1}{2}}(4N)$$

be the half-integral weight modular form corresponding to f_i (in the sense of [4]).

We assume that the set $\Delta(f_1, f_2)$ is nonempty and cardinality of $\Delta(f_1, f_2)$ is finite. We shall show that this assumption of finiteness on $\Delta(f_1, f_2)$ will lead us to a contradiction to a already known result about the number of zeros of $L(f_1 \times f_2, s)$.

Write $H(\tau) = \overline{F_1(\tau)}F_2(\tau)$. Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$,

$$H(\gamma z) = \overline{(cz+d)}^{k_1+\frac{1}{2}}(cz+d)^{k_2+\frac{1}{2}}H(\tau). \tag{6}$$

Let $\kappa_1 = \infty, \kappa_2, \dots, \kappa_h$ be the non-equivalent cusps of $\Gamma_0(4N)$, then for each cusp κ_i there exists $g_i \in SL_2(\mathbb{Q})$ such that $g_i\infty = \kappa_i$ and $\Gamma_i := g_i\Gamma_\infty g_i^{-1}$, where Γ_i and Γ_∞ are the stabilizer group of the cusps κ_i $(i > 1)$ and ∞ respectively. We define the Eisenstein series corresponding to each cusp κ_i as follows;

$$E_i(\tau, s, k) := \sum_{\gamma \in \Gamma_i \backslash \Gamma_0(4N)} j(g_i^{-1}\gamma, \tau)^k Im(g_i^{-1}\gamma \cdot \tau)^s, \tag{7}$$

where $j(\gamma, \tau) = \overline{(c\tau+d)}(c\tau+d)^{-1}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

These Eisenstein series converge absolutely for $Re(s) > 1$ and has analytic continuation to the whole complex plane. If we write

$$\vec{E}(\tau, s, k) = \begin{bmatrix} E_1(\tau, s, k) \\ E_2(\tau, s, k) \\ \dots\dots\dots\dots \\ E_h(\tau, s, k) \end{bmatrix}.$$

Then these Eisenstein series together satisfy a functional equation given by

$$\vec{E}(\tau, s, k) = \Phi(s)\vec{E}(\tau, 1-s, k), \tag{8}$$

where $\Phi(s)$ is a $h \times h$ matrix, called scattering matrix and satisfies the functional equation

$$\Phi(s)\Phi(1-s) = I_{h\times h}.$$

Observe that for any $\alpha, \beta \in SL_2(\mathbb{Z})$, $j(\alpha\beta, \tau) = J(\alpha, \beta\cdot\tau)j(\beta, \tau)$. Consequently we have,

$$E_i(\eta\cdot\tau, s, k) = \overline{(c\tau+d)}^{-k}(c\tau+d)^k E_i(\tau, s, k), \quad \eta \in \Gamma_0(4N). \tag{9}$$

From the equations (6) and (9), it follows that the function

$$H(\tau)E_i\left(\tau, s, \frac{k_1-k_2}{2}\right)y^{\frac{k_1+k_2+1}{2}}$$

is invariant under $\Gamma_0(4N)$. Therefore, we can consider the Rankin-Selberg integral

$$R_i(s) = \int_{\Gamma_0(4N)\backslash\mathcal{H}} H(\tau)E_i\left(\tau, s, \frac{k_1-k_2}{2}\right)Im(\tau)^{\frac{k_1+k_2+1}{2}}\frac{dudv}{v^2}, \quad (\tau = u+iv \in \mathcal{H}).$$

Theorem 3.1 ([1]). *If $R_i(s)$ is as above, then $R_i(s)$ has a meromorphic continuation to all s, the only possible poles being at $s = 0, 1, \alpha_{ij}, 1-\alpha_{ij}$ and $\rho/2$, where ρ's are the nontrivial zeros of the Riemann zeta function. Further we have the following functional equation*

$$\vec{R}(s) = \begin{bmatrix} R_1(s) \\ R_2(s) \\ \dots\dots \\ R_h(s) \end{bmatrix} = \Phi(s)\begin{bmatrix} R_1(1-s) \\ R_2(1-s) \\ \dots\dots\dots \\ R_h(1-s) \end{bmatrix} = \Phi(s)\vec{R}(1-s). \tag{10}$$

We consider the integral

$$R_i(s) = \int_{\Gamma_0(4N)\backslash\mathcal{H}} H(\tau)E_i\left(\tau, s, \frac{(k_1-k_2)}{2}\right) Im(\tau)^{\frac{k_1+k_2+1}{2}} \frac{dudv}{v^2},$$

$$= \int_{\Gamma_0(4N)\backslash\mathcal{H}} H(\tau) \sum_{\gamma\in\Gamma_i\backslash\Gamma_0(4N)} j(g_i^{-1}\gamma,\tau)^{\frac{k_1-k_2}{2}} Im(g_i^{-1}\gamma\cdot\tau)^s\, Im(\tau)^{\frac{k_1+k_2+1}{2}} \frac{dudv}{v^2},$$

$$= \int_{\Gamma_0(4N)\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_i\backslash\Gamma_0(4N)} H(\tau) j(g_i^{-1}\gamma,\tau)^{\frac{k_1-k_2}{2}} Im(g_i^{-1}\gamma\cdot\tau)^s\, Im(\tau)^{\frac{k_1+k_2+1}{2}} \frac{dudv}{v^2}.$$

Interchanging the sum and integration in the above equation we get

$$R_i(s) = \sum_{\gamma\in\Gamma_i\backslash\Gamma_0(4N)} \int_{\Gamma_0(4N)\backslash\mathcal{H}} H(\tau) j(g_i^{-1}\gamma,\tau)^{\frac{k_1-k_2}{2}} Im(g_i^{-1}\gamma\cdot\tau)^s\, Im(\tau)^{\frac{k_1+k_2+1}{2}} \frac{dudv}{v^2}.$$

Using the change of variable $\tau \to g_i\cdot\tau$, we get

$$R_i(s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4N)} \int_{g_i^{-1}\cdot\Gamma_0(4N)\backslash\mathcal{H}} H(g_i\cdot\tau) j(g_i^{-1}\gamma, g_i\cdot\tau)^{\frac{k_1-k_2}{2}} Im(g_i^{-1}\gamma g_i\cdot\tau)^s\, Im(g_i\cdot\tau)^{\frac{k_1+k_2+1}{2}} \frac{dudv}{v^2}.$$

Now using the Rankin unfolding argument, we have

$$R_i(s) = \int_{\Gamma_\infty\backslash\mathcal{H}} j(g_i,\tau)^{\frac{-(k_1-k_2)}{2}} H(g_i\cdot\tau) Im(g_i\cdot\tau)^{\frac{k_1+k_2+1}{2}} Im(\tau)^s \frac{dudv}{v^2}.$$

Therefore,

$$R_\infty(s) = \int_{\Gamma_\infty\backslash\mathcal{H}} H(\tau) Im(\tau)^{\frac{k_1+k_2+1}{2}} Im(\tau)^s \frac{dudv}{v^2},$$

$$= \int_{\Gamma_\infty\backslash\mathcal{H}} \overline{F_1(\tau)} F_2(\tau) Im(\tau)^{\frac{k_1+k_2+1}{2}} Im(\tau)^s \frac{dudv}{v^2}.$$

Now replacing F_1 and F_2 by their Fourier series expansions, we have

$$R_\infty(s) = \int_{\Gamma_\infty\backslash\mathcal{H}} \left(\sum_{n=1}^{\infty} \overline{A_1(n)} e(-n\overline{\tau})\right)\left(\sum_{m=1}^{\infty} A_2(m) e(m\tau)\right) Im(\tau)^{\frac{k_1+k_2+1}{2}+s} \frac{dudv}{v^2}. \tag{11}$$

It is well known that the set $\{\tau = u + iv : u \in [0,1], v \in [0,\infty)\}$ is a fundamental domain for the action of Γ_∞ on $\mathcal{H}$. Integrating (11) over this region $R_\infty(s)$ equals

$$R_\infty(s) = \int_0^1\int_0^\infty \left(\sum_{n=1}^{\infty} \overline{A_1(n)} e(-n\overline{\tau})\right)\left(\sum_{m=1}^{\infty} A_2(m) e(m\tau)\right) Im(\tau)^{\frac{k_1+k_2+1}{2}+s} \frac{dudv}{v^2}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \overline{A_1(n)} A_2(m) \int_0^1\int_0^\infty e(-n\overline{\tau}) e(m\tau) Im(\tau)^{\frac{k_1+k_2+1}{2}+s} \frac{dudv}{v^2}$$

$$= \sum_{n=1}^{\infty} \overline{A_1(n)} A_2(n) \int_0^\infty e^{-4\pi nv}\, v^{\frac{k_1+k_2+1}{2}+s-2} dv$$

$$= \frac{\Gamma(s+k^*)}{(4\pi)^{(s+k^*)}} \sum_{n=1}^{\infty} \frac{\overline{A_1(n)} A_2(n)}{n^{s+k^*}}, \quad k^* = \frac{k_1+k_2-1}{2}.$$

We now consider the Dirichlet series defined by

$$D(s) := \sum_{n=1}^{\infty} \frac{\overline{A_1(n)} A_2(n)}{n^{s+k^*}},$$

which appeared in the the Rankin-Selberg integral R_∞. Now, we want to relate $D(s)$ with the normalized Rankin-Selberg L-function $L(f_1 \times f_2, s)$, where

$$L(f_1 \times f_2, s) := \sum_{n=1}^{\infty} \frac{a_1(n) a_2(n)}{n^{k_1+k_2+s-1}}.$$

Using the formula (3) and decomposition (5), we have

$$\overline{A_1(n)} A_2(n) = \overline{A_1(|d|)} A_2(|d|) \prod_{i=1,2} \sum_{\substack{\delta_i | m \\ (\delta_i, N)=1}} \mu(\delta_i) \chi_d(\delta_i) {\delta_i}^{k_i-1} a_i(m/\delta_i).$$

Therefore,

$$D(s) = \sum_{d \in \Delta(f_1, f_2)} \frac{\overline{A_1(|d|)} A_2(|d|)}{|d|^{s+k^*}} \sum_{m=1}^{\infty} \frac{1}{m^{2(s+k^*)}} \prod_{i=1,2} \sum_{\substack{\delta_i | m \\ (\delta_i, N)=1}} \mu(\delta_i) \chi_d(\delta_i) {\delta_i}^{k_i-1} a_i(m/\delta_i).$$

Since f_i for $i = 1, 2$ are normalized Hecke eigenforms, therefore the functions $\sum_{\substack{\delta_i | m \\ (\delta_i, N)=1}} \mu(\delta_i) \chi_d(\delta_i) {\delta_i}^{k_i-1} a_i(m/\delta_i)$ are multiplicative in m, we get

$$\sum_{m=1}^{\infty} \frac{1}{m^{2\omega}} \prod_{i=1,2} \sum_{\substack{\delta_i | m \\ (\delta_i, N)=1}} \mu(\delta_i) \chi_d(\delta_i) {\delta_i}^{k_i-1} a_i(m/\delta_i) = \prod_p L_p(w, d),$$

where $w = s + k^*$ and

$$L_p(w, d) = 1 + \sum_{l=1}^{\infty} \frac{1}{p^{2lw}} \prod_{i=1,2} \sum_{\substack{\delta_i | p^l \\ (\delta_i, N)=1}} \mu(\delta_i) \chi_d(\delta_i) {\delta_i}^{k_i-1} a_i(p^l/\delta_i).$$

Also,

$$\sum_{\delta | p^l} \mu(\delta) \chi_d(\delta) \delta^{k-1} a(p^l/\delta) = a(p^l) - \chi_d(p) p^{k-1} a(p^{l-1}),$$

consequently we have

$$L_p(w, d) = 1 + \sum_{l=1}^{\infty} \frac{1}{p^{2lw}} \left[\left(a_1(p^l) - \chi_d(p) p^{k_1-1} a_1(p^{l-1}) \right) \left(a_2(p^l) - \chi_d(p) p^{k_2-1} a(p^{l-1}) \right) \right].$$

Observe that the local Euler factor of $L_p(f_1 \otimes f_2, s)$ is given by $1 + \sum_{l=1}^{\infty} \frac{a_1(p^l) a_2(p^l)}{p^{ls}}$ and the local Euler factor of $L(f_i \otimes \chi_d, s)$ is given by $(1 - \chi_d(p) a_i(p) p^{-s} + \chi_N(p) p^{2k_i-1-2s})^{-1}$, where $\chi_N(p)$ is given by

$$\chi_N(p) = \begin{cases} 1 & \text{if } p \nmid N \\ 0 & \text{if } p | N. \end{cases}$$

Executing the sum over l and using the Hecke relation for the fourier coefficients $a_1(p^l)$ and $a_2(p^l)$ we obtain

$$L_p(w; d) = \frac{L_p(f_1 \times f_2, 2s)}{L_p(f_1 \otimes \chi_d, 2s + 1/2) L_p(f_2 \otimes \chi_d, 2s + 1/2)} E_p(s; d),$$

where the first factor on the right-hand side is the local Euler factor for the normalized Rankin-Selberg L-function $L(f_1 \times f_2, 2s)$ (the center of $L(f_1 \times f_2, s)$ is 1/2), and the last factor is such that the Euler product $\prod_p E_p(s, d)$ is absolutely convergent for $\sigma > 1/8$. Therefore,

$$D(s) = L(f_1 \times f_2, 2s)\Xi(s), \tag{12}$$

where

$$\Xi(s) = \sum \frac{\overline{A_1(|d|)A_2(|d|)}}{|d|^{s+k^*}} l(s; d),$$

with

$$l(s; d) = \frac{E(s; d)}{L(f_1 \otimes \chi_d, 2s + 1/2)L(f_2 \otimes \chi_d, 2s + 1/2)}$$

and $E(s; d)$ is an Euler product which converges absolutely in the half plane $\sigma > 1/8$.

Recall our assumption that $\Delta(f_1, f_2)$ is nonempty but finite, say

$$\Delta(f_1, f_2) = \{d_1, \ldots, d_m\}, \text{ with } |d_1| < |d_2| < d \ldots |d_m|.$$

Then $\Xi(s)$ is meromorphic in the half plane $\sigma > \frac{1}{8}$, and it is holomorphic in the half plane $\sigma \geq \frac{1}{4}$. Following the argument of Munshi [6] (section 6), if we consider

$$R = \{s = \sigma + it : 1/3 \leq \sigma \leq \alpha, T \leq t \leq T + H\}$$

and

$$f(s) = \frac{|d_1|^{s+k^*}\Xi(s)}{\overline{A_1(|d_1|)A_2(|d_1|)}l(s; d_1)} = 1 + \sum_{m=2}^{\infty} \alpha(i)\frac{l(s; d_i)}{l(s; d_1)}\left(\frac{d_1}{d_i}\right)^{s+k^*}$$

Then by applying the Littlewood lemma which states that

$$\int_{1/3}^{\alpha} \nu(\sigma)d\sigma = \frac{-1}{2\pi i}\int_{\partial R} \log g(s)ds,$$

where ∂R denotes the boundary of R and $\nu(\sigma)$ denotes the number of zeroes minus the number of poles of meromorphic function g in the region R, we get

$$\sum_{\substack{\rho=\beta+i\gamma\in R,\\ f(\rho)=0}} \beta - 1/3 = \frac{1}{2\pi}\int_T^{T+H} \log|f(1/3 + it)|dt - \frac{1}{2\pi}\int_T^{T+H} \log|f(\alpha + it)|dt$$

$$+ \frac{1}{2\pi}\int_{1/3}^{\alpha} arg(f(\sigma + i(T + H)))d\sigma - \frac{1}{2\pi}\int_{1/3}^{\alpha} arg(f(\sigma + iT))d\sigma.$$

Also for large α, $\log|f(\alpha + it)| = O_\Delta(1)$ and as $\sigma \to \infty$, we have

$$f(\sigma + it) = 1 + O_\Delta(e^{-\sigma(\log|d_2| - \log|d_1|)}),$$

and finally

$$\sum_{\substack{\rho=\beta+i\gamma\in R\\ f(\rho)=0}} \beta - 1/3 = O_\Delta(H).$$

If we denote by $N(T, \Xi)$ the number of zeros of $\Xi(s)$ (counted with multiplicity) in the region $\{s : \sigma \geq 1/2, |t| < T\}$, we obtain

$$N(T, \Xi) = O_\Delta(T).$$

Now using the functional equation satisfied by $\overrightarrow{R}(s)$, as given in (10), we get the following functional equation

$$D(s) = G(s)D(1-s),$$

where $G(s)$ is determined by the scattering matrix $\Phi(s)$, and it involves Gamma functions. Using (12), we get

$$L(f_1 \otimes f_2, s) = G(s)L(f_1 \otimes f_2, 2-s)\frac{\Xi(1-s/2)}{\Xi(s/2)} \tag{13}$$

We now look at the number of zeros of $L(f_1 \otimes f_2, s)$ and the function in the right hand side of the functional equation (13) in the ractangle

$$R = \{S = \sigma + it : 1/2 \leq \sigma \leq 1, |t| \leq T\}.$$

It is well known that (see [2]) number of zeros $N(T, f_1 \otimes f_2)$ of $L(f_1 \otimes f_2, s)$ is of order $cT \log T$, where c is a non zero constant. Now, $G(s)$ will have atmost $O(1)$ possible zeros in R, and $L(f_1 \otimes f_2, 2-s)$ will not have any zero in R as $2 - Re(s) \geq 1$. Also $\Xi(s/2)$ has no poles in the region, the major contribution to zeros of right hand side is coming from $\Xi(1-s/2)$, which is of order $O_\Delta(T)$ and hence we get a contradiction.

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