p-adic Asai transfer

Baskar Balasubramanyam[1] and Dipramit Majumdar[2]

[1]*Department of Mathematics, Indian Institute of Science Education and Research Pune, Dr. Homi Bhabha Road, Pashan, Pune 411 008, India*
e-mail: baskar@iiserpune.ac.in

[2]*Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India*
e-mail: dipramit@iitm.ac.in

Communicated by: Prof. Florian Herzig

Abstract. Let $K/\mathbb{Q}$ be a real quadratic field. Given an automorphic representation π for GL_2/K, let $As^{\pm}(\pi)$ denote the plus/minus Asai transfer of π to an automorphic representation for $GL_4/\mathbb{Q}$. In this paper, we construct a rigid analytic map from a subvariety of the universal eigenvariety of GL_2/K to an eigenvariety of $GL_4/\mathbb{Q}$, which at nice classical points interpolate this Asai transfer.

2010 Mathematics Subject Classification: 11F41, 11F55, 11F33, 11F85, 11R39, 14G22.

1. Introduction

Let K be a real quadratic field. Let $k \geq 2$ be an integer. Let f be a Hilbert modular form over K of weight k (i.e., of parallel weight (k, k)) and level 1. Further assume that f is an eigenform for all the Hecke operators with eigenvalues $c(\mathfrak{a}, f)$, where $\mathfrak{a}$ runs over integral ideals of $\mathcal{O}_K$ (the ring of integers in K). The standard L-function for f is constructed from $c(\mathfrak{a}, f)$ as a Dirichlet series over K.

In [1], Asai introduced the following L-function which is constructed only from the Hecke eigenvalues for ideals that come from $\mathbb{Q}$. This is now referred to as the Asai L-function. More specifically, the Asai L-function is defined as

$$G(s, f) = \zeta(2s - 2k + 2) \sum_{m=1}^{\infty} \frac{c(m\mathcal{O}_K, f)}{m^s}.$$

A priori, this function converges for s in a certain right half-plane. It is known that this L-function has an Euler product expansion, analytic continuation to all of $\mathbb{C}$ and a functional equation.

Let π be the automorphic representation associated to the form f. Then the L-function $G(s, f)$ is (up to a shift) a certain automorphic L-function, denoted by $L(s, \pi, As^+)$, associated to π. The principle of Langlands functoriality suggests, in this case, that this automorphic L-function is the standard L-function of an automorphic representation for $GL_4/\mathbb{Q}$. The details of this Asai transfer are recalled in §2. Fix a prime p that is unramified in K. The aim of this article is to construct a p-adic version of this Asai transfer.

Specifically, when p splits in K, we construct a rigid analytic map from a closed subvariety of the eigenvariety attached to GL_2/K to the universal eigenvariety attached to $GL_4/\mathbb{Q}$ that interpolate the classical Asai transfer on a dense subset of classical points. But when p is inert, we are only able to construct a map to an eigenvariety which may be viewed as a quotient of the universal eigenvariety attached to $GL_4/\mathbb{Q}$. For the purpose of simplification, we have only constructed p-adic Asai transfer for tame level 1 eigenvarieties. There should be no added difficulty to extend this arbitrary tame level eigenvarieties.

Historically, the study of Langlands' functoriality in families of automorphic forms can be traced back to the work of Hida [9], where a Λ-adic Jacquet–Langlands' transfer is constructed between families of Hilbert modular

forms and of quaternionic automorphic forms. In the setting of eigenvarieties, Chenevier [7] constructed p-adic Jacquet–Langlands transfer, which at classical points interpolate the classical Jacquet–Langlands' transfer for GL_2. Following Chenevier's method, other instances of p-adic Langlands' functoriality have been established, see [8,12–15,19].

This article is organized as follows. In §2, we recall some of the basic properties of the Asai transfer. In § 3, we recall the construction due to Hansen of universal eigenvarieties attached to certain connected reductive groups. In § 4, we explicitly describe the eigenvarieties that are relevant to our construction of the p-adic Asai transfer map. In § 5, we finally construct the p-adic Asai transfer map between eigenvarieties using a comparison theorem obtained in section 3.

Notations. Throughout this article, p will denote a fixed odd integer prime, and K a real quadratic extension in which p is unramified. The ring of adeles over $\mathbb{Q}$ will be denoted by $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$. We will denote by $\mathbb{A}_f$ and $\mathbb{A}_\infty$ the finite adeles and the adeles at infinity, respectively. For a set of places S of $\mathbb{Q}$, we will denote by $\mathbb{A}_S$ the adeles supported at S and by $\mathbb{A}^S$ the adeles supported away from S. We will denote by $\mathbb{A}_K = \mathbb{A} \otimes_{\mathbb{Q}} K$ the adeles over K. We similarly define $\mathbb{A}_{K,f}$, $\mathbb{A}_{K,\infty}$, $\mathbb{A}_K^S$ and $\mathbb{A}_{K,S}$ when S is a set of places of K.

Acknowledgements

2. Asai transfer

Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field. We now recall the Langlands functorial transfer from automorphic representations for GL_2 over K to automorphic representations of GL_4 over $\mathbb{Q}$ that is called the Asai transfer.

2.1 Map between L-groups

Let G_1 denote the algebraic group GL_2 over K and let $G_1 = \mathrm{Res}_{K/\mathbb{Q}} G_1$ denote the Weil restriction of G_1 from K to $\mathbb{Q}$. The Langlands dual group for G_1 is defined as

$$^L G_1 = (GL_2(\mathbb{C}) \times GL_2(\mathbb{C})) \rtimes \mathrm{Gal}(K/\mathbb{Q})$$

where the nontrivial element $c \in \mathrm{Gal}(K/\mathbb{Q})$ acts on the tuple of matrices via permutation. Let G_2 denote the algebraic group GL_4 over $\mathbb{Q}$. The L-group of G_2 is given by $^L G_2 = GL_4(\mathbb{C})$. Let $\mathrm{As}^\pm$ denote the following representation of $^L G_1$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$ given by

$$\mathrm{As}^\pm[(A, B)](x \otimes y) = Ax \otimes By, \quad \text{for } A, B \in GL_2(\mathbb{C})$$

$$\mathrm{As}^\pm[c](x \otimes y) = \pm(y \otimes x).$$

We view these representations as maps between the L-groups $\mathrm{As}^\pm : {}^L G_1 \to {}^L G_2$. The main results of [11] and [16] show that the Asai transfer is automorphic.

2.2 The Asai motive

We now recall some basic facts about Hilbert modular forms and the associated Asai motive. Let $\sigma_1, \sigma_2 : K \to \mathbb{R}$ denote the two embeddings of K into $\mathbb{R}$. The weights of Hilbert modular forms are elements of the lattice $\mathbb{Z}[\sigma] := \mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2$. Specifically, the weight of a Hilbert modular form will correspond to a pair (n, v) with

$n, v \in \mathbb{Z}[\sigma]$ such that $n + 2v$ is parallel. By a parallel weight, we mean that $n + 2v = mt$ where $t = \sigma_1 + \sigma_2$ and $m \in \mathbb{Z}$. Writing $n = n_1\sigma_1 + n_2\sigma_2$, this condition implies that the parity of n_1 and n_2 are the same. The parity assumption is necessary for the existence of Hilbert modular forms of a particular weight. Let $k = n + 2t$ and assume further that $k_i \geq 2$.

Let $\mathfrak{n}$ be an integral ideal in K. Let f be a Hilbert cusp form of weight (n, v) and level $\mathfrak{n}$. Suppose that f is a primitive eigenform, then the motive M attached to f is a pure simple rank 2 motive defined over K. The Hodge types of the motive at σ_i are $\{(n_i + 1 + v_i, v_i), (v_i, n_i + 1 + v_i)\}$. If $\mathfrak{a}$ is an integral ideal in K, let $c(\mathfrak{a}, f)$ denote the Hecke eigenvalue for the Hecke operator $T(\mathfrak{a})$. Let $^c f$ be the Hilbert modular form whose Hecke eigenvalues are given by $c(\mathfrak{a}, {}^c f) = c(\mathfrak{a}^c, f)$. Let $^c M$ denote the motive associated to $^c f$. The Hodge type of this conjugate motive at σ_i will be the Hodge type at $\sigma_i \circ c$.

The motives $\mathrm{As}^{\pm}(M)$ associated to the Asai transfer will be pure simple rank 4 motives defined over $\mathbb{Q}$. The Hodge types at the infinite place of $\mathbb{Q}$ are

$$(n_1 + n_2 + v_1 + v_2 + 2, v_1 + v_2),$$

$$(n_1 + 1 + v_1 + v_2, n_2 + 1 + v_1 + v_2),$$

$$(n_2 + 1 + v_1 + v_2, n_1 + 1 + v_1 + v_2),$$

$$(v_1 + v_2, n_1 + n_2 + v_1 + v_2 + 2).$$

Note that the weight of this motive is $n_1 + n_2 + 2(v_1 + v_2) + 2 = 2m + 2$. Note also that when $n_1 = n_2$, the Asai motive has a middle (i.e., (p, p)) Hodge type and is hence not cohomological. Henceforth, we assume that $n_1 > n_2$.

2.3 Weight of the Asai transfer

Let G_1 and G_2 be as above. A weight for G_1 is a tuple $\lambda = (\lambda_1, \lambda_2)$ where $\lambda_i \in \mathbb{Z}^2$. We say that the weight λ is dominant if $\lambda_i = (a_i, b_i)$ with $a_i \geq b_i$. We can relate these weights to the ones discussed above by taking $a_i = n_i + v_i$ and $b_i = v_i$.

Let π be an automorphic representation for G_1. Let λ be a dominant weight for G_1. We say that π is cohomological of weight λ if

$$H^*(\mathfrak{g}_{1,\infty}, K_{1,\infty}; \pi_\infty \otimes \mathscr{L}_\lambda) \neq 0,$$

where $\mathfrak{g}_{1,\infty}$ is the Lie algebra of $G_{1,\infty} = G_1(\mathbb{R})$ and $K_{1,\infty}$ is the maximal compact modulo the centre in $G_{1,\infty}$, and $\mathscr{L}_\lambda$ is the highest weight representation associated to λ. In order for λ to support cohomological automorphic representations, we require that λ be pure, i.e., $a_1 + b_1 = a_2 + b_2$.

A weight for G_2 is a tuple $\mu \in \mathbb{Z}^4$. We say that $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ is dominant if $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$. If Π is an automorphic representation for G_2 and μ a dominant weight for G_2, we define the notion of Π being cohomological of weight μ in a similar fashion. Similarly, for μ to support cohomological automorphic representations, we require that μ is pure, i.e., $\mu_1 + \mu_4 = \mu_2 + \mu_3$.

Let π be the automorphic representation over $G_1(\mathbb{A})$ attached to the primitive eigenform f of weight (n, v). Then $\pi_\infty = \pi_1 \otimes \pi_2$, where π_∞ is the representation at infinity and π_i are discrete series representations up to twists by powers of the determinant. The Langlands parameter of π_i is given by

$$\tau(\pi_i) = z^{\frac{1}{2} - v_i} \bar{z}^{-n_i - v_i - \frac{1}{2}} + z^{-n_i - v_i - \frac{1}{2}} \bar{z}^{\frac{1}{2} - v_i}.$$

One calculates that the Langlands parameter of the Asai transfer is

$$\tau(\mathrm{As}^{\pm}(\pi)) = z^{1 - v_1 - v_2} \bar{z}^{-n_1 - n_2 - v_1 - v_2 - 1} + z^{-n_1 - n_2 - v_1 - v_2 - 1} \bar{z}^{1 - v_1 - v_2}$$

$$+ z^{-n_2 - v_1 - v_2} \bar{z}^{-n_1 - v_1 - v_2} + z^{-n_1 - v_1 - v_2} \bar{z}^{-n_2 - v_1 - v_2}.$$

As the exponents in the Langlands parameter are not half-integers, $\mathrm{As}^{\pm}(\pi)$ is not cohomological. However, if we normalize the Asai transfer to be $\mathrm{As}^{\pm}(\pi) \otimes |\det|^{1/2}$, then the Langlands parameter becomes

$$z^{\frac{3}{2} - v_1 - v_2} \bar{z}^{-n_1 - n_2 - v_1 - v_2 - \frac{1}{2}} + z^{-n_1 - n_2 - v_1 - v_2 - \frac{1}{2}} \bar{z}^{\frac{3}{2} - v_1 - v_2}$$

$$+ z^{\frac{1}{2} - n_2 - v_1 - v_2} \bar{z}^{\frac{1}{2} - n_1 - v_1 - v_2} + z^{\frac{1}{2} - n_1 - v_1 - v_2} \bar{z}^{\frac{1}{2} - n_2 - v_1 - v_2}.$$

One can verify that this representation is supported in cohomology of weight $\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \in \mathbb{Z}^4$ which we now describe. The weight μ is pure (in the sense of Clozel) with purity weight $w = 2m - 1$, i.e., $\mu_1 + \mu_4 = w = \mu_2 + \mu_3$ (here $m = n_1 + 2v_1$). We also have

$$\mu_1 = \frac{n_1 + n_2}{2} + m - 1$$

$$\mu_2 = \frac{n_1 - n_2}{2} + m - 1.$$

We easily calculate

$$\mu_3 = m - \frac{n_1 - n_2}{2}$$

$$\mu_4 = m - \frac{n_1 + n_2}{2}$$

from the purity condition.

Remark 2.3.1. We note from this calculation that when $n_1 = n_2$, the weight μ is not dominant. This shows that Asai transfer for such automorphic representations do not contribute to the cuspidal cohomology. This applies in particular to those that arise from parallel weight Hilbert cusp forms. Hence, we do not include parallel weights in our sets of classical weights, see (5.1.1) and (5.2.2).

2.4 Local Asai transfer of unramified representations

In this section, we describe the local Asai transfer of unramified representations at almost all finite places in terms of unramified Langlands' parameters. Let k be any local field, let W_k denote the Weil group for k. Let ℓ be an integer prime that is unramified in K. We have two cases depending on whether ℓ splits or is inert in K.

First we assume that ℓ is split in K. Say, $(\ell) = \mathfrak{l}\mathfrak{l}^c$. Since $G_1 = \mathrm{Res}_{K/\mathbb{Q}}(G_1)$, we have $(G_1)_\ell = G_1/\mathbb{Q}_\ell = (G_1)_\mathfrak{l} \times (G_1)_{\mathfrak{l}^c}$. We know that $^L(G_1)_\mathfrak{l} = {}^L(G_1)_{\mathfrak{l}^c} = \mathrm{GL}_2(\mathbb{C})$ and $^L(G_1)_\ell = {}^L(G_1)_\mathfrak{l} \times {}^L(G_1)_{\mathfrak{l}^c}$. By an L-parameter for $(G_1)_\mathfrak{l}$, we mean a continuous morphism

$$\varphi : W_{K_\mathfrak{l}} \to {}^L(G_1)_\mathfrak{l}$$

such that $\varphi(x)$ is semi-simple for all $x \in W_{K_\mathfrak{l}}$. Similarly take an L-parameter for $(G_1)_{\mathfrak{l}^c}$ denoted by

$$\varphi^c : W_{K_{\mathfrak{l}^c}} \to {}^L(G_1)_{\mathfrak{l}^c}.$$

Now let $\mathrm{Frob}_\mathfrak{l}$, $\mathrm{Frob}_{\mathfrak{l}^c}$ and Frob_ℓ denote the Frobenius elements associated to $\mathfrak{l}$, $\mathfrak{l}^c$ and ℓ respectively. We now assume that the parameters φ and φ^c are unramified; that is, the maps factors through inertia subgroup and hence are determined by the values $\varphi(\mathrm{Frob}_\mathfrak{l})$ and $\varphi^c(\mathrm{Frob}_{\mathfrak{l}^c})$.

Identifying $W_{K_\mathfrak{l}}$ with $W_{\mathbb{Q}_\ell}$ and using φ and φ^c, we now construct an L-parameter for $(G_1)_\ell$ as

$$\tilde{\varphi} = \varphi \times \varphi^c : W_{\mathbb{Q}_\ell} \to {}^L(G_1)_\ell.$$

Note that $\tilde{\varphi}$ is determined by the value $\tilde{\varphi}(\mathrm{Frob}_\ell)$. We now describe the Asai transfer for unramified L-parameters below. Suppose that the

$$\varphi(\mathrm{Frob}_\mathfrak{l}) = \begin{bmatrix} \alpha_\mathfrak{l} & \\ & \beta_\mathfrak{l} \end{bmatrix} \quad \text{and} \quad \varphi^c(\mathrm{Frob}_{\mathfrak{l}^c}) = \begin{bmatrix} \alpha_{\mathfrak{l}^c} & \\ & \beta_{\mathfrak{l}^c} \end{bmatrix},$$

then we see that

$$\tilde{\varphi}(\mathrm{Frob}_\ell) = \left(\begin{bmatrix} \alpha_\mathfrak{l} & \\ & \beta_\mathfrak{l} \end{bmatrix}, \begin{bmatrix} \alpha_{\mathfrak{l}^c} & \\ & \beta_{\mathfrak{l}^c} \end{bmatrix} \right).$$

Applying the Asai transfer $\mathrm{As}^{\pm}$, we see that

$$\mathrm{As}^{\pm}\tilde{\varphi}(\mathrm{Frob}_{\ell}) = \begin{bmatrix} \alpha_{\ell}\alpha_{\ell^c} & & & \\ & \alpha_{\ell}\beta_{\ell^c} & & \\ & & \beta_{\ell}\alpha_{\ell^c} & \\ & & & \beta_{\ell}\beta_{\ell^c} \end{bmatrix}.$$

The calculation above shows that the local Asai transfer for f at a split place $\ell = \mathfrak{l}\mathfrak{l}^c$ is same as local Rankin–Selberg transfer for $f \times {}^c f$ at ℓ.

Now we assume that ℓ is inert in K. Let K_{ℓ} denote the completion of K at ℓ. This is an unramified quadratic extension of $\mathbb{Q}_{\ell}$. In this case, ${}^L(G_1)_{\ell} = ({}^L(G_1)_{\ell} \times {}^L(G_1)_{\ell}) \rtimes \mathrm{Gal}(K_{\ell}/\mathbb{Q}_{\ell})$ where ${}^L(G_1)_{\ell} = \mathrm{GL}_2(\mathbb{C})$. We also know that $W_{K_{\ell}}$ is an index 2 subgroup of $W_{\mathbb{Q}_{\ell}}$. We fix a Frobenius element $\mathrm{Frob}_{\mathbb{Q},\ell} \in W_{\mathbb{Q}_{\ell}}\backslash W_{K_{\ell}}$ for the ideal ℓ in $\mathbb{Q}_{\ell}$ and take $\mathrm{Frob}_{K,\ell} = \mathrm{Frob}_{\mathbb{Q},\ell}^2$ to be the Frobenius element for the ideal ℓ in K_{ℓ}.

We now consider an L-parameter

$$\varphi : W_{K_{\ell}} \to {}^L(G_1)_{\ell}$$

of $(G_1)_{\ell}$. There is an extension of φ to an L-parameter of $(G_1)_{\ell}$ constructed as follows. Pick any $j \in W_{\mathbb{Q}_{\ell}}\backslash W_{K_{\ell}}$. Define a map

$$\tilde{\varphi} : W_{\mathbb{Q}_{\ell}} \to {}^L(G_1)_{\ell}$$

by sending $x \in W_{K_{\ell}}$ to $(\varphi(x), \varphi(jxj^{-1})) \times 1$ and by sending j to $(\mathrm{Id}, \varphi(j^2)) \times c$.

We now assume that φ is unramified; that is, it is φ factors through the inertia subgroup and is hence determined by the value $\varphi(\mathrm{Frob}_{K,\ell})$. We now construct $\tilde{\varphi}$ by taking $j = \mathrm{Frob}_{\mathbb{Q},\ell}$ and it is clear that $\tilde{\varphi}$ is completely determined by the value $\tilde{\varphi}(\mathrm{Frob}_{\mathbb{Q},\ell})$. Suppose that

$$\varphi(\mathrm{Frob}_{K,\ell}) = \begin{bmatrix} \alpha_{\ell} & \\ & \beta_{\ell} \end{bmatrix},$$

then we see that

$$\tilde{\varphi}(\mathrm{Frob}_{\mathbb{Q},\ell}) = \left(\mathrm{Id}, \begin{bmatrix} \alpha_{\ell} & \\ & \beta_{\ell} \end{bmatrix} \right) \times c.$$

Applying the Asai transfer map, we get

$$\mathrm{As}^{\pm}\tilde{\varphi}(\mathrm{Frob}_{\mathbb{Q},\ell}) = \begin{bmatrix} \pm\alpha_{\ell} & & & \\ & \pm\beta_{\ell} & & \\ & & \pm\alpha_{\ell} & \\ & & & \pm\beta_{\ell} \end{bmatrix}.$$

This last matrix is equivalent to

$$\begin{bmatrix} \pm\alpha_{\ell} & & & \\ & \pm\beta_{\ell} & & \\ & & \pm\sqrt{\alpha_{\ell}\beta_{\ell}} & \\ & & & \mp\sqrt{\alpha_{\ell}\beta_{\ell}} \end{bmatrix}.$$

Unramified L-parameters are in bijection with unramified automorphic representations [4, Proposition 1.12.1]. Hence this calculation completely determines the Asai transfer of unramified representations at unramified places.

Remark 2.4.1.

(1) Given $\pi = \otimes_v \pi_v$, the local Langlands correspondence gives the local Asai transfer $\mathrm{As}^{\pm}(\pi_v)$. If $\mathrm{As}^{\pm}(\pi) = \otimes_v \mathrm{As}^{\pm}(\pi_v)$, the global Asai transfer (i.e., the automorphy of $\mathrm{As}^{\pm}(\pi)$) is proved using the converse theorem. The analytic properties for certain L-functions needed to apply the converse theorem are proved using the Rankin–Selberg method (in [16]) or the Langlands–Shahidi method (in [11]).

(2) We have only described the Asai transfer away from a finite set of bad places. However, since the global Asai transfer is known, by multiplicity one for GL_4 this is sufficient to completely determine the representation $\mathrm{As}^{\pm}(\pi)$.

3. Brief overview of eigenvarieties due to Hansen

In this section, we first recall the notion of eigenvariety datum and the construction of an eigenvariety from such a datum. We then recall the construction of universal eigenvariety due to Hansen for certain reductive groups G. Finally, we recall the comparison theorem that allows us to construct rigid analytic maps between eigenvarieties. Our main reference for this section will be [8], and we adopt much of its notation.

3.1 Eigenvariety data

Let p be an odd prime. An eigenvariety datum is defined as a tuple $\mathscr{D} = (\mathscr{W}, \mathscr{Z}, \mathscr{M}, \mathbb{T}, \psi)$. We describe below each of the terms appearing in this definition.

The space $\mathscr{W}$ is a separated, reduced, relatively factorial rigid analytic space and is called the weight space. In our context, the space $\mathscr{W}$ will parametrize homomorphisms from the maximal torus of a reductive group G. The weight space contains as a dense subset a set of classical weights that support classical automorphic forms on G.

Let $\mathbf{A}^1$ denote the rigid analytic affine line. The spectral variety $\mathscr{Z} \subset \mathscr{W} \times \mathbf{A}^1$ is a Fredholm hypersurface, i.e., a closed immersion that is cut out by a Fredholm series. See [8, Definition 4.1.1] for the precise definition. Projection on the first coordinate induces a map $w : \mathscr{Z} \to \mathscr{W}$ called the weight map.

The sheaf $\mathscr{M}$ is a coherent sheaf on $\mathscr{Z}$. This sheaf is usually constructed from a suitable graded module M^* of overconvergent cohomology classes of a complex whose cohomology yields M^*. The Hecke algebra $\mathbb{T}$ will be a commutative $\mathbb{Q}_p$-algebra equipped with an action $\psi : \mathbb{T} \to \mathrm{End}_{\mathcal{O}_{\mathscr{Z}}}(\mathscr{M})$. The variety $\mathscr{Z}$, in fact, will parametrize eigenvalues of an operator $U \in \mathbb{T}$ acting on the complex whose cohomology is M^*.

The following theorem gives us the eigenvariety associated to an eigenvariety datum. This follows from Buzzard's eigenvariety machine [5].

Theorem 3.1.1 ([8, Theorem 4.2.2]). *Given an eigenvariety datum $\mathscr{D}$, there exists a separated rigid analytic space $\mathscr{X}$ together with a finite morphism $\pi : \mathscr{X} \to \mathscr{Z}$, a morphism $w : \mathscr{X} \to \mathscr{W}$, an algebra homomorphism $\phi_{\mathscr{X}} : \mathbb{T} \to \mathcal{O}(\mathscr{X})$, and a coherent sheaf $\mathscr{M}^\dagger$ on $\mathscr{X}$ together with a canonical isomorphism $\mathscr{M} \cong \pi_* \mathscr{M}^\dagger$ compatible with the actions of $\mathbb{T}$ on $\mathscr{M}$ and $\mathscr{M}^\dagger$ (via ψ and $\phi_{\mathscr{X}}$, respectively). The points of $\mathscr{X}$ lying over $z \in \mathscr{Z}$ are in bijection with the generalized eigenspaces for the action of $\mathbb{T}$ on $\mathscr{M}(z)$ (the stalk at z).*

3.2 Universal eigenvariety

Throughout this section, let G denote a connected reductive algebraic group over $\mathbb{Q}$ that is restriction of scalars from a number field to $\mathbb{Q}$ of an algebraic group H split at all primes above p. Let B, N, Z and T denote a choice of a Borel subgroup, unipotent subgroup, the centre and maximal torus respectively. We also fix a p-integral model for H. This allows us to define $\mathbb{Z}_p$-valued points of G. Let I denote the Iwahori subgroup of $G(\mathbb{Z}_p)$ associated to the choice of B. In this subsection, we recall the definition and basic properties of the eigenvariety associated to G.

The weight space $\mathscr{W}_G$ associated to G is a rigid analytic space whose $\overline{\mathbb{Q}}_p$ points are given by $\mathscr{W}_G(\overline{\mathbb{Q}}_p) = \mathrm{Hom}_{\mathrm{cts}}(T(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$. For any open compact subgroup $K^p \subset G(\mathbb{A}_{\mathbb{Q}}^p)$, let $\mathscr{W} = \mathscr{W}_G(K^p)$ denote weight space of level K^p which parametrizes continuous homomorphisms from the torus that are trivial on the closure of $Z(K^p I) \cap G(\mathbb{Q}) \subset T(\mathbb{Z}_p)$.

A Hecke pair consists of a monoid $\Delta \subset G(\mathbb{A}_f)$ and a subgroup $K_f \subset \Delta$ such that K_f and $\delta K_f \delta^{-1}$ are commensurable for all $\delta \in \Delta$. We will denote by $\mathbb{T}(\Delta, K_f)$ the $\mathbb{Q}_p$-algebra generated by double cosets $T_\delta = [K_f \delta K_f]$ under the convolution product.

The algebra $\mathbb{T}$ is the eigenvariety data will be of the form

$$\mathbb{T} = \mathbb{T}_G(K^p) = \mathscr{A}_p^+ \otimes \mathbb{T}^{\mathrm{unr}}(K^p),$$

where $\mathscr{A}_p^+$ is a certain subalgebra of the Iwahori Hecke algebra $\mathbb{T}(G(\mathbb{Q}_p), I)$ and the unramified Hecke algebra is a commutative algebra given by

$$\mathbb{T}^{\mathrm{unr}}(K^p) = \bigotimes_{v \notin S}' \mathbb{T}(G(\mathbb{Q}_v), K_v^p)$$

for a finite set of primes S. In practice, K_v^p will be a hyperspecial maximal subgroup of $G(\mathbb{Q}_v)$, which ensures the commutativity of the Hecke algebra. There is a subclass of operators in $\mathscr{A}_p^+$ which are called controlling operators which play a crucial role in the construction of the eigenvariety.

We now proceed to describe the construction of the coherent sheaf $\mathscr{M}$ along with the action of the Hecke algebra. Let $\bar{B}$ and $\bar{N}$ respectively denote the opposite Borel and unipotent subgroup to B and N. For any $s \in \mathbb{Z}_{\geq 0}$, define $\bar{B}^s = \{b \in \bar{B}(\mathbb{Z}_p) \mid b \equiv 1 \in G(\mathbb{Z}_p/p^s\mathbb{Z}_p)\}$. Similarly, define $T^s = T(\mathbb{Z}_p) \cap \bar{B}^s$, $\bar{N}^s = \bar{N}(\mathbb{Z}_p) \cap \bar{B}^s$ and $I^s = I \cap \operatorname{Ker}\{G(\mathbb{Z}_p) \to G(\mathbb{Z}_p/p^s\mathbb{Z}_p)\}$. Furthermore, we define $I_1^s = \{g \in I \mid g \bmod p^s \in \bar{N}(\mathbb{Z}/p^s/\mathbb{Z})\}$.

Let $\Omega \subset \mathscr{W}$ be an admissible affinoid open subset. For Ω, the tautological character induced from $id \in \mathscr{W}(\mathscr{W})$ is denoted by

$$\chi_\Omega : T(\mathbb{Z}_p) \to \mathcal{O}(\Omega)^\times.$$

Let $s[\Omega]$ denote the smallest integer for which $\chi_\Omega|_{T^{s[\Omega]}}$ is analytic. For $s \geq s[\Omega]$, define

$$A_\Omega^s = \left\{ f : I \to \mathcal{O}(\Omega) \middle| \begin{array}{l} f \text{ is analytic on each } I^s\text{-cosets and} \\ f(gtn) = \chi_\Omega(t)f(g) \text{ for all } n \in N(\mathbb{Z}_p), t \in T(\mathbb{Z}_p), g \in I \end{array} \right\}.$$

Via the map $f \mapsto f|_{\bar{N}^1}$, we identify A_Ω^s with the space of s-locally analytic $\mathcal{O}(\Omega)$ valued functions on $\bar{N}^1$. Hence A_Ω^s is endowed with a Banach $\mathcal{O}[\Omega]$-module structure. We have natural injective, compact transition maps $A_\Omega^s \to A_\Omega^{s+1}$ and taking direct limit with respect to these transition maps, we define

$$\mathcal{A}_\Omega = \varinjlim_s A_\Omega^s.$$

The corresponding distribution space is defined as the continuous $\mathcal{O}(\Omega)$-linear dual of $\mathcal{A}_\Omega$,

$$\mathcal{D}_\Omega = \{\mu : \mathcal{A}_\Omega \to \mathcal{O}(\Omega) \mid \mu \text{ is } \mathcal{O}(\Omega)\text{-linear and continuous}\}.$$

If $\lambda \in \mathscr{W}$ is any point, we similarly define the modules A_λ^s, $\mathcal{A}_\lambda$ and $\mathcal{D}_\lambda$.

Fix a controlling operator $U \in \mathscr{A}_p^+$. For Ω as above, there exist complexes $C_\bullet(K^pI, \mathcal{A}_\Omega)$ and $C^\bullet(K^pI, \mathcal{D}_\Omega)$ that admit an extension $\tilde{U}$ of the controlling operator U. We remark that the controlling operator acts as compact operator on $C_\bullet(K^pI, \mathcal{A}_\Omega)$ and $C^\bullet(K^pI, \mathcal{D}_\Omega)$. Let $f_\Omega(X)$ denote the Fredholm series associated to this action on $C_\bullet(K^pI, \mathcal{A}_\Omega)$. Then these functions patch together to give Fredholm series $f \in \mathcal{O}(\mathscr{W})\{\{X\}\}$. The spectral variety $\mathscr{Z} = \mathscr{Z}_G$ is defined as the Fredholm hypersurface $\mathscr{Z}_f$ associated to this series.

Given an affinoid open subset $\Omega \subset \mathscr{W}$ and $h \in \mathbb{Q}$, there is exists a corresponding affinoid open subset $\mathscr{Z}_{\Omega,h} \subset \mathscr{Z}$. We call this slope-adapted if $f_\Omega = f|_{\mathcal{O}(\Omega)\{\{X\}\}}$ has a slopge-$\leq h$ decomposition. The slope-adapted affinoids form an admissible open cover of $\mathscr{Z}$. We also know that if Ω is slope adapted for h, then $C_\bullet(K^pI, \mathcal{D}_\Omega)$ admits a slope-$\leq h$ decomposition.

Moreover, there exists a unique complex of coherent analytic sheaves, $\mathscr{K}^\bullet$, on $\mathscr{Z}$ such that $\mathscr{K}^\bullet(\mathscr{Z}_{\Omega,h}) = C^\bullet(K^pI, \mathcal{D}_\Omega)_{\leq h}$ for any slope-adapted $\mathscr{Z}_{\Omega,h}$. Taking cohomology of $\mathscr{K}^\bullet$, we get a graded sheaf $\mathscr{M}^*$ on $\mathscr{Z}$, such that $\mathscr{M}^*(\mathscr{Z}_{\Omega,h}) = H^*(K^pI, \mathcal{D}_\Omega)_{\leq h}$. This sheaf comes equipped with a Hecke action, which we denote by $\psi : \mathbb{T} \to \operatorname{End}_{\mathcal{O}_{\mathscr{Z}}}(\mathscr{M}^*)$. Finally, we take our coherent sheaf $\mathscr{M}$ in the eigenvariety data to be the graded sheaf $\mathscr{M}^*$.

To summarise, the eigenvariety datum given by

$$\mathscr{D} = (\mathscr{W}, \mathscr{Z}, \mathscr{M}, \mathbb{T}, \psi)$$

gives rise to the eigenvariety associated to G via Buzzard's machinery. The eigenvariety $\mathscr{X} = \mathscr{X}(\mathscr{D})$ thus constructed is a separated rigid analytic variety along with

(i) a finite morphism $\pi : \mathscr{X} \to \mathscr{Z}$,

(ii) a morphism (weight map) $w : \mathscr{X} \to \mathscr{W}$,

(iii) an algebra homomorphism $\phi_{\mathscr{X}} : \mathbb{T} \to \mathcal{O}(\mathscr{X})$, and

(iv) a coherent sheaf $\mathscr{M}^\dagger$ on $\mathscr{X}$ together with a canonical isomorphism $\mathscr{M} \cong \pi_*\mathscr{M}^\dagger$ compatible with the action of $\mathbb{T}$.

3.3 Points on the eigenvariety and refinements

The points on the eigenvariety lying over $z \in \mathcal{Z}$ are in bijection with the generalized eigenspaces for the action of $\mathbb{T}$ on $\mathcal{M}(z)$.

Definition 3.3.1. *A finite-slope eigenpacket of weight $\lambda \in \mathcal{W}(\overline{\mathbb{Q}}_p)$ and level K^p is a algebra homomorphism* $\phi : \mathbb{T} \to \overline{\mathbb{Q}}_p$ *such that the space*

$$\{v \in H^*(K^p I, \mathcal{D}_\lambda) \otimes_{k_\lambda} \overline{\mathbb{Q}}_p \mid T \cdot v = \phi(T)v \text{ for all } T \in \mathbb{T} \text{ and } \phi(U) \neq 0\}$$

is nonzero, that is, there exists a non-zero eigenvector acting by the homomorphism ϕ. Here U is the controlling operator fixed earlier.

Another way to define finite-slope eigenpacket is as follows. We denote by $\mathbb{T}_{\lambda,h}(K^p)$ the subalgebra of $\mathrm{End}_{k_\lambda}(H^*(K^p I, \mathcal{D}_\lambda)_{\leq h})$ generated by the image of $\mathbb{T} \otimes_{\mathbb{Q}_p} k_\lambda$. We define the algebra $\mathbb{T}_\lambda(K^p) = \varprojlim_h \mathbb{T}_{\lambda,h}(K^p)$. Finite-slope eigenpackts of weight λ and level K^p can be identified with algebra homomorphisms $\phi : \mathbb{T}_\lambda(K^p) \to \overline{\mathbb{Q}}_p$.

Given any point $x \in \mathcal{X}(\overline{\mathbb{Q}}_p)$ on the eigenvariety, we can naturally define an algebra homomorphism

$$\phi_x : \mathbb{T} \xrightarrow{\phi_{\mathcal{X}}} \mathcal{O}(\mathcal{X}) \longrightarrow \mathcal{O}_{\mathcal{X},x} \longrightarrow k_x$$

called the eigenpacket parametrized by x. The points $x \in \mathcal{X}(\overline{\mathbb{Q}}_p)$ lying over a given weight $\lambda \in \mathcal{W}(\overline{\mathbb{Q}}_p)$ are in bijection with the finite-slope eigenpackets for G of weight λ and level K^p obtained by sending $x \mapsto \phi_x$.

We say a point x in the eigenvariety $\mathcal{X}(\overline{\mathbb{Q}}_p)$ is classical if ϕ_x is the eigenpacket corresponding to a cohomological automorphic representation of G of weight λ and level K^p.

Before we discuss the notion of refinements, we need to make explicit the Hecke operators that we consider at p. Let Φ^+ denote the positive roots for the choice of the Borel subgroup B. Define the semi-group

$$T^+ = \{t \in T(\mathbb{Q}_p) \mid v_p(\alpha(t)) \leq 0, \forall \alpha \in \Phi^+\},$$

and similarly

$$T^{++} = \{t \in T(\mathbb{Q}_p) \mid v_p(\alpha(t)) < 0, \forall \alpha \in \Phi^+\}.$$

For the monoid $\Delta = \Delta_p = I T^+ I$, we consider the Hecke operators $\mathcal{A}_p^+ = \mathbb{T}(\Delta, I)$. Moreover, we define the space of Atkin-Lehner operators $\mathcal{A}_p \subset \mathbb{T}(G(\mathbb{Q}_p), I)$ as the subalgebra of the Iwahori Hecke algebra generated by U_t and U_t^{-1}, where $U_t = [ItI]$ denotes the double coset operators for $t \in T^+$. Furthermore, we take our controlling operators to be of the form U_t where $t \in T^{++}$. See [8, §2] for further details.

We also briefly clarify the action of Δ on the cohomology groups in sight. There is a natural right action of I on A_Ω^s. Indeed, given $f \in A_\Omega^s$ and $\gamma \in I$, we define $f|_\gamma (g) = f(\gamma g)$. This action induces an action on $\mathcal{A}_\Omega$ and by duality on $\mathcal{D}_\Omega$. This action extends to Δ as follows: identifying N^1 with $I/B(\mathbb{Z}_p)$, we define $t \star gB(\mathbb{Z}_p) = tgt^{-1}B(\mathbb{Z}_p)$ for $t \in T^+$ and $g \in I$; the action of Δ on A_Ω^s, $\mathcal{A}_\Omega$ and $\mathcal{D}_\Omega$ as before. The induced action of the Hecke algebra on cohomology groups will be denoted as the $\star$-action.

Now suppose that locally G is of the form $\mathrm{GL}_n/\mathbb{Q}_p$. Then, we take the following generators for $\mathcal{A}_p^+$ and $\mathcal{A}_p$. We denote by $U_{p,i}$ the element in $\mathcal{A}_p^+$ given by the diagonal matrix

$$(1, \ldots, 1, p, \ldots, p)$$

where p occurs i times. We let $u_{p,i} = U_{p,i-1}^{-1} U_{p,i} \in \mathcal{A}_p$.

Let π_p be an unramified irreducible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ defined over L. Denote by $r : \mathrm{WD}(\mathbb{Q}_p) \to \mathrm{GL}_n(L)$ the Weil–Deligne representation associated to π_p. Here $\mathrm{WD}(\mathbb{Q}_p)$ denotes the Weil–Deligne group of $\mathbb{Q}_p$. Let $\phi_1, \ldots, \phi_n$ be any ordering of eigenvalues of $r(\mathrm{Frob}_p)$. This ordering of eigenvalues gives rise to a character χ of $\mathcal{A}_p$ by the formula $\chi(u_{p,i}) = p^{1-i}\phi_i$. The character χ is called a refinement of π_p.

There exists a vector $0 \neq v \in \pi_p^I$, such that $\mathcal{A}_p$ acts on v by χ. If π is a classical automorphic representation on G such that π_p is unramified and if $x \in \mathcal{X}(\overline{\mathbb{Q}}_p)$ corresponds to π, then we obtain a refinement of π_p by considering $\phi_x|_{\mathcal{A}_p}$. That is, each classical automorphic representation appears roughly $n!$ times in $\mathcal{X}$. Hence, we often denote classical points in the eigenvariety as a tuple (π, χ).

3.4 Control theorem

In this section, we define the notion of arithmetic weight and the space of classical automorphic forms above such weights. We then state the control theorem due to Ash–Stevens and Urban, which relates overconvergent automorphic forms with small slopes and classical automorphic forms.

Let X^* denote the integral weight lattice for G and let $X^*_+ \subset X^*$ denote the subset of B-dominant weights. We call a weight $\lambda \in \mathcal{W}$ arithmetic if $\lambda = \lambda_1 \epsilon$ where ϵ is a finite order character of $T(\mathbb{Z}_p)$ and $\lambda_1 \in X^*$. Let $s[\epsilon]$ be the smallest integer such that ϵ is trivial on $T^{s[\epsilon]}$. Moreover, we say λ is dominant arithmetic if $\lambda_1 \in X^*_+$. For the dominant weight λ_1, let $\mathscr{L}_{\lambda_1}$ denote the highest weight representation and let f_{λ_1} denote the highest weight vector associated to λ_1.

The control theorem due to Ash–Stevens [2] and Urban [18] which is a generalization of control theorem due to Stevens [17] and Chenevier [3,6] relates the space of overconvergent automorphic forms $H^*(K^p I, \mathcal{D}_\lambda)$ and the classical automorphic forms $H^*(K^p I^s_1, \mathscr{L}_{\lambda_1})$, where $I^s_1 \subset I$ are the subgroups defined earlier.

The standard action of the Hecke algebra $\mathscr{A}^+_p$ on $H^*(K^p I^s_1, \mathscr{L}_{\lambda_1})$ is modified as follows. For $t \in T^{++}$ and $m \in H^*(K^p I^s_1, \mathscr{L}_{\lambda_1})$, we define the $\star$-action in weight λ_1 by

$$U_t \star_{\lambda_1} m = \lambda_1(t)^{-1} U_t m.$$

Next, we note that, for $g \in G$ and $i \in I$ the function $f_{\lambda_1}(gi)$ defines an element in $\mathscr{L}_{\lambda_1} \otimes \mathcal{A}_\lambda$ and pairing it with $\mu \in \mathcal{D}_\lambda$ we obtain a map $i_\lambda : \mathcal{D}_\lambda \to \mathscr{L}_{\lambda_1}$, which we symbolically write as

$$i_\lambda(\mu)(g) = \int f_{\lambda_1}(gi)\mu(i).$$

Then i_λ induces a morphism

$$i_\lambda : H^*(K^p I, \mathcal{D}_\lambda) \to H^*(K^p I^s_1, \mathscr{L}_{\lambda_1})$$

for any $s \geq s[\epsilon]$. This map is an intertwining operator for the action of the Hecke algebra $\mathbb{T}$, where we take the $\star$-action on the source and the $\star$-action in weight λ_1 on the target.

Let W denote the Weyl group of the torus T. For a controlling operator U, we call $h \in \mathbb{Q}$ small slope for a dominant arithmetic weight $\lambda = \lambda_1 \epsilon$ if

$$h < \inf_{w \in W \setminus \{1\}} v_p(w \cdot \lambda_1(U)) - v_p(\lambda_1(U)),$$

where v_p denotes the p-adic valuation. Recall that the Weyl group W acts on weights by the rule $w \cdot \mu = (\mu + \rho)^w - \rho$, where $\rho \in X^* \otimes_{\mathbb{Z}} \frac{1}{2}\mathbb{Z}$ denotes half the sum of positive roots with respect to B.

Theorem 3.4.1 ([8, Theorem 3.2.5]). *For the controlling operator U, if h is a small slope for a dominant arithmetic weight $\lambda = \lambda_1 \epsilon$, then there exists a natural isomorphism of Hecke modules*

$$H^*(K^p I, \mathcal{D}_\lambda)_{\leq h} \cong H^*(K^p I^s_1, \mathscr{L}_{\lambda_1})_{\leq h}^{T(\mathbb{Z}/p^s\mathbb{Z})=\epsilon}.$$

for all $s \geq s[\epsilon]$.

3.5 Comparison theorem

In this section we prove a similar version of a comparison theorem Johanssen and Newton [10, Theorem 3.2.1] for universal eigenvarieties (which is a generalisation of a comparison theorem Hansen [8, Theorem 5.1.6] which intern is a generalisation of a theorem due to Chenevier [7, Proposition 4.5]) enabling us to construct rigid analytic maps between eigenvarieties.

Lemma 3.5.1. *Let*

$$\mathcal{D} = (\mathcal{W}, \mathcal{Z}, \mathcal{M}, \mathbb{T}, \psi)$$

be an eigenvariety datum and let $\mathcal{X}$ the associated universal eigenvariety. Let $j : \mathcal{W}' \to \mathcal{W}$ be a closed immersion and $\mathcal{W}'$ is reduced. Let us denote by $p_1 : \mathcal{Z} \times_{\mathcal{W}} \mathcal{W}' \to \mathcal{Z}$ be the natural projection map. Define an eigenvariety datum,

$$\mathcal{D}' := (\mathcal{W}', \mathcal{Z} \times_{\mathcal{W}} \mathcal{W}', p_1^*\mathcal{M}, \mathbb{T}, \psi')$$

where ψ' is composition of ψ with natural map $\mathrm{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{M}) \to \mathrm{End}_{\mathcal{O}_{\mathcal{Z} \times_{\mathcal{W}} \mathcal{W}'}}(p_1^*\mathcal{M})$. Let $\mathcal{X}'$ be the universal eigenvariety associated to $\mathcal{D}'$. Then we have a closed immersion

$$\mathcal{X}' \hookrightarrow \mathcal{X}.$$

Proof. The proof is essentially contained in Johanssen-Newton [10, §3.1], we give a brief sketch. Let $U \subset \mathcal{Z}$ be an affinoid open, then $U' := p_1^{-1}(U)$ is affinoid open in $\mathcal{Z} \times_{\mathcal{W}} \mathcal{W}'$. Let $A = \mathcal{O}_{\mathcal{Z}}(U)$ and I the ideal cut out by U'. Let $M = \mathcal{M}(U)$ and $T_U = Im(\mathbb{T} \otimes A \to \mathrm{End}_A(M))$. Then $p_1^*\mathcal{M}(U') = M/IM$ and $T'_{U'} = Im(\mathbb{T} \otimes A \to \mathrm{End}_{A/I}(M/IM))$. Thus we have a natural surjective map $\phi : T_U \to T'_{U'}$. Since $Spec(T_U)$s glues together to form $\mathcal{X}$, we have a natural closed immersion $\mathcal{X}' \hookrightarrow \mathcal{X}$. □

Lemma 3.5.2. *Let*

$$\mathcal{D} = (\mathcal{W}, \mathcal{Z}, \mathcal{M}, \mathbb{T}, \psi)$$

be an eigenvariety datum and let $\mathcal{X}$ the associated universal eigenvariety. Let $\imath : \mathcal{Z}' \to \mathcal{Z}$ be a closed immersion. Define an eigenvariety datum,

$$\mathcal{D}'' := (\mathcal{W}, \mathcal{Z}', \imath^*\mathcal{M}, \mathbb{T}, \psi_{\mathcal{Z}'})$$

where $\psi_{\mathcal{Z}'}$ is composition of ψ with natural map $\mathrm{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{M}) \to \mathrm{End}_{\mathcal{O}_{\mathcal{Z}'}}(\imath^*\mathcal{M})$. Let $\mathcal{X}''$ be the universal eigenvariety associated to $\mathcal{D}'$. Then we have a closed immersion

$$\mathcal{X}'' \hookrightarrow \mathcal{X}.$$

Proof. We note that if $U \subset \mathcal{Z}$ is an affinoid open, then $\imath^{-1}(U)$ is affinoid open in $\mathcal{Z}'$ and the proof follows as in Lemma 3.5.1. This lemma is essentially contained in [10, §3.1]. □

Theorem 3.5.3. *Let*

$$\mathcal{D}_i = (\mathcal{W}_i, \mathcal{Z}_i, \mathcal{M}_i, \mathbb{T}_i, \psi_i) \text{ for } i = 1, 2$$

be two eigenvariety datum associated with two eigenvariety $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively. Moreover, assume all the following conditions hold.

(1) *There is a closed immersion of weight spaces $j : \mathcal{W}_1 \hookrightarrow \mathcal{W}_2$.*

(2) *We have an $\mathbb{Q}_p$ algebra homomorphism $\sigma : \mathbb{T}_2 \to \mathbb{T}_1$.*

(3) *There is a subset $\mathcal{X}^{cl} \subset Max(\mathcal{X}_1)$ with $j(\pi_1(x)) \in \mathcal{Z}_2$ for all $x \in \mathcal{X}^{cl}$ such that the $\mathbb{T}_1$ eigenspace of x composed with σ appears in $\mathcal{M}_2(j(\pi_1(x)))$.*

Let $\overline{\mathcal{X}}$ denotes the Zariski closure of $\mathcal{X}^{cl}$ in $\mathcal{X}_1$ with its induced reduced structure. Then, we have a canonical finite morphism

$$i : \overline{\mathcal{X}} \to \mathcal{X}_2$$

lying over $j : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\phi_{\overline{\mathcal{X}}} \circ \sigma = i^* \circ \phi_{\mathcal{X}_2}$.

Proof. This is a restatement of comparison theorem for extended eigenvarieties due to Johanssen-Newton [10, Theorem 3.2.1] for universal eigenvarities. Basic idea is to reduce the problem to the case when $\mathscr{W}_1 = \mathscr{W}_2$, $\mathscr{Z}_1 = \mathscr{Z}_2$ and $\mathbb{T}_1 = \mathbb{T}_2$ by applying Lemma 3.5.1 and Lemma 3.5.2 repeatedly. We give a sketch of their proof in the set up of universal eigenvarieties.

Consider the eigenvariety datum

$$\mathscr{D}_1^\sigma = (\mathscr{W}_1, \mathscr{Z}_1, \mathscr{M}_1, \mathbb{T}_2, \psi_1 \circ \sigma)$$

and let us denote the associated eigenvariety by $\mathscr{X}_1^\sigma$. Since $im(\psi_1 \circ \sigma) \subseteq im(\psi_1)$, we have a finite morphism $f : \mathscr{X}_1 \to \mathscr{X}_1^\sigma$. For $x \in \mathscr{X}^{cl}$, choose $x^\sigma \in \{(\pi_1^\sigma)^{-1}(\pi_1(x))\}$. Define $\mathscr{X}^{cl,\sigma} \subset \mathrm{Max}(\mathscr{X}_1^\sigma)$ which contains x^σ for all $x \in \mathscr{X}^{cl}$ and has the property that for all $y \in \mathscr{X}^{cl,\sigma}$ we have $j(\pi_1^\sigma(y)) \in \mathscr{Z}_2$ and the $\mathbb{T}_2$ eigenspace of y appears in $\mathscr{M}_2(j(\pi_1^\sigma(y)))$. Let us denote the reduced Zariski closure of $\mathscr{X}^{cl,\sigma}$ in $\mathscr{X}_1^\sigma$ by $\overline{\mathscr{X}}^\sigma$. Then it is obvious that $f(\overline{\mathscr{X}}) \subseteq \overline{\mathscr{X}}^\sigma$. Thus it is enough to construct a morphism from $\overline{\mathscr{X}}^\sigma$ to $\mathscr{X}_2$, that is we may assume $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{T}$.

Next consider the eigenvariety datum

$$\mathscr{D}_2' = (\mathscr{W}_1, \mathscr{Z}_2 \times_{\mathscr{W}_2} \mathscr{W}_1, p_1^* \mathscr{M}_2, \mathbb{T}_2, \psi_2'),$$

comparing it with the eigenvariety datum $\mathscr{D}_2$ using lemma 3.5.1, we see that there is a closed immersion $\mathscr{X}_2' \to \mathscr{X}_2$. Hence it is enough to construct a morphism from $\overline{\mathscr{X}}^\sigma$ to $\mathscr{X}_2'$, that is we may assume $\mathscr{W}_1 = \mathscr{W}_2 = \mathscr{W}$.

Let $\overline{\mathscr{Z}}^\sigma$ denotes the Zariski closure of $\pi_1^\sigma(\mathscr{X}^{cl,\sigma})$ in $\mathscr{Z}_1$. Then we have $\iota : \overline{\mathscr{Z}}^\sigma \subset \mathscr{Z}_1$. Now consider the eigenvariety datum

$$\mathscr{D}_{1,\overline{\mathscr{Z}}^\sigma}^\sigma = (\mathscr{W}_1, \overline{\mathscr{Z}}^\sigma, \iota^* \mathscr{M}_1, \mathbb{T}_2, (\psi_1 \circ \sigma)_{\overline{\mathscr{Z}}^\sigma}).$$

By lemma 3.5.2, we have a closed immersion $\mathscr{X}_{1,\overline{\mathscr{Z}}^\sigma} \hookrightarrow \mathscr{X}_1^\sigma$, moreover since π_1^σ is finite and $\overline{\mathscr{X}}^\sigma$ is reduced, $\pi_1^\sigma|_{\overline{\mathscr{X}}^\sigma}$ factors through $\overline{\mathscr{Z}}^\sigma$. Thus it is enough to construct a morphism from $\mathscr{X}_{1,\overline{\mathscr{Z}}^\sigma}$ to $\mathscr{X}_2'$, that is we can replace $\mathscr{Z}_1$ by $\overline{\mathscr{Z}}^\sigma$. Next consider the eigenvariety datum

$$\mathscr{D}_2'' = (\mathscr{W}_1, \overline{\mathscr{Z}}^\sigma, j^*(p_1^* \mathscr{M}_2), \mathbb{T}_2, (\psi_2')_{\overline{\mathscr{Z}}^\sigma}),$$

where j is the induced closed immersion from $\overline{\mathscr{Z}}^\sigma \to \mathscr{Z}_2 \times_{\mathscr{W}_2} \mathscr{W}_1$. Applying lemma 3.5.2, we obtain a closed immersion from $\mathscr{X}_2'' \to \mathscr{X}_2'$. Hence it is enough to construct a morphism from $\mathscr{X}_{1,\overline{\mathscr{Z}}^\sigma}$ to $\mathscr{X}_2''$, that is we may assume $\mathscr{Z}_1 = \mathscr{Z}_2 = \overline{\mathscr{Z}}^\sigma = \mathscr{Z}$.

Finally consider the eigenvariety datum

$$\mathscr{D}_3 = (\mathscr{W}, \mathscr{Z}, \mathscr{M}_3 = \iota^* \mathscr{M}_1 \oplus j^*(p_1^* \mathscr{M}_2), \mathbb{T}, \psi_3 = ((\psi_1 \circ \sigma)_{\mathscr{Z}}, (\psi_2')_{\mathscr{Z}})),$$

then the associated eigenvariety $\mathscr{X}_3$ contains both $\mathscr{X}_{1,\overline{\mathscr{Z}}^\sigma}$ and $\mathscr{X}_2''$ as Zariski closed subspaces. Then it is enough to show that $\mathscr{X}^{cl,\sigma} \subset \mathscr{X}_2''$, but it obvious from the definition of $\mathscr{X}^{cl,\sigma}$. □

4. Eigenvarieties attached to G_i and an auxiliary eigenvariety

For the fixed odd prime p that is unramified in K, let $\mathcal{O}_p = \mathcal{O}_K \otimes \mathbb{Z}_p$. If p splits as $\mathfrak{p}\mathfrak{p}^c$ in K, then $\mathcal{O}_p = \mathcal{O}_{K,\mathfrak{p}} \times \mathcal{O}_{K,\mathfrak{p}^c}$. If $\alpha \in \mathcal{O}_p$, then write $\alpha = (\alpha_1, \alpha_2) \in \mathcal{O}_{K,\mathfrak{p}} \times \mathcal{O}_{K,\mathfrak{p}^c}$. On the other hand, if p is inert in K, then $\mathcal{O}_p$ is a degree 2 extension of $\mathbb{Z}_p$. Given $\alpha \in \mathcal{O}_p$, let α_1 and α_2 denote its Galois conjugates.

4.1 Weight spaces

We first start with a description of the weight spaces for $G_1 = \mathrm{Res}_{K/\mathbb{Q}} \mathbf{G}_1$. Let $\mathbf{B}_1$ and $\mathbf{T}_1$ denote the standard Borel subgroup and maximal torus in $\mathbf{G}_1$. Let $B_1 = \mathrm{Res}_{K/\mathbb{Q}} \mathbf{B}_1$ and $T_1 = \mathrm{Res}_{K/\mathbb{Q}} \mathbf{T}_1$ denote the corresponding subgroups in G_1. Let Z_1 denote the center of G_1 and let I_1 denote the Iwahori subgroup of G_1 (with respect to B_1).

We take our level structure to be $K_1 = \prod_v \mathrm{GL}_2(\mathcal{O}_{K,v})$, where the product runs over all the non-archimedean places of K. For any $\mathbb{Q}_p$ affinoid algebra A, we have

$$\mathscr{W}_1(A) := \mathscr{W}_{K_1^p, G_1}(A) = \{\kappa : T_1(\mathbb{Z}_p) \to A^\times \mid \kappa \text{ is trivial on } \overline{Z_1(\mathbb{Q}) \cap K_1^p I_1}\}.$$

Suppose that $n = (n_1, n_2)$ and $v = (v_1, v_2)$ are weights with $n + 2v = mt$ as before. Let $\kappa : T_1(\mathcal{O}_p) \to \overline{\mathbb{Q}}_p^\times$ be the map, $\kappa(\alpha, \beta) = \prod_{i=1}^2 \alpha_i^{n_i + v_i} \beta_i^{v_i}$. The unit group $\mathcal{O}_K^\times$ sits in $T_1(\mathcal{O}_p)$ diagonally as $\gamma \mapsto (\gamma, \gamma)$. Then κ is in the weight space $\mathscr{W}_1$ if κ is trivial on $\mathcal{O}_K^\times$ (not just on the totally positive units). If $k = n + 2t$ and $w = v + n + t$, then Hilbert modular forms of weight (k, w) (in the sense of Hida) has weight κ in the weight space $\mathscr{W}_1$.

Note that we will be working with a weight space that 2-dimensional, rather than the 3-dimensional space $\mathscr{W}_1$.

Remark 4.1.1. Note that we adopt a slightly different normalization, than usual, in our definition of weight spaces. The usual normalization sends (n, v) to κ that maps $(\alpha, \beta) \mapsto \prod_{i=1}^2 \alpha_i^{n_i} \beta_i^{v_i}$ and the units $\mathcal{O}_K^\times$ embeds as $\gamma \mapsto (\gamma, \gamma^2)$. See Buzzard [5], for example.

We now describe the weight space for the group $G_2 = \mathrm{GL}_4/\mathbb{Q}$. With our notation as before, let B_2 and T_2 denote the standard Borel and maximal torus of G_2. Let Z_2 denote the center of G_2 and let I_2 denote the Iwahori subgroup with respect to B_2.

We take our level structure to be $K_2 = \prod_\ell \mathrm{GL}_4(\mathbb{Z}_\ell)$, where the product runs over all integer primes ℓ. And the weight space $\mathscr{W}_2$ is defined similarly. For any $\mathbb{Q}_p$ affinoid algebra A, we have

$$\mathscr{W}_2(A) := \mathscr{W}_{K_2^p, G_2}(A) = \left\{ \chi : T_2(\mathbb{Z}_p) \to A^\times \ \middle| \ \begin{array}{l} \chi \text{ is trivial on the closure} \\ \text{of } \overline{Z_2(\mathbb{Q}) \cap K_2^p I_2} \end{array} \right\}.$$

4.2 Hecke algebras

Let S denote the set of primes of $\mathbb{Q}$ which ramify in K and let $\tilde{S}$ be the set of places of K lying above primes in S. Let S_p denotes the set of places in K above p.

For the group G_1, we define our unramified Hecke algebra as the commutative algebra

$$\mathbb{T}_1^{\mathrm{unr}} = \bigotimes_{v \notin S \cup \{p\}}' \mathbb{T}(G_1(\mathbb{Q}_v), G_1(\mathbb{Z}_v)).$$

Note that we are omitting Hecke operators at primes that are ramified in the quadratic extension K. At the prime p, we define a subrings $\mathscr{A}_{1,p}^+ \subset \mathscr{A}_{1,p} \subset \mathbb{T}(G_1(\mathbb{Q}_p), I_1)$, as before. Specifically, let Φ_1^+ denote the set of positive roots for B_1. We define two semigroups T_1^+ and T_1^{++} inside $T_1(\mathbb{Q}_p)$ as

$$T_1^+ = \{t \in T_1(\mathbb{Q}_p) \mid v_p(\alpha(t)) \le 0 \text{ for all } \alpha \in \Phi_1^+\}, \quad \text{and}$$

$$T_1^{++} = \{t \in T_1(\mathbb{Q}_p) \mid v_p(\alpha(t)) < 0 \text{ for all } \alpha \in \Phi_1^+\}.$$

Suppose for the moment that p splits in K and suppose that $t = (t_1, t_2) \in T_1(\mathbb{Q}_p)$, where $t_1 = \mathrm{diag}(p^{a_1}, p^{a_2})$ and $t_2 = \mathrm{diag}(p^{b_1}, p^{b_2})$. Then $t \in T_1^+$ if and only if $a_1 \le a_2$ and $b_1 \le b_2$. The same t belongs to T_1^{++} if and only if all the above inequalities are strict. We have similar conditions when p is inert in K.

For any $t \in T_1^+ \cap G_1(\mathbb{Z}_p)$, the double coset operators $U_t = [I_1 t I_1]$ generate the algebra $\mathscr{A}_{1,p}^+$. The Atkin–Lehner algebra $\mathscr{A}_{1,p}$ is a commutative subalgebra of $\mathbb{T}(G_1(\mathbb{Q}_p), I_1)$ generated by U_t and U_t^{-1} with $t \in T_1^+ \cap G_1(\mathbb{Z}_p)$. We can naturally identify

$$\mathscr{A}_{1,p}^+ \cong \mathbb{Q}_p[T_1^+ \cap G_1(\mathbb{Z}_p)] \quad \text{and} \quad \mathscr{A}_{1,p} \cong \mathbb{Q}_p(T_1(\mathbb{Q}_p)/T_1(\mathbb{Z}_p)).$$

We will call an operator $U_t \in \mathscr{A}_{1,p}^+$ a controlling operator if $t \in T_1^{++}$. Finally, we define the Hecke algebra as

$$\mathbb{T}_1 := \mathscr{A}_{1,p}^+ \otimes_{\mathbb{Q}_p} \mathbb{T}_1^{\mathrm{unr}}.$$

We also view the unramified Hecke Algebra $\mathbb{T}_1^{unr}$ as a product of local Hecke algebras as

$$\mathbb{T}_1^{unr} = \otimes'_{\mathfrak{l} \notin \tilde{S} \cup S_p} \mathbb{T}_{1,\mathfrak{l}},$$

where the local Hecke algebra $\mathbb{T}_{1,\mathfrak{l}} = \mathbb{T}(\mathbf{G}_1(K_\mathfrak{l}), \mathbf{G}_1(\mathcal{O}_{K,\mathfrak{l}}))$. For a place $\mathfrak{l}$ of K not in $\tilde{S} \cup S_p$, let $\varpi_\mathfrak{l}$ denote the uniformizer at $\mathfrak{l}$. We denote by $T_\mathfrak{l}$ and $S_\mathfrak{l}$ the double coset operators $\left[GL_2(\mathcal{O}_{K,\mathfrak{l}}) \begin{bmatrix} \varpi_\mathfrak{l} & \\ & 1 \end{bmatrix} GL_2(\mathcal{O}_{K,\mathfrak{l}}) \right]$ and

$\left[GL_2(\mathcal{O}_{K,\mathfrak{l}}) \begin{bmatrix} \varpi_\mathfrak{l} & \\ & \varpi_\mathfrak{l} \end{bmatrix} GL_2(\mathcal{O}_{K,\mathfrak{l}}) \right]$ respectively. The operators $T_\mathfrak{l}$ and $S_\mathfrak{l}$ generates the local Hecke algebra $\mathbb{T}_{1,\mathfrak{l}}$. Let

$\mathfrak{p} \in S_p$. Let $\varpi_\mathfrak{p}$ denotes the uniformizer at $\mathfrak{p}$. We denote by $U_\mathfrak{p}$ and $S_\mathfrak{p}$ the double coset operators $\left[\mathbf{I}_1 \begin{bmatrix} 1 & \\ & \varpi_\mathfrak{p} \end{bmatrix} \mathbf{I}_1 \right]$

and $\left[\mathbf{I}_1 \begin{bmatrix} \varpi_\mathfrak{p} & \\ & \varpi_\mathfrak{p} \end{bmatrix} \mathbf{I}_1 \right]$, respectively. Here $\mathbf{I}_1$ is the Iwahori subgroup with respect to the Borel subgroup $\mathbf{B}_1$. Then $U_\mathfrak{p}$

and $S_\mathfrak{p}$ for all $\mathfrak{p} \in S_p$ generates the algebra $\mathscr{A}^+_{1,p}$.

Standard choice of controlling operator is

$$U_p = \prod_{v \mid p} U_v.$$

However we will work with different choice of controlling operators, depending on whether p splits or remains inert K.

If $p = \mathfrak{p}\mathfrak{p}^c$ splits in K, we take $U_{spl} := U_\mathfrak{p}^4 U_{\mathfrak{p}^c}^2$ to be the controlling operator. We denote the corresponding eigenvariety by $\mathscr{X}_1^{spl}$.

If $p = p$ remains inert in K, we take $U_{in} := -U_p^4$ to be the controlling operator. We denote the corresponding eigenvariety by $\mathscr{X}_1^{in}$.

A point in $\mathscr{X}_1^\dagger(\overline{\mathbb{Q}}_p)$, where $\dagger \in \{spl, in\}$, is called classical if the eigenpacket ϕ_x corresponds to a cohomological Hilbert cusp form of weight $\kappa = (n, v)$ and tame level 1.

We now come to the group G_2, where our definitions are similar. The unramified Hecke algebra can be written as

$$\mathbb{T}_2^{unr} = \otimes'_{\ell \notin S \cup \{p\}} \mathbb{T}_{2,\ell};$$

where the local Hecke algebra $\mathbb{T}_{2,\ell} = \mathbb{T}(G_2(\mathbb{Q}_\ell), G_2(\mathbb{Z}_\ell))$ is generated by operators $T_{\ell,i}$ corresponding to the double coset of matrix

$$\mathrm{diag}(\underbrace{\ell, \dots, \ell}_{i}, 1, \dots, 1),$$

where $i = 1, \dots, 4$. We define $\mathscr{A}^+_{2,p}$ and $\mathscr{A}_{2,p}$ similarly. For each $i = 1, \dots, 4$, we denote by $U_{p,i}$, the element of $\mathscr{A}^+_{2,p}$ corresponding to the matrix

$$\mathrm{diag}(1, \dots, 1, \underbrace{p, \dots, p}_{i}).$$

The operators $U_{p,i}$ generates the algebra $\mathscr{A}^+_{2,p}$. The operators $u_{p,i} := U_{p,i}U_{p,i-1}^{-1} \in \mathscr{A}_{2,p}$ generates the Atkin–Lehner algebra $\mathscr{A}_{2,p}$. Finally, we take

$$U_p = U_{p,1}U_{p,2}U_{p,3} \in \mathscr{A}^+_{2,p}$$

as our choice for the controlling operator. We denote this eigenvariety by $\mathscr{X}_2$.

4.3 The eigenvariety $\mathscr{X}$

We have previously constructed the eigenvarieties $\mathscr{X}_i$ associated to the groups G_i. In this section, we construct an auxiliary eigenvariety $\mathscr{X}$, which plays a role in the construction of the p-adic Asai transfer map when p is inert.

Let $\mathscr{D}_2 = (\mathscr{W}_2, \mathscr{X}_2, \mathscr{M}_2, \mathbb{T}_2, \psi_2)$ denote the eigenvariety datum associated to the eigenvariety $\mathscr{X}_2$. In order to construct $\mathscr{X}$, we only modify the Hecke algebra and keep the other objects the same as in $\mathscr{X}_2$.

Let $\tilde{T}_2^+$ denotes the subgroup T_2^+ whose elements are

$$\tilde{T}_2^+ = \left\{ \begin{bmatrix} p^{a_1} & & & \\ & p^{a_2} & & \\ & & p^{a_3} & \\ & & & p^{a_4} \end{bmatrix} \middle| \begin{array}{l} a_i \in \mathbb{N} \cup \{0\}, a_1 \leq a_2 \leq a_3 \leq a_4 \\ \text{and} \quad a_3 - a_2 \in 2\mathbb{N} \cup \{0\} \end{array} \right\}.$$

Analogously, define

$$\tilde{T}_2^{++} = \{t \in \tilde{T}_2^+ \mid a_1 < a_2 < a_3 < a_4\}.$$

Let $\tilde{\mathscr{A}}_{2,p}^+$ (resp. $\tilde{\mathscr{A}}_{2,p}$) denotes the $\mathbb{Q}_p$ algebra generated by U_t with $t \in \tilde{T}_2^+$ (resp. by U_t, U_t^{-1} with $t \in \tilde{T}_2^+$). We call $U_t \in \tilde{\mathscr{A}}_{2,p}^+$ a controlling operator if $t \in \tilde{T}_2^{++}$.

We denote by $\tilde{U}_{p,1}, \tilde{U}_{p,2}, \tilde{U}_{p,3}$ and $\tilde{U}_{p,4}$ the double coset operators corresponding to the following matrices

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p^2 & \\ & & & p^2 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix} \text{ and } \begin{bmatrix} p & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix},$$

respectively. Then they generates the algebra $\tilde{\mathscr{A}}_{2,p}^+$. We have a natural choice for the controlling operator

$$\tilde{U}_p = \tilde{U}_{p,1}\tilde{U}_{p,2}\tilde{U}_{p,3} = \begin{bmatrix} I_2 & \begin{bmatrix} 1 & & & \\ & p & & \\ & & p^3 & \\ & & & p^4 \end{bmatrix} & I_2 \end{bmatrix}.$$

Define $\tilde{u}_{p,1} = \tilde{U}_{p,1}, \tilde{u}_{p,2} = \tilde{U}_{p,2}(\tilde{U}_{p,1}^{-1})^2, \tilde{u}_{p,3} = \tilde{U}_{p,3}\tilde{U}_{p,2}^{-1}\tilde{U}_{p,1}$ and $\tilde{u}_{p,4} = \tilde{U}_{p,4}\tilde{U}_{p,3}^{-1}$, then integral powers of $\tilde{u}_{p,i}$ generate the algebra $\tilde{\mathscr{A}}_{2,p}$.

We define Hecke algebra $\tilde{\mathbb{T}}_2$ as

$$\tilde{\mathbb{T}}_2 := \tilde{\mathscr{A}}_{2,p}^+ \otimes_{\mathbb{Q}_p} \mathbb{T}_2^{unr},$$

where $\mathbb{T}_2^{unr}$ is the same as before. Let $id : \tilde{\mathbb{T}}_2 \hookrightarrow \mathbb{T}_2$ denote the natural injection of Hecke algebras. Let $\mathscr{D}$ denotes the eigenvariety datum

$$\mathscr{D} = (\mathscr{W}_2, \mathscr{X}_2, \mathscr{M}_2, \tilde{\mathbb{T}}_2, \psi_2|_{\tilde{\mathbb{T}}_2} = \psi_2 \circ id)$$

and $\mathscr{X}$ denote the associated eigenvariety. We note that, since $im(\psi_2|_{\tilde{\mathbb{T}}_2}) \subset im(\psi_2)$, we have a finite morphism $f : \mathscr{X}_2 \to \mathscr{X}$ as in the proof of Theorem 3.5.3.

Let π_p be an automorphic representation of $G_2(\mathbb{Q}_p)$. We call a character $\tilde{\chi}$ of $\tilde{\mathscr{A}}_{2,p}$ an accessible refinement of π_p in $\mathscr{X}$, if there exists a character χ of $\mathscr{A}_{2,p}$, such that $\tilde{\chi} = \chi|_{\tilde{\mathscr{A}}_{2,p}}$ and χ is a refinement of π_p appearing in $\mathscr{X}_2$.

5. p-adic Asai transfer

In this section, we construct a rigid analytic map between the eigenvarieties attached to GL_2/K and $GL_4/\mathbb{Q}$. This map is constructed using the comparison theorem 3.5.3 described in the section 3. In order to apply the theorem, we need to construct compatible maps at the level of weight spaces and Hecke algebras.

We first describe the map between the weight spaces. For the weight spaces $\mathscr{W}_1$ and $\mathscr{W}_2$ defined in the previous section, we construct the map $j : \mathscr{W}_1 \hookrightarrow \mathscr{W}_2$ as follows. For a weight $\kappa \in \mathscr{W}_1$, define

$$j(\kappa)(t_1, t_2, t_3, t_4) = (t_1 t_2)^{-1} \kappa(t_1 t_2, t_3 t_4, t_1 t_3, t_2 t_4). \tag{5.0.1}$$

We note that, if $\kappa(t_1, t_2, t_3, t_4) = t_1^{n_1+v_1} t_2^{v_1} t_3^{n_2+v_2} t_4^{v_2}$, then,

$$j(\kappa)(t_1, t_2, t_3, t_4) = t_1^{n_1+n_2+v_1+v_2-1} t_2^{n_1+v_1+v_2-1} t_3^{n_2+v_1+v_2-1} t_4^{v_1+v_2}$$

$$= t_1^{m-1+\frac{n_1+n_2}{2}} t_2^{m-1+\frac{n_1-n_2}{2}} t_3^{m-\frac{n_1-n_2}{2}} t_4^{m-\frac{n_1+n_2}{2}},$$

where, $m = n_1 + 2v_1 = n_2 + 2v_2$.

The map between the Hecke algebras and the set of classical points depend on whether p is split or inert. We will consider these two cases separately. In fact, when p is inert we only construct a map to the auxiliary eigenvariety $\mathscr{X}$ attached to $GL_4/\mathbb{Q}$.

5.1 The case where $p = \mathfrak{p}\mathfrak{p}^c$ is split in K

Recall that in the split case for G_1 we are working with the eigenvariety $\mathscr{X}_1^{spl}$, that is the eigenvariety associated to the datum $\mathscr{D}_1 = (\mathscr{W}_1, \mathscr{Z}_1, \mathscr{M}_1, \mathbb{T}_1, \psi_1)$ with respect to the controlling operator $U_{spl} := U_{\mathfrak{p}}^4 U_{\mathfrak{p}^c}^2$ and for G_2 we are working with the eigenvariety $\mathscr{X}_2$ associated to the $\mathscr{D}_2 = (\mathscr{W}_2, \mathscr{Z}_2, \mathscr{M}_2, \mathbb{T}_2, \psi_2)$ with respect to the controlling operator $U_p := U_{p,1} U_{p,2} U_{p,3}$. We want to use the comparison theorem (Theorem 3.5.3) to construct this map. The map j between weight spaces is given in (5.0.1). Below we construct the map of the Hecke algebras and give a set of classical points in the universal eigenvariety $\mathscr{X}_1^{spl}$.

We first construct the map between Hecke algebras attached to G_1 and G_2. The Hecke algebra $\mathbb{T}_2$ is generated by the elements $T_{\ell,i}$ (for $\ell \neq p$ and unramified in K) and $U_{p,i}$ for $i = 1, \ldots, 4$. In $\mathbb{T}_1$, we also have the standard Hecke operators $T_{\mathfrak{l}}$ and $S_{\mathfrak{l}}$ for $\mathfrak{l}$ away from p and $U_{\mathfrak{p}}$ and $S_{\mathfrak{p}}$ for $\mathfrak{p}|p$.

We define a map $\sigma^{\pm} : \mathbb{T}_2 \to \mathbb{T}_1$ as follows:

When $\ell = \mathfrak{l}\mathfrak{l}^c$	When ℓ is inert
$T_{\ell,1} \mapsto T_{\mathfrak{l}} T_{\mathfrak{l}^c}$	$T_{\ell,1} \mapsto \pm T_{\ell}$
$T_{\ell,2} \mapsto T_{\mathfrak{l}}^2 S_{\mathfrak{l}^c} + S_{\mathfrak{l}} T_{\mathfrak{l}^c}^2 - 2\ell S_{\mathfrak{l}} S_{\mathfrak{l}^c}$	$T_{\ell,2} \mapsto 0$
$T_{\ell,3} \mapsto \ell^{-1} T_{\mathfrak{l}} S_{\mathfrak{l}} T_{\mathfrak{l}^c} S_{\mathfrak{l}^c}$	$T_{\ell,3} \mapsto \mp \ell^{-1} T_{\ell} S_{\ell}$
$T_{\ell,4} \mapsto \ell^{-2} S_{\mathfrak{l}}^2 S_{\mathfrak{l}^c}^2$	$T_{\ell,4} \mapsto -\ell^{-2} S_{\ell}^2$

$$U_{p,1} \mapsto U_{\mathfrak{p}} U_{\mathfrak{p}^c}$$

$$U_{p,2} \mapsto U_{\mathfrak{p}}^2 S_{\mathfrak{p}^c}$$

$$U_{p,3} \mapsto p^{-1} U_{\mathfrak{p}} S_{\mathfrak{p}} U_{\mathfrak{p}^c} S_{\mathfrak{p}^c}$$

$$U_{p,4} \mapsto p^{-2} S_{\mathfrak{p}}^2 S_{\mathfrak{p}^c}^2.$$

We now justify the definition of $\sigma^{\pm}$. Let π be an automorphic representation coming from a Hilbert modular form f of weight $\kappa = (n, v)$ as before. Let us choose a refinement of f such that $U_{\mathfrak{p}} f = \alpha_{\mathfrak{p}} f$ and $U_{\mathfrak{p}^c} f = \alpha_{\mathfrak{p}^c} f$ and let $x \in \mathscr{X}_1^{spl}$ be the corresponding point. We denote the eigenpacket associated to x by $\phi_{\pi,\{\alpha_{\mathfrak{p}}, \alpha_{\mathfrak{p}^c}\}}$. Assume that $n_1 > n_2$, then by our normalization $As^{\pm}(\pi)$ is cohomological of weight $j(\kappa)$. For any refinement χ of $As^{\pm}(\pi)$, for primes $\ell \nmid p$, $As^{\pm}(\pi)_{\ell}$ is an unramified representation and the Hecke operators $T_{\ell,i}$ act on the spherical vector via the scalar $(\phi_{\pi,\{\alpha_{\mathfrak{p}}, \alpha_{\mathfrak{p}^c}\}})(\sigma(T_{\ell,i}))$.

First suppose that ℓ splits as $\mathfrak{l}\mathfrak{l}^c$ in K. Denote by $\alpha_{\mathfrak{l}}$ and $\beta_{\mathfrak{l}}$, the $\text{Frob}_{\mathfrak{l}}$ eigenvalues of f. Then the characteristic polynomial for $\text{Frob}_{\mathfrak{l}}$ is given by

$$X^2 - T_{\mathfrak{l}} X + N(\mathfrak{l}) S_{\mathfrak{l}} = (X - \alpha_{\mathfrak{l}})(X - \beta_{\mathfrak{l}}).$$

Hence $T_{\mathfrak{l}}$ acts by $\alpha_{\mathfrak{l}} + \beta_{\mathfrak{l}}$ and $N(\mathfrak{l}) S_{\mathfrak{l}}$ acts by $\alpha_{\mathfrak{l}} \beta_{\mathfrak{l}}$. Similarly, the characteristic polynomial for $\text{Frob}_{\mathfrak{l}^c}$ is given by

$$X^2 - T_{\mathfrak{l}^c} X + N(\mathfrak{l}) S_{\mathfrak{l}^c} = (X - \alpha_{\mathfrak{l}^c})(X - \beta_{\mathfrak{l}^c})$$

where $\alpha_{\mathfrak{l}^c}$ and $\beta_{\mathfrak{l}^c}$ are the Frob$_{\mathfrak{l}^c}$ eigenvalues of f. On the other hand, characteristic polynomial for Frob$_\ell$ corresponding to As$^\pm(\pi)_\ell$ is given by

$$X^4 - T_{\ell,1}X^3 + \ell T_{\ell,2}X^2 - \ell^3 T_{\ell,3}X + \ell^6 T_{\ell,4}.$$

From our earlier calculation, we know that the Frob$_\ell$ eigenvalues on As$^\pm(\pi)_\ell$ are $\alpha_{\mathfrak{l}}\alpha_{\mathfrak{l}^c}$, $\alpha_{\mathfrak{l}}\beta_{\mathfrak{l}^c}$, $\beta_{\mathfrak{l}}\alpha_{\mathfrak{l}^c}$ and $\beta_{\mathfrak{l}}\beta_{\mathfrak{l}^c}$. Thus

$$X^4 - T_{\ell,1}X^3 + \ell T_{\ell,2}X^2 - \ell^3 T_{\ell,3}X + \ell^6 T_{\ell,4} = (X - \alpha_{\mathfrak{l}}\alpha_{\mathfrak{l}^c})(X - \alpha_{\mathfrak{l}}\beta_{\mathfrak{l}^c})(X - \beta_{\mathfrak{l}}\alpha_{\mathfrak{l}^c})(X - \beta_{\mathfrak{l}}\beta_{\mathfrak{l}^c}).$$

From this, we see that $T_{\ell,1}$ acts by the eigenvalue $\alpha_{\mathfrak{l}}\alpha_{\mathfrak{l}^c} + \alpha_{\mathfrak{l}}\beta_{\mathfrak{l}^c} + \beta_{\mathfrak{l}}\alpha_{\mathfrak{l}^c} + \beta_{\mathfrak{l}}\beta_{\mathfrak{l}^c} = (\alpha_{\mathfrak{l}} + \beta_{\mathfrak{l}})(\alpha_{\mathfrak{l}^c} + \beta_{\mathfrak{l}^c})$. On the other hand we know that $T_{\ell,1}$ acts by $\phi_{\pi,\{\alpha_{\mathfrak{p}},\alpha_{\mathfrak{p}^c}\}}(\sigma(T_{\ell,1}))$. Hence our definition

$$\sigma(T_{\ell,1}) = T_{\mathfrak{l}}T_{\mathfrak{l}^c}.$$

The calculations for $T_{\ell,i}$ when $i = 2, 3, 4$ are similar.

Now assume that ℓ is inert in K. Let α_ℓ and β_ℓ denote the Frob$_\ell$ eigenvalues of f. Then Frob$_\ell$ eigenvalues on As$^\pm(\pi)$ are given by $\pm\alpha_\ell$, $\pm\sqrt{\alpha_\ell\beta_\ell}$, $\mp\sqrt{\alpha_\ell\beta_\ell}$ and $\pm\beta_\ell$. Thus

$$X^4 - T_{\ell,1}X^3 + \ell T_{\ell,2}X^2 - \ell^3 T_{\ell,3}X + \ell^6 T_{\ell,4} = (X \mp \alpha_\ell)(X \mp \sqrt{\alpha_\ell\beta_\ell})(X \pm \sqrt{\alpha_\ell\beta_\ell})(X \mp \beta_\ell).$$

We see that $T_{\ell,1}$ acts by $\pm(\alpha_\ell + \beta_\ell)$, $T_{\ell,2}$ by 0, $T_{\ell,3}$ by $\mp\ell^{-3}\alpha_\ell\beta_\ell(\alpha_\ell + \beta_\ell)$ and $T_{\ell,4}$ acts by $-\ell^{-6}(\alpha_\ell\beta_\ell)^2$. Hence our definition of $\sigma^\pm$ above.

The following lemma will justify the definition of $\sigma^\pm$ for Hecke operators supported at p. Note that, $N(\mathfrak{p})S_{\mathfrak{p}}$ (resp. $N(\mathfrak{p}^c)S_{\mathfrak{p}^c}$) acts via the eigenvalue $\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}}$ (resp. $\alpha_{\mathfrak{p}^c}\beta_{\mathfrak{p}^c}$).

Lemma 5.1.1. *The module* As$^\pm(\pi)_p^{l_2}$ *contains a vector* $v^\pm$ *on which* $\mathscr{A}_{2,p}$ *acts via the character associated to the tuple* $(p^{-3}\beta_{\mathfrak{p}}\beta_{\mathfrak{p}^c}, p^{-2}\beta_{\mathfrak{p}}\alpha_{\mathfrak{p}^c}, p^{-1}\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}^c}, \alpha_{\mathfrak{p}}\alpha_{\mathfrak{p}^c})$. *In particular,* U_p *acts via the scalar* $p^{3m-1}\alpha_{\mathfrak{p}}^4\alpha_{\mathfrak{p}^c}^2$.

Proof. The proof of this lemma is similar to [8, Lemma 5.5.2]. We give a brief sketch here. For this particular refinement, we see that $u_{p,1} = U_{p,1}$ acts by $\alpha_{\mathfrak{p}}\alpha_{\mathfrak{p}^c}$, $u_{p,2} = U_{p,2}U_{p,1}^{-1}$ acts by $p^{-1}\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}^c}$, thus $U_{p,2}$ acts by $p^{-1}\alpha_{\mathfrak{p}}^2\alpha_{\mathfrak{p}^c}\beta_{\mathfrak{p}^c}$. Similarly $u_{p,3} = U_{p,3}U_{p,2}^{-1}$ acts by $p^{-2}\beta_{\mathfrak{p}}\alpha_{\mathfrak{p}^c}$ and hence $U_{p,3}$ acts by $p^{-3}\alpha_{\mathfrak{p}}^2\alpha_{\mathfrak{p}^c}^2\beta_{\mathfrak{p}}\beta_{\mathfrak{p}^c}$. Finally $u_{p,4} = U_{p,4}U_{p,3}^{-1}$ acts by $p^{-3}\beta_{\mathfrak{p}}\beta_{\mathfrak{p}^c}$ and $U_{p,4}$ acts by $p^{-6}\alpha_{\mathfrak{p}}^2\alpha_{\mathfrak{p}^c}^2\beta_{\mathfrak{p}}^2\beta_{\mathfrak{p}^c}^2$.

From the characteristic polynomials of $U_{\mathfrak{p}}$ and $U_{\mathfrak{p}^c}$, we see that $\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}} = \alpha_{\mathfrak{p}}\beta_{\mathfrak{p}^c} = p^{m+1}$. For the computation of the U_p operator, notice that $U_p = U_{p,1}U_{p,2}U_{p,3}$ and hence acts via

$$(p^{-2}\beta_{\mathfrak{p}}\alpha_{\mathfrak{p}^c})(p^{-1}\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}^c})^2(\alpha_{\mathfrak{p}}\alpha_{\mathfrak{p}^c})^3 = p^{-4}(\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}})(\alpha_{\mathfrak{p}^c}\beta_{\mathfrak{p}^c})^2\alpha_{\mathfrak{p}}^4\alpha_{\mathfrak{p}^c}^2 = p^{3m-1}\alpha_{\mathfrak{p}}^4\alpha_{\mathfrak{p}^c}^2.$$

This completes the proof of the lemma. □

The $\star$-action of $\mathscr{A}_{2,p}$ on As$^\pm(\pi)_p^{l_2}$ is the usual action, rescaled by $j(\kappa)(1, p, p^2, p^3)^{-1}$, where $j(\kappa)$ corresponds to the highest weight vector

$$\mu = \left(\frac{n_1 + n_2}{2} + m - 1, \frac{n_1 - n_2}{2} + m - 1, m - \frac{n_1 - n_2}{2}, m - \frac{n_1 + n_2}{2}\right).$$

By our assumption that $n_1 > n_2$, the weight μ is a dominant integral weight. We compute that

$$\mu(1, p, p^2, p^3) = p^{\frac{n_1 - n_2}{2} + m - 1 + 2\left(m - \frac{n_1 - n_2}{2}\right) + 3\left(m - \frac{n_1 + n_2}{2}\right)}$$

$$= p^{6m - 1 - 2n_1 - n_2}$$

$$= p^{3m - 1 + 4v_1 + 2v_2}.$$

Define the set of classical weights as

$$\mathscr{W}_{spl}^{cl} = \{(n, v) \in \mathscr{W}_1 \mid n_i, v_i \in \mathbb{Z}, \ n_1 > n_2 \geq 0 \text{ and } 2v_1 + v_2 = 0\}. \tag{5.1.1}$$

We remark that $\overline{\mathscr{W}}_{spl}$ the Zariski closure of $\mathscr{W}_{spl}^{cl}$ in $\mathscr{W}_1$ is a 2-dimensional subspace of the full weight space. Moreover, if f has weight $(n, v) \in \mathscr{W}_{spl}^{cl}$, then the eigenvalue of $\star$-action of U_p on As$^\pm(\pi)_p^{l_2}$ is $\alpha_{\mathfrak{p}}^4\alpha_{\mathfrak{p}^c}^2$.

Now we compute small slope h for the weight $j(\kappa) = \mu$. By definition, we have

$$h = \inf_{w \in S_4 \setminus \{1\}} v_p((w \cdot \mu)(1, p, p^2, p^3)) - v_p(\mu(1, p, p^2, p^3)),$$

where $w \cdot \mu = (\mu + \rho)^w - \rho$. We note that $\rho = (3, 2, 1, 0)$ and

$$v_p((w \cdot \mu)(1, p, p^2, p^3)) - v_p(\mu(1, p, p^2, p^3))$$

$$= (\mu_{w^{-1}(2)} - \mu_2 + \rho_{w^{-1}(2)} - \rho_2) + 2(\mu_{w^{-1}(3)} - \mu_3 + \rho_{w^{-1}(3)} - \rho_3)$$

$$+ 3(\mu_{w^{-1}(4)} - \mu_4 + \rho_{w^{-1}(4)} - \rho_4).$$

Hence $v_p((w \cdot \mu)(1, p, p^2, p^3)) - v_p(\mu(1, p, p^2, p^3))$ is a non-negative integer linear combination of

$$\mu_1 - \mu_2 + \rho_1 - \rho_2 = n_2 + 1,$$

$$\mu_1 - \mu_3 + \rho_1 - \rho_3 = n_1 + 1,$$

$$\mu_1 - \mu_4 + \rho_1 - \rho_4 = n_1 + n_2 + 2,$$

$$\mu_2 - \mu_3 + \rho_2 - \rho_3 = n_1 - n_2,$$

$$\mu_2 - \mu_4 + \rho_2 - \rho_4 = n_1 + 1,$$

$$\mu_3 - \mu_4 + \rho_3 - \rho_4 = n_2 + 1.$$

By taking $w = (1\ 2)$ and $w = (2\ 3)$, we see that

$$h = \min\{n_1 - n_2, n_2 + 1\}.$$

Proposition 5.1.3. *If $\alpha_{\mathfrak{p}}$ and $\alpha_{\mathfrak{p}^c}$ satisfy*

$$v_p(\alpha_{\mathfrak{p}}^4 \alpha_{\mathfrak{p}^c}^2) < \min\{n_1 - n_2, n_2 + 1\},$$

then $H^*(K_2^p I_2, \mathcal{D}_{j(\kappa)})$ contains a nonzero vector $v^{\pm}$ such that every $T \in \mathbb{T}_2$ acts on $v^{\pm}$ through the scalar $\phi_{\pi, \{\alpha_{\mathfrak{p}}, \alpha_{\mathfrak{p}^c}\}}(\sigma^{\pm}(T))$.

Proof. Since $n_1 > n_2$, we have $\mu = j(\kappa)$ is dominant and $\mathrm{As}^{\pm}(\pi)$ is cohomological of weight $j(\kappa)$, see [16]. By construction of the map $\sigma^{\pm}$ we see that $T_{\ell,i}$ acts by $\phi_{f, \{\alpha_{\mathfrak{p}}, \alpha_{\mathfrak{p}^c}\}}(\sigma^{\pm}(T_{\ell,i}))$ on the line $\mathrm{As}^{\pm}(\pi)_{\ell}^{\mathrm{GL}_4(\mathbb{Z}_\ell)}$. From the choice of our refinement and the $*$-action of U_p operator (for G_2), we see that $\mathrm{As}^{\pm}(\pi)_p^{I_2}$ contains a vector on which U_p acts by $\alpha_{\mathfrak{p}}^4 \alpha_{\mathfrak{p}^c}^2$. Finally we note that for U_p, any $h < h_0$ is a small slope for the dominant weight μ. Applying Theorem 3.4.1 we obtain an isomorphism

$$H^*(K_2^p I_2, \mathcal{D}_{j(\kappa)})_{<h} \cong H^*(K_2^p I_2, \mathscr{L}_{j(\kappa)})_{<h},$$

and the target contains a vector satisfying the claim of the proposition. $\square$

Let $\mathscr{Z}_{spl}$ be the set of points in $\mathscr{X}_1^{spl}$ of the form $(n_1, n_2, v_1, v_2, \alpha^{-1})$ such that, $(n_1, n_2, v_1, v_2) \in \mathscr{W}_{spl}^{cl}$ and $v_p(\alpha) < \min\{n_1 - n_2, n_2 + 1\}$. If $z = (\kappa, \alpha^{-1})$ and if $\underline{k} = (n_1 + 2, n_2 + 2)$ and $\underline{w} = (n_1 + v_1 + 1, n_2 + v_2 + 1)$, then $\mathscr{M}_1(z) \cong H^*(K_1^p I_1, \mathscr{L}_\kappa)_{<a}$. On the other hand, $\mathscr{M}_2(j(z)) \cong H^*(K_2^p I_2, \mathcal{D}_{j(\kappa)})_{<a} \cong H^*(K_2^p I_2, \mathscr{L}_{j(\kappa)})_{<a}$. We define $\mathscr{X}_{spl}^{cl}$ to be the set of classical points on $x \in \mathscr{X}_1^{spl}$, such that $\pi_1(x) \in \mathscr{Z}_{spl}$. Then, by Proposition 5.1.3, the $\mathbb{T}_1$ eigenspace of x composed with $\sigma^{\pm}$ appears in $\mathscr{M}_2(j(z))$.

Let $\mathscr{X}_{spl}$ denote the Zariski closure of $\mathscr{X}_{spl}^{cl}$ in $\mathscr{X}_1$. Then, applying the comparison theorem (Theorem 3.5.3), we obtain a rigid analytic map of eigenvarieties $\mathscr{X}_{spl} \to \mathscr{X}_2$ which p-adically interpolates Asai transfer.

Theorem 5.1.4 (p-adic Asai transfer: split case). *There exists a rigid analytic map*

$$\phi^{\pm} : \mathscr{X}_{spl} \to \mathscr{X}_2$$

which sends the point $(\pi, \{\alpha_{\mathfrak{p}}, \alpha_{\mathfrak{p}^c}\})$ to the point $(\mathrm{As}^{\pm}(\pi), \chi)$, where χ is the refinement given by

$$\chi(u_{p,1}) = \alpha_{\mathfrak{p}} \alpha_{\mathfrak{p}^c}, \chi(u_{p,2}) = p^{-1} \alpha_{\mathfrak{p}} \beta_{\mathfrak{p}^c}, \chi(u_{p,3}) = p^{-2} \beta_{\mathfrak{p}} \alpha_{\mathfrak{p}^c} \text{ and } \chi(u_{p,4}) = p^{-3} \beta_{\mathfrak{p}} \beta_{\mathfrak{p}^c}.$$

5.2 The case where p is inert in K

Assume now that p is inert in K. Recall that in this case for G_1 we are working with the eigenvariety $\mathscr{X}_1^{in}$, that is the eigenvariety associated to the datum $\mathscr{D}_1 = (\mathscr{W}_1, \mathscr{X}_1, \mathscr{M}_1, \mathbb{T}_1, \psi_1)$ with respect to the controlling operator $U_{in} := -U_p^4$ and for G_2 we are working with the eigenvariety $\mathscr{X}$ associated to the $\mathscr{D} = (\mathscr{W}_2, \mathscr{X}_2, \mathscr{M}_2, \mathbb{T}_2, \psi_2|_{\tilde{\mathbb{T}}_2})$ with respect to the controlling operator $\tilde{U}_p := \tilde{U}_{p,1}\tilde{U}_{p,2}\tilde{U}_{p,3}$. We want to use the comparison theorem (Theorem 3.5.3) to construct this map. The map j between weight spaces is given in (5.0.1). Below we construct the map of the Hecke algebras and give a set of classical points in the spectral variety $\mathscr{X}_1$.

First we define a map $\tilde{\sigma}^{\pm} : \tilde{\mathbb{T}}_2 \to \mathbb{T}_1$ as follows. On the unramified part of the Hecke algebra, the maps $\tilde{\sigma}^{\pm}$ agrees with $\sigma^{\pm}$. At p, we define $\tilde{\sigma}^{\pm}$ on $\tilde{\mathscr{A}}_{2,p}^+$ by sending

$$\tilde{U}_{p,1} \mapsto \pm U_p$$

$$\tilde{U}_{p,2} \mapsto U_p^2 S_p$$

$$\tilde{U}_{p,3} \mapsto \mp p^{-1} U_p S_p$$

$$\tilde{U}_{p,4} \mapsto -p^{-2} S_p^2.$$

The following lemma will justify the definition of $\tilde{\sigma}^{\pm}$ for Hecke operators supported at p. For a Hilbert modular form f we choose the refinement such that, the U_p eigenvalue of f is α_p. Note that, $N_{K/\mathbb{Q}}(p)S_p$ acts via the eigenvalue $\alpha_p \beta_p$.

Lemma 5.2.1. *The module* $\mathrm{As}^{\pm}(\pi)_p^{l_2}$ *contains a vector v on which $\tilde{\mathscr{A}}_{2,p}$ acts via the character χ associated to the tuple* $(\pm p^{-3}\beta_p, \pm p^{-2}\sqrt{\alpha_p \beta_p}, \mp p^{-1}\sqrt{\alpha_p \beta_p}, \pm \alpha_p)$. *As a consequence, the character $\tilde{\chi} = \chi|_{\tilde{\mathscr{A}}_{2,p}}$ of $\tilde{\mathscr{A}}_{2,p}$ is a refinement associated to* $\mathrm{As}^{\pm}(\pi)_p^{l_2}$ *in $\mathscr{X}$. In particular, $\tilde{U}_p$ acts via the scalar* $-p^{4m-1}\alpha_p^4$.

Proof. The proof of this lemma is similar to Lemma 4.4.1. We give a brief sketch here.

First, from the character χ of $\mathscr{A}_p$, we get an explicit description of the character $\tilde{\chi}$ of $\tilde{\mathscr{A}}_{2,p}$. We easily compute that

$$\tilde{\chi}(\tilde{u}_{p,1}) = \chi(u_{p,1}) = \pm \alpha_p$$

$$\tilde{\chi}(\tilde{u}_{p,2}) = \chi(u_{p,2})^2 = p^{-2}\alpha_p \beta_p$$

$$\tilde{\chi}(\tilde{u}_{p,3}) = \chi(u_{p,3})\chi(u_{p,2})^{-1} = -p^{-1}$$

$$\tilde{\chi}(\tilde{u}_{p,4}) = \chi(u_{p,4}) = \pm p^{-3}\beta_p.$$

Observe that, $\tilde{U}_{p,1} = \tilde{u}_{p,1}$, $\tilde{U}_{p,2} = \tilde{u}_{p,2}\tilde{U}_{p,1}^2$, $\tilde{U}_{p,3} = \tilde{u}_{p,3}\tilde{U}_{p,2}\tilde{U}_{p,1}^{-1}$ and $\tilde{U}_{p,4} = \tilde{u}_{p,4}\tilde{U}_{p,3}$; hence, we get that $\tilde{U}_{p,1}, \tilde{U}_{p,2}, \tilde{U}_{p,3}$ and $\tilde{U}_{p,4}$ acts by $\pm \alpha_p, p^{-2}\alpha_p^3 \beta_p, \mp p^{-3}\alpha_p^2 \beta_p$ and $-p^{-6}\alpha_p^2 \beta_p^2$ respectively.

From the characteristic polynomial of U_p, we see that $\alpha_p \beta_p = p^{2m+2}$. By definition, our controlling operator $\tilde{U}_p = \tilde{U}_{p,1}\tilde{U}_{p,2}\tilde{U}_{p,3}$. We compute the action of $\tilde{U}_p$ as

$$(\pm \alpha_p)(p^{-2}\alpha_p^3 \beta_p)(\mp p^{-3}\alpha_p^2 \beta_p) = -p^{-5}(\alpha_p \beta_p)^2 \alpha_p^4 = -p^{4m-1}\alpha_p^4.$$

This completes the proof of the lemma.   □

The $\star$-action of $\tilde{\mathscr{A}}_{2,p}^+$ on $\mathrm{As}^{\pm}(\pi)_p^{l_2}$ is the usual action, rescaled by $j(\kappa)(1, p, p^3, p^4)^{-1}$, where $j(\kappa)$ corresponds to the highest weight vector

$$\mu = \left(\frac{n_1 + n_2}{2} + m - 1, \frac{n_1 - n_2}{2} + m - 1, m - \frac{n_1 - n_2}{2}, m - \frac{n_1 + n_2}{2}\right).$$

By our assumption that $n_1 > n_2$, the weight μ is a dominant integral weight. We compute that

$$\mu(1, p, p^2, p^3) = p^{\frac{n_1-n_2}{2}+m-1+3\left(m-\frac{n_1-n_2}{2}\right)+4\left(m-\frac{n_1+n_2}{2}\right)}$$

$$= p^{8m-1-3n_1-n_2}$$

$$= p^{4m-1+6v_1+2v_2}.$$

Define the set of classical weights as

$$\mathcal{W}_{in}^{cl} = \{(n, v) \in \mathcal{W}_1 \mid n_i, v_i \in \mathbb{Z}, \; n_1 > n_2 \geq 0 \quad \text{and} \quad 3v_1 + v_2 = 0\}. \tag{5.2.2}$$

We remark that $\overline{\mathcal{W}}_{in}$ the Zariski closure of $\mathcal{W}_{in}^{cl}$ in $\mathcal{W}_1$ is a 2-dimensional subspace of the full weight space. Moreover, if f has weight $(n, v) \in \mathcal{W}_{in}^{cl}$, then the eigenvalue of $\star$-action of U_p on $\text{As}^{\pm}(\pi)_p^{l_2}$ is $-\alpha_p^4$.

Now we compute small slope h for the weight $j(\kappa) = \mu$. By definition, we have

$$h = \inf_{w \in S_4 \backslash \{1\}} v_p((w \cdot \mu)(1, p, p^2, p^3)) - v_p(\mu(1, p, p^2, p^3)).$$

As in the split case, we note that

$$v_p((w \cdot \mu)(1, p, p^2, p^3)) - v_p(\mu(1, p, p^2, p^3))$$

$$= (\mu_{w^{-1}(2)} - \mu_2 + \rho_{w^{-1}(2)} - \rho_2) + 3(\mu_{w^{-1}(3)} - \mu_3 + \rho_{w^{-1}(3)} - \rho_3)$$

$$+ 4(\mu_{w^{-1}(4)} - \mu_4 + \rho_{w^{-1}(4)} - \rho_4).$$

By taking $w = (1\;2)$ and $w = (2\;3)$, we see that

$$h = \min\{n_2 + 1, 2(n_1 - n_2)\}.$$

Proposition 5.2.3. *If*

$$v_p(\alpha_p) < \min\left\{\frac{n_2+1}{4}, \frac{n_1-n_2}{2}\right\},$$

then $H^*(K_2^p I_2, \mathcal{D}_{j(\kappa)})$ contains a nonzero vector $v^{\pm}$ such that every $T \in \tilde{\mathbb{T}}_2$ acts on $v^{\pm}$ through the scalar $\phi_{f,\alpha_p}(\tilde{\sigma}^{\pm}(T))$.

Proof. The proof follows from the control theorem, Theorem 3.4.1, exactly as in Proposition 5.1.3. □

Let $\mathcal{X}_{in}$ be the set of points in $\mathcal{X}_1^{in}$ of the form $(n_1, n_2, v_1, v_2, a^{-1})$ such that, $(n_1, n_2, v_1, v_2) \in \mathcal{W}_{in}^{cl}$ and $v_p(a) < \min\left\{\frac{n_2+1}{4}, \frac{n_1-n_2}{2}\right\}$. If $z = (\kappa, a^{-1})$ and if $\underline{k} = (n_1 + 2, n_2 + 2)$ and $\underline{w} = (n_1 + v_1 + 1, n_2 + v_2 + 1)$, then $\mathcal{M}_1(z) \cong H^*(K_1^p I_1, \mathcal{L}_\kappa)_{<a}$. On the other hand, $\mathcal{M}_2(j(z)) \cong H^*(K_2^p I_2, \mathcal{D}_{j(\kappa)})_{<a} \cong H^*(K_2^p I_2, \mathcal{L}_{j(\kappa)})_{<a}$. We define $\mathcal{X}_{in}^{cl}$ to be the set of classical points on $x \in \mathcal{X}_1^{in}$, such that $\pi_1(x) \in \mathcal{X}_{in}$. Then, by Proposition 5.2.3, the $\mathbb{T}_1$ eigenspace of x composed with $\sigma^{\pm}$ appears in $\mathcal{M}_2(j(z))$.

Let $\mathcal{X}_{in}$ denote the Zariski closure of $\mathcal{X}_{in}^{cl}$ in $\mathcal{X}_1^{in}$. Then, applying the comparison theorem (Theorem 3.5.3), we obtain a rigid analytic map of eigenvarieties $\mathcal{X}_{in} \to \mathcal{X}$ which p-adically interpolates the classical Asai transfer.

Theorem 5.2.4 (p-adic Asai transfer: inert case). *There exists a rigid analytic map*

$$\phi^{\pm} : \mathcal{X}_{in} \to \mathcal{X}$$

which sends the point (f, α_p) to the point $(\text{As}^{\pm}(\pi), \tilde{\chi})$ where $\tilde{\chi}$ is the refinement given by

$$\tilde{\chi}(\tilde{u}_{p,1}) = \pm\alpha_p, \; \tilde{\chi}(\tilde{u}_{p,2}) = p^{-2}\alpha_p\beta_p, \; \tilde{\chi}(\tilde{u}_{p,3}) = -p^{-1} \text{ and } \tilde{\chi}(\tilde{u}_{p,4}) = \pm p^{-3}\beta_p$$

as in Lemma 5.2.1.

References

[1] T. Asai, On certain Dirichlet series associated with Hilbert modular forms and Rankin's method, *Math. Ann.*, **226** no. 1, (1977) 81–94.

[2] A. Ash and G. Stevens, p-adic deformations of arithmetic cohomology, Preprint (2008).

[3] J. Bellaïche and G. Chenevier, Families of Galois representations and Selmer groups. Astérisque, **324** (2009).

[4] D. Blasius and J. D. Rogawski, Zeta functions of Shimura varieties, Motives (Seattle, WA, 1991), *Proc. Sympos. Pure Math., Amer. Math. Soc.*, Providence, RI, **55** (1994) 525–571.

[5] K. Buzzard, Eigenvarieties. In L-functions and Galois representations, London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge **320** (2007) 59–120.

[6] G. Chenevier, Familles p-adiques de formes automorphes pour GL_n, *J. Reine Angew. Math.*, **570** (2004) 143–217.

[7] G. Chenevier, Une correspondance de Jacquet–Langlands p-adique, *Duke Math. J.*, **126** no. 1, (2005) 161–194.

[8] D. Hansen, Universal eigenvarieties, trianguline Galois representations, and p-adic Langlands functoriality, *J. Reine Angew. Math.*, to appear.

[9] H. Hida, On Abelian Varieties with Complex Multiplication as Factors of the Jacobians of Shimura Curves, *American J. Math.*, **103** no. 4, (1981) 727–776.

[10] C. Johansson and J. Newton, Irreducible components of extended eigenvarieties and interpolating Langlands functoriality, arXiv:1703.03568.

[11] M. Krishnamurthy, The Asai transfer to GL_4 via the Langlands–Shahidi method, *Int. Math. Res. Not.*, no. 41, (2003) 2221–2254.

[12] J. Ludwig, p-adic functoriality for inner forms of unitary groups in three variables, *Math. Res. Let.*, **21**(1) (2014) 141–148.

[13] J. Ludwig, A p-adic Labesse-Langlands transfer, *Manuscripta Mathematica*, **154**(1–2) (2017) 23–57.

[14] D. Majumdar, Endoscopic transfer between eigenvarieties for definite unitary groups, *J. Ramanujan Math. Soc.*, **31** no. 3, (2016) 235–255.

[15] J. Newton, Completed cohomology of Shimura curves and a p-adic Jacquet–Langlands correspondence, *Math. Ann.*, **355**(2) (2013) 729–763.

[16] D. Ramakrishnan, Modularity of solvable Artin representations of GO(4)-type, *Int. Math. Res. Not.*, no. 1, (2002) 1–54.

[17] G. Stevens, Rigid analytic modular symbols, (Unpublished) preprint (1994).

[18] E. Urban, Eigenvarieties for reductive groups, *Ann. of Math. (2)*, **174** no. 3, (2011) 1685–1784.

[19] P.-J. White, p-adic Langlands functoriality for the definite unitary group, *J. Reine Angew. Math.*, **691** (2014) 1–27.