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Rationality of cycles over function field of exceptional projective homogeneous varieties

Raphael Fino
UPMC Sorbonne Universités, Institut de Mathématiques de Jussieu, Paris, France
e-mail: fino@math.jussieu.fr

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Abstract. In this article we prove a result comparing rationality of algebraic cycles over the function field of a projective homogeneous variety under a linear algebraic group of type F_4 or E_8 and over the base field, which can be of any characteristic.

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1. Introduction

The purpose of this note is to prove the following theorem dealing with rationality of algebraic cycles over function field of some exceptional projective homogeneous varieties.

Theorem 1.1. *Let G be a linear algebraic group of type F_4 or E_8 over a field F and let X be a projective homogeneous G-variety. For any equidimensional variety Y, the change of field homomorphism*

$$\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(X)}),$$

where Ch *is the Chow group modulo p, with $p = 3$ when G is of type F_4 and $p = 5$ when G is of type E_8, is surjective in codimension $< p + 1$.*

It is also surjective in codimension $p + 1$ for a given Y provided that $1 \notin \deg \mathrm{Ch}_0(X_{F(\zeta)})$ for each generic point $\zeta \in Y$.

In this note, a *projective homogeneous G-variety* is a twisted form of G_0/P, where G_0 is a split linear algebraic group of the same type as G and P is a parabolic subgroup. The proof of Theorem 1.1 is given in section 5.

In previous papers ([3], [4], after the so-called Main Tool Lemma by A. Vishik, see [19], [20]), similar issues about rationality of cycles, with *quadrics* instead of exceptional projective homogeneous varieties, have been treated. The above statement is to put in relation with the result [11, Theorem 4.3] by N. Karpenko and A. Merkurjev, where *generic splitting varieties* have been considered.

In characteristic 0, Theorem 1.1 is contained in [11, Theorem 4.3]. In an earlier paper (see [21, Corollary 1.4]), K. Zainoulline proved the first conclusion of Theorem 1.1 (modulo torsion) in characteristic 0 if G is of type F_4. Our result is valid in any characteristic.

The method of proof is basically the method used to prove [11, Theorem 4.3] combined with a motivic decomposition result for *generically split* projective homogeneous varieties due to V. Petrov, N. Semenov and K. Zainoulline (see [16, Theorem 5.17]) and involving the *Rost motive*. This is described in section 3.

In section 4, we present some properties about Chow groups of the Rost motive of groups of strongly inner type (e.g F_4 and E_8) with maximal J-invariant. Those properties make the method particularly suitable for groups of type F_4 and E_8.

The method also relies on a linkage between the γ-filtration on the Grothendieck ring of projective homogeneous varieties and Chow groups, in the spirit of [6].

In the aftermath of Theorem 1.1, we get the following statement dealing with integral Chow groups (see [11, Theorem 4.5]).

Corollary 1.2. *We use notation introduced in Theorem 1.1 and we write* CH *for the integral Chow group. If $p \in \deg \mathrm{CH}_0(X)$, then for any equidimensional variety Y, the change of field homomorphism*

$$\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(X)})$$

is surjective in codimension $< p+1$.

It is also surjective in codimension $p+1$ for a given Y provided that $1 \notin \deg \mathrm{Ch}_0(X_{F(\zeta)})$ *for each generic point $\zeta \in Y$.*

Remark 1.3. Our method of proof for Theorem 1.1 works for groups of type G_2 as well (with $p=2$). However, the case of G_2 can be treated in a more elementary way if $\mathrm{char}(F)=0$.

Indeed, it is known that to each group G of type G_2 one can associate a 3-fold Pfister quadratic form ρ such that, denoting by X_ρ the Pfister quadric associated with ρ, the variety X has a rational point over $F(X_\rho)$ and vice-versa. Thus, for any equidimensional variety Y, one has the commutative diagram

$$\begin{array}{ccc} \mathrm{Ch}(Y) & \longrightarrow & \mathrm{Ch}(Y_{F(X)}) \\ \downarrow & & \downarrow \\ \mathrm{Ch}(Y_{F(X_p)}) & \longrightarrow & \mathrm{Ch}(Y_{F(X_p \times X)}) \end{array}$$

where the right and the bottom maps are isomorphisms. Furthermore, as suggested in [20, Remark on Page 665] (where the assumption $\mathrm{char}(F) = 0$ is required), the change of field homomorphism $\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(X_p)})$ is surjective in codimension < 3.

2. Filtrations on Grothendieck ring of projective homogeneous varieties

In this section, we prove two propositions which play a crucial role in the proof of Theorem 1.1.

First of all, we recall that for any smooth variety X over a field F (in this paper, an F-*variety* is a separated scheme of finite type over F), one can consider two particular filtrations on the Grothendieck ring $K(X)$ (see [6, §1.A]), namely the γ-filtration and the topological filtration, whose respective terms of codimension i are given by

$$\gamma^i(X) = \langle c_{n_1}(a_1), \ldots, c_{n_m}(a_m) | n_1 + \cdots + n_m \geq i \quad \text{and} \quad a_1, \ldots, a_m \in K(X) \rangle$$

and

$$\tau^i(X) = \langle [\mathcal{O}_Z] \mid Z \hookrightarrow X \quad \text{and} \quad \mathrm{codim}(Z) \geq i \rangle,$$

where c_n is the n-th Chern Class with values in $K(X)$ and $[\mathcal{O}_Z]$ is the class in $K(X)$ of the structure sheaf of a closed subvariety Z. For any i, one has $\gamma^i(X) \subset \tau^i(X)$ and one even has $\gamma^i(X) = \tau^i(X)$ for $i \leq 2$. We write $\gamma^{i/i+1}(X)$ and $\tau^{i/i+1}(X)$ for the respective quotients. We denote by pr^i the canonical surjection

$$\begin{aligned} \mathrm{CH}^i(X) &\twoheadrightarrow \tau^{i/i+1}(X) \\ [Z] &\longmapsto [\mathcal{O}_Z]. \end{aligned}$$

Note that for any prime p, one can also consider the γ-filtration γ_p and the topological filtration τ_p on the ring $K(X)/pK(X)$ by replacing $K(X)$ by $K(X)/pK(X)$ in the previous definitions.

The method of proof of the following proposition is largely inspired by the proof of [10, Theorem 6.4 (2)].

Proposition 2.1. *Let G_0 be a split connected semisimple linear algebraic group over a field F and let B be a Borel subgroup of G_0. There exist an extension E/F and a cocycle $\xi \in H^1(E, G_0)$ such that the topological filtration and the γ-filtration on $K({}_\xi(G_0/B))$ coincide.*

Proof. Let n be an integer such that $G_0 \subset \mathbf{GL}_n$ and let us set $S := \mathbf{GL}_n$ and $E := F(S/G_0)$. We denote by $\mathbf{T}$ the E-variety $S \times_{S/G_0} \operatorname{Spec}(E)$ given by the generic fiber of the projection $S \to S/G_0$. Note that since $\mathbf{T}$ is clearly a G_0-torsor over E, there exists a cocycle $\xi \in H^1(E, G_0)$ such that the smooth projective variety $X := \mathbf{T}/B_E$ is isomorphic to ${}_\xi(G_0/B)$. We claim that the Chow ring $\mathrm{CH}(X)$ is generated by Chern classes.

Indeed, the morphism $h : X \to S/B$ induced by the canonical G_0-equivariant morphism $\mathbf{T} \to S$ being a localization, the associated pull-back

$$h^* : \mathrm{CH}(S/B) \longrightarrow \mathrm{CH}(X)$$

is surjective. Furthermore, the ring $\mathrm{CH}(S/B)$ itself is generated by Chern classes: by [10, §6,7] there exists a morphism

$$\mathbb{S}(T^*) \longrightarrow \mathrm{CH}(S/B), \tag{2.2}$$

(where $\mathbb{S}(T^*)$ is the symmetric algebra of the group of characters T^* of a split maximal torus $T \subset B$) with its image generated by Chern classes. Moreover, the morphism (2.2) is surjective by [10, Proposition 6.2]. Since h^* is surjective and Chern classes commute with pull-backs, the claim is proved.

We show now that the two filtrations on $K(X)$ coincide by induction on codimension. Let $i \geq 0$ and assume that $\tau^{i+1}(X) = \gamma^{i+1}(X)$. Since for any $j \geq 0$, one has $\gamma^j(X) \subset \tau^j(X)$, the induction hypothesis implies that

$$\gamma^{i/i+1}(X) \subset \tau^{i/i+1}(X).$$

Thus, the ring $\mathrm{CH}(X)$ being generated by Chern classes, one has $\gamma^{i/i+1}(X) = \tau^{i/i+1}(X)$ by [9, Lemma 2.16]. Therefore one has $\tau^i(X) = \gamma^i(X)$ and the proposition is proved. □

Note that this result remains true when one consider a *special* parabolic subgroup P instead of B.

Now, we prove a result which will be used in section 5 to get the second conclusion of Theorem 1.1.

We recall that for any smooth variety X over a field, for any prime p, and for any $i < p + 1$, the canonical surjection $pr_p^i : \mathrm{Ch}^i(X) \twoheadrightarrow \tau_p^{i/i+1}(X)$ is an isomorphism by the Riemann-Roch Theorem without denominators (see [6, §1.A] for example). The following proposition extends this fact to $i = p+1$ provided that X is a projective homogeneous variety under a certain class of linear algebraic group (containing F_4 and E_8) and $p > 2$.

Proposition 2.3. *Let X be a projective homogeneous variety under a semi-simple adjoint algebraic group G of inner type whose Tits algebras are trivial, then for any prime $p > 2$ the canonical surjection*

$$\mathrm{Ch}^{p+1}(X) \twoheadrightarrow \tau_p^{p+1/p+2}(X),$$

is injective.

That proposition is obtained by combining the two following lemmas.

Lemma 2.4. *Let X be a smooth variety and $p > 2$ be a prime. If the inclusion $E_\infty^{1,-2}(X) \subset E_2^{1,-2}(X)$ given by the Brown-Gersten-Quillen spectral sequence is an equality, then the epimorphism $\mathrm{Ch}^{p+1}(X) \twoheadrightarrow \tau_p^{p+1/p+2}(X)$ is an isomorphism.*

Proof. For any smooth variety X and any $i \geq 1$, the epimorphism pr^i coincides with the edge homomorphism of the spectral Brown-Gersten-Quillen structure $E_2^{i,-i}(X) \Rightarrow K(X)$ (see [17, §7]), that is to say

$$pr^i : \mathrm{CH}^i(X) \simeq E_2^{i,-i}(X) \twoheadrightarrow \cdots \twoheadrightarrow E_{i+1}^{i,-i}(X) = \tau^{i/i+1}(X).$$

In particular, for any prime p, the map pr_p^{p+1} is the composite of the surjections

$$q_r : E_r^{p+1,-p-1}(X) \pmod p \twoheadrightarrow \frac{E_r^{p+1,-p-1}(X)}{\mathrm{Im}(\delta_r)} \pmod p,$$

for r from 2 to $p+1$, where δ_r is the differential starting from $E_r^{p+1-r,-p-2+r}(X)$.

Moreover, by [13, Theorem 3.4], every prime divisor l of the order of δ_r is such that $l-1$ divides $r-1$. Hence, for $r \leq p-1$, the order of δ_r is coprime to p and this implies that q_r is an isomorphism. For $r = p+1$, one has $l = 2$ ou $l = p+1$ and in both cases l is coprime to p (since $p > 2$).

Therefore, we have shown that pr_p^{p+1} is injective if and only if q_p is an isomorphism. Let us consider the following inclusions given by the BGQ-structure

$$E_\infty^{1,-2}(X) \subset \cdots \subset E_3^{1,-2}(X) \subset E_2^{1,-2}(X).$$

By the very definition, one has $E_\infty^{1,-2}(X) = E_2^{1,-2}(X)$ if and only if for any $r \geq 2$ the differential starting from $E_r^{1,-2}(X)$ is zero. In particular, the equality $E_\infty^{1,-2}(X) = E_2^{1,-2}(X)$ implies that $\delta_p = 0$ and the lemma is proved. □

Lemma 2.5. *Let G be a semisimple adjoint algebraic group of inner type whose Tits algebras are trivial. Then for any projective homogeneous G-variety X, the inclusion $E_\infty^{1,-2}(X) \subset E_2^{1,-2}(X)$ given by the Brown-Gersten-Quillen spectral sequence is an equality.*

Proof. On the one hand, by the very defintion, the group $E_{\infty}^{1,-2}(X)$ is the first quotient $K_1^{(1/2)}(X)$ of the topological filtration on $K_1(X)$. On the other hand, one has $E_2^{1,-2}(X) = H^1(X, K_2)$ (for any integers p and q, one has $E_2^{p,q}(X) = H^p(X, K_{-q})$).

First, we claim that the natural map

$$H^0(X, K_1) \otimes \mathrm{CH}^1(X) \to H^1(X, K_2) \tag{2.6}$$

is an isomorphism. Indeed, since G has only trivial Tits algebras, by [12, Theorem], one has

$$H^1(X, K_2) \simeq H^1(X_{\mathrm{sep}}, K_2)^{\Gamma},$$

where Γ is the absolute Galois group of F. Moreover, since the variety X_{sep} is cellular, by [12, Proposition 1], one has

$$H^1(X_{\mathrm{sep}}, K_2) \simeq K_1 F_{\mathrm{sep}} \otimes \mathrm{CH}^1(X_{\mathrm{sep}}).$$

Note that since X is smooth, the Picard group $\mathrm{Pic}(X_{\mathrm{sep}})$ is identified with $\mathrm{CH}^1(X_{\mathrm{sep}})$. Furthermore, any projective homogeneous variety under a semi-simple adjoint group of inner type whose Tits algebras are trivial has a rational Picard group (see [14]). Therefore one has $\mathrm{CH}^1(X) \simeq \mathrm{CH}^1(X_{\mathrm{sep}})$ and since $(K_1 F_{\mathrm{sep}})^{\Gamma} = K_1 F = H^0(X, K_1)$, one has $H^0(X, K_1) \otimes \mathrm{CH}^1(X) \simeq H^1(X, K_2)$ and the claim is proved.

Now, it is known that $\mathrm{CH}^1(X_{\mathrm{sep}})$ is a free abelian group of finite rank (see [18, §2] for example) and it follows that there exists an integer $k \geq 0$ such that $\mathrm{CH}^1(X) = \mathbb{Z}^{\oplus k}$. Let us denote by φ the isomorphism

$$(F^{\times})^{\oplus k} \longrightarrow H^1(X, K_2)$$

such that for any $a \in (F^{\times})^{\oplus k}$ the element $\varphi(a)$ corresponds by (2.6) to $\sum_{i=0}^{k} \pi_i(a) \otimes e_i$ in $H^0(X, K_1) \otimes \mathrm{CH}^1(X)$, where $(e_i)_{1 \leq i \leq k}$ is the canonical basis of $\mathbb{Z}^{\oplus k}$ and $\pi_i : (F^{\times})^{\oplus k} \to F^{\times}$ is the standard projection.

Then it suffices to find a homomorphism $\psi : (F^{\times})^{\oplus k} \to K_1^{(1/2)}(X)$ such that the diagram (see [8, §4])

$$\begin{array}{ccc} K_1^{(1/2)}(X) & \hookrightarrow & H^1(X, K_2) \\ & {\scriptstyle\psi}\nwarrow \quad \nearrow{\scriptstyle\varphi} & \\ & (F^{\times})^{\oplus k} & \end{array}$$

is commutative to get the conclusion. The homomorphism ψ defined as follow is suitable (and ψ is necessarily defined this way). For every $i = 0, \ldots, k$, let $j_i : Z_i \subset X$ be a subvariety of codimension 1 such that $[Z_i] = e_i$ in

$\mathrm{CH}^1(X)$ and let p_i be the structure morphism $Z_i \to \mathrm{Spec}(F)$. Then we set $\psi = \sum_{i=1}^{k} \psi_i$, with

$$\psi_i : (F^\times)^{\oplus k} \xrightarrow{\pi_i} F^\times \xrightarrow{p_i{}^*} K_1(Z_i) \xrightarrow{j_i{}^*} K_1^1(X) \longrightarrow K_1^{1/2}(X). \qquad \square$$

Remark 2.7. Assume that G_0 is of *strongly inner* type (e.g F_4 and E_8, see [6, §3] for instance) and consider an extension E/F and a cocycle $\xi \in H^1(E, G_0)$. By the result [15, Theorem 2.2.(2)] of I. Panin, the change of field homomorphism

$$K({}_\xi(G_0/B)_E) \to K({}_\xi(G_0/B)_{\overline{E}}) \simeq K(G_0/B),$$

with $\overline{E}$ an algebraic closure of E, is an isomorphism. Therefore, since the γ-filtration is defined in terms of Chern classes and the latter commute with pull-backs, the quotients of the γ-filtration on $K({}_\xi(G_0/B)_E)$ do not depend nor on the extension E/F neither on the choice of $\xi \in H^1(E, G_0)$.

3. Generically split projective homogeneous varieties

In this section, we introduce in a more general context the basis of the method we will use in section 5 to prove Theorem 1.1.

The method of proof largely relies on the following proposition, which is a version of the result [2, Lemma 88.5] slightly altered to fit our situation (see also the proof of [11, Proposition 2.8]).

Proposition 3.1 (Karpenko, Merkurjev). *Let X be a smooth variety over a field F and Y an equidimensional F-variety. Given an integer k such that for any i and any point $y \in Y$ of codimension i the change of field homomorphism*

$$\mathrm{CH}^{k-i}(X) \longrightarrow \mathrm{CH}^{k-i}(X_{F(y)})$$

is surjective, the change of field homomorphism

$$\mathrm{CH}^k(Y) \longrightarrow \mathrm{CH}^k(Y_{F(X)})$$

is also surjective.

Note that this statement remains true for any prime p when one considers the group Ch with $\mathbb{Z}/p\mathbb{Z}$-coefficients instead of CH.

Now let X be a projective homogeneous variety under a semisimple linear algebraic group G of inner type. Assume furthermore that the F-variety X is *generically split*, i.e the group G splits over the generic point of X (e.g any projective homogeneous variety X under a group G of type F_4 or E_8 admitting a splitting field of degree 3 or 5 respectively). Then one can apply the motivic decomposition result [16, Theorem 5.17] to X and get that for any prime p, the Chow motive $\mathcal{M}(X, \mathbb{Z}/p\mathbb{Z})$ decomposes as a sum of twists of an indecomposable motive $\mathcal{R}_p(G)$ (in the same way as (4.3)), called *Rost motive*. Note that the quantity and the value of those twists do not depend on the base field. In particular, we get that for any extension L/F and any integer k, the group $\mathrm{Ch}^k(X_L)$ is isomorphic to a direct sum of groups $\mathrm{Ch}^{k-i}(\mathcal{R}_p(G)_L)$ with $0 \leq i \leq k$.

Consequently, combining this with Proposition 3.1, one get the following statement.

Proposition 3.2. *Let G be a semisimple linear algebraic group of inner type over a field F. Let p be a prime and $\mathcal{R}_p(G)$ the associated Rost motive of G. If for any extension L/F, the change of field*

$$\mathrm{Ch}(\mathcal{R}_p(G)) \longrightarrow \mathrm{Ch}(\mathcal{R}_p(G)_L)$$

is surjective in codimension $< k$ then for any equidimensional variety Y and for any generically split projective homogeneous G-variety X, the change of field

$$\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(X)})$$

is surjective in codimension $< k$.

4. Maximal J-invariant

In this section, G is a simple linear algebraic group of strongly inner type. Let G_0 be a split connected linear algebraic group of the same type as the type of G and let $\xi \in H^1(F, G_0)$ be a cocycle such that G is isomorphic to the twisted form ${}_\xi G_0$. We write $\mathfrak{B}$ for the Borel variety of G (one has $\mathfrak{B} \simeq {}_\xi(G_0/B)$, where B is a Borel subgroup of G_0).

For any torsion prime p of G, we write $J_p(G) = (j_1, \ldots, j_r)$ for the J-invariant modulo p of G and we say that $J_p(G)$ is *maximal* if for every $i = 1, \ldots, r$, one has $j_i = k_i$, where k_i is the p-primary power of the ith p-exceptional degree of G_0 (see [16, §4]). Note that for any extension L/F, one has $J_p(G_L) \leq J_p(G)$ by [16, Example 4.7].

In this section, we present some properties about Chow groups of the Rost motive of simple linear algebraic groups of strongly inner type (e.g F_4 and E_8) with maximal J-invariant modulo some torsion prime. In the next section, we will combine those properties with the method described in §3 to prove Theorem 1.1.

Lemma 4.1. *Let G be a simple linear algebraic group of strongly inner type such that its J-invariant $J_p(G)$ is maximal. Then one has*

(i) $p = 3$ *or* 5;

(ii) $\mathrm{Ch}^2(\mathcal{R}_p(G)) = \mathbb{Z}/p\mathbb{Z}$ *and* $\mathrm{Ch}^3(\mathcal{R}_p(G)) = 0$.

Proof. Since $J_p(G)$ is maximal, by [7, Example 5.3], the cocycle $\xi \in H^1(F, G_0)$ corresponds to a *generic* G_0-torsor in the sense of [7]. Thus, by [6, Proposition 3.2] and [5, pp. 31, 133], one has $\mathrm{Tors}_p\mathrm{CH}^2(\mathfrak{B}) \neq 0$ (we need the assumption strongly inner to use material from [6, § 3]). The conclusion is given by [6, Proposition 5.4]. □

Lemma 4.2. *Let G be a simple linear algebraic group of strongly inner type such that its J-invariant $J_p(G)$ is maximal and let L/F be an extension such that $J_p(G_L) = J_p(G)$. Then one has*

(i) $\mathrm{Ch}^2(\mathcal{R}_p(G)_L) = \mathbb{Z}/p\mathbb{Z}$ *and* $\mathrm{Ch}^3(\mathcal{R}_p(G)_L) = 0$*;*

(ii) *the change of field* $\mathrm{Ch}^2(\mathfrak{B}) \to \mathrm{Ch}^2(\mathfrak{B}_L)$ *is an isomorphism.*

Proof. Since $J_p(G_L)$ is maximal then by Lemma 4.1 one has $\mathrm{Ch}^2(\mathcal{R}_p(G_L)) = \mathbb{Z}/p\mathbb{Z}$ and $\mathrm{Ch}^3(\mathcal{R}_p(G_L)) = 0$. Moreover, since $J_p(G_L) = J_p(G)$, one has $\mathcal{R}_p(G_L) \simeq \mathcal{R}_p(G)_L$ (see [16, Proposition 5.18 (i)]) and (i) is proved.

We show now that the change of field $\mathrm{Ch}^2(\mathfrak{B}) \to \mathrm{Ch}^2(\mathfrak{B}_L)$ is an isomorphism. We use material and notation introduced in section 2. Since $J_p(G) = J_p(G_L)$ is maximal, the cocycles ξ and ξ_L correspond to generic G_0-torsors and one consequently has $\gamma^3(\mathfrak{B}) = \tau^3(\mathfrak{B})$ and $\gamma^3(\mathfrak{B}_L) = \tau^3(\mathfrak{B}_L)$ (see [6, Theorem 3.1(ii)]). In particular, it follows that

$$\gamma_p^{2/3}(\mathfrak{B}) = \tau_p^{2/3}(\mathfrak{B}) \quad \text{and} \quad \gamma_p^{2/3}(\mathfrak{B}_L) = \tau_p^{2/3}(\mathfrak{B}_L).$$

Therefore, since $2 < p + 1$, the homomorphism $\mathrm{Ch}^2(\mathfrak{B}) \to \mathrm{Ch}^2(\mathfrak{B}_L)$ coincides with

$$\mathrm{Ch}^2(\mathfrak{B}) \simeq \gamma_p^{2/3}(\mathfrak{B}) \to \gamma_p^{2/3}(\mathfrak{B}_L) \simeq \mathrm{Ch}^2(\mathfrak{B}_L)$$

and the center arrow is an isomorphism by Remark 2.7. □

Recall that by [16, Theorem 5.13], one has the motivic decomposition

$$\mathcal{M}(\mathfrak{B}, \mathbb{Z}/p\mathbb{Z}) \simeq \bigoplus_{i\geq 0} \mathcal{R}_p(G)(i)^{\oplus a_i}, \tag{4.3}$$

where $\Sigma_{i\geq 0} a_i t^i = P(\mathrm{CH}(\overline{\mathfrak{B}}), t)/P(\mathrm{CH}(\overline{\mathcal{R}_p(G)}), t)$, with $P(-, t)$ the *Poincaré polynomial.* Thus, for any integer k and any extension L/F, we get the following decomposition concerning Chow groups

$$\mathrm{Ch}^k(\mathfrak{B}_L) \simeq \bigoplus_{i\geq 0} \mathrm{Ch}^{k-i}(\mathcal{R}_p(G)_L)^{\oplus a_i}. \tag{4.4}$$

Lemma 4.5. *In this statement, one has $p = 5$. Let G be a simple linear algebraic group of strongly inner type such that its J-invariant $J_5(G)$ is maximal and let L/F be an extension such that $J_5(G_L) = J_5(G)$. Then one has*

$$\mathrm{Ch}^4(\mathcal{R}_5(G)_L) = 0 \quad \textit{and} \quad \mathrm{Ch}^5(\mathcal{R}_5(G)_L) = 0.$$

Proof. Since $J_5(G_L) = J_5(G)$ one has $\mathcal{R}_5(G)_L = \mathcal{R}_5(G_L)$ and it suffices to prove that $Ch^4(\mathcal{R}_5(G)) = Ch^5(\mathcal{R}_5(G)) = 0$.

By Proposition 2.1 there exist an extension E/F and a cocycle $\xi' \in H^1(E, G_0)$ such that the topological filtration and the γ-filtration on $K(\mathfrak{B}')$, with $\mathfrak{B}' = {}_{\xi'}(G_0/B)$, coincide. Let us set $G' = {}_{\xi'}G_0$.

We claim that $J_5(G') \neq (0, \dots, 0)$. Indeed, assume that $J_5(G') = (0, \dots, 0)$. In that case, one has $R_5(G') = \mathbb{Z}/5\mathbb{Z}$ (Tate motive) by [16, Corollary 6.7] and the isomorphism (4.4) gives that $\mathrm{Ch}^2(\mathfrak{B}') = \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$. Since $2 < p+1$, it implies that $\gamma_5^{2/3}(\mathfrak{B}') = \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$, and consecutively $\gamma_5^{2/3}(\mathfrak{B}) = \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$ by Remark 2.7. However, we have $\gamma_5^{2/3}(\mathfrak{B}) = \tau_5^{2/3}(\mathfrak{B})$ (because $\gamma^3(\mathfrak{B}) = \tau^3(\mathfrak{B})$ since $\xi \in H^1(F, G_0)$ is generic). Thus, we have $\mathrm{Ch}^2(\mathfrak{B}) = \mathbb{Z}/5\mathbb{Z}^{\oplus a_2}$ which contradicts $\mathrm{Ch}^2(\mathcal{R}_5(G)) = \mathbb{Z}/5\mathbb{Z}$ and the claim is proved (we recall that for any $i < 6 = p+1$, one has $\tau_5^{i/i+1}(X) \simeq \mathrm{Ch}^i(X)$).

We now compute the groups $\gamma_5^{i/i+1}(\mathfrak{B}')$ for $i = 3, 4, 5$. Note that since G is of strongly inner type one has $K(\mathfrak{B}') \simeq K(G_0/B)$ by Remark 2.7. Furthermore, the description of the free group $K(G_0/B)$ in terms of generators does not depend on the characteristic of the base field (see [1, Lemma 13.3(4)]). Thus, in order to compute the groups $\gamma_5^{i/i+1}(\mathfrak{B}')$ for $i = 3, 4, 5$, since $J_5(G') \neq (0, \dots, 0)$, one can use the following theorem (adapted from [11, Theorem RM.10] to our situation)

Theorem 4.6 (Karpenko, Merkurjev). *Let H be a semisimple linear algebraic group of inner type over a field of characteristic* 0 *and let p be a torsion prime of H. If $J_p(H) \neq (0, \dots, 0)$ then*

$$\mathrm{Ch}^j(\mathcal{R}_p(H)) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } j=0 \text{ or } j=k(p+1)-p+1,\ 1 \le k \le p-1 \\ 0 & \text{otherwise,} \end{cases}$$

which combined with (4.4) gives that

$$\gamma_5^{i/i+1}(\mathfrak{B}') \simeq \mathrm{Ch}^i(\mathfrak{B}') = \mathbb{Z}/5\mathbb{Z}^{\oplus(a_{i-2}+a_i)} \quad \text{for} \quad i = 3, 4, 5$$

(where the isomorphism is due to $i < p+1$). Therefore, we get

$$\gamma_5^{i/i+1}(\mathfrak{B}) = \mathbb{Z}/5\mathbb{Z}^{\oplus(a_{i-2}+a_i)} \quad \text{for} \quad i = 3, 4, 5.$$

Thus, since $\tau_5^{3/4}(\mathfrak{B}) \simeq \mathrm{Ch}^3(\mathfrak{B})$, the isomorphism (4.4) for $k = 3$ gives that $\tau_5^{3/4}(\mathfrak{B}) \simeq \gamma_5^{3/4}(\mathfrak{B})$. Since the γ-filtration is contained in the topological one, we get

$$\tau_5^4(\mathfrak{B}) = \gamma_5^4(\mathfrak{B}),$$

which implies the existence of an exact sequence

$$0 \to (\tau_5^5(\mathfrak{B})/\gamma_5^5(\mathfrak{B})) \to \gamma_5^{4/5}(\mathfrak{B}) \to \tau_5^{4/5}(\mathfrak{B}) \to 0.$$

Thus, since $\tau_5^{4/5}(\mathfrak{B}) \simeq \mathrm{Ch}^4(\mathfrak{B})$, by applying the isomorphism (4.4) for $k = 4$, we get a surjection

$$\mathbb{Z}/5\mathbb{Z}^{\oplus(a_2+a_4)} \twoheadrightarrow \mathrm{Ch}^4(\mathcal{R}_5(G)) \oplus \mathbb{Z}/5\mathbb{Z}^{\oplus(a_2+a_4)},$$

which implies that $\mathrm{Ch}^4(\mathcal{R}_5(G)) = 0$.

We prove that $\mathrm{Ch}^5(\mathcal{R}_5(G)) = 0$ by proceeding in exactly the same way. □

5. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Remark 5.1. Let G be a semisimple linear algebraic group over a field F and let X be a projective homogeneous G-variety. The F-variety X is *A-trivial* in the sense of [11, Definition 2.3] (see [11, Example 2.5]), i.e for any extension L/F with $X(L) \neq \emptyset$, the degree homomorphism $\deg : \mathrm{CH}_0(X_L) \to \mathbb{Z}$ is an isomorphism.

Since by [11, Lemma 2.9], any A-trivial variety X with $1 \in \deg \mathrm{Ch}_0(X)$ is such that for any equidimensional variety Y the change of field homomorphism $\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(X)})$ is an isomorphism (in any codimension, with Ch the Chow group modulo p, for any prime p), one can assume that $1 \notin \deg \mathrm{Ch}_0(X)$ in order to prove Theorem 1.1.

Now, we know from [16, Table 4.13] that if G is of type F_4 or E_8 then the J-invariant $J_p(G)$ of G is equal to (0) or (1) (in the latter case, the J-invariant modulo p is maximal), with $p = 3$ if G is of type F_4 and $p = 5$ if G is of type E_8. However, the assumption $J_p(G) = (0)$ is equivalent to the existence of a splitting field K/F of G of degree coprime to p (see [16, Corollary 6.7]). In that case one has $\mathrm{Ch}_0(X) \simeq \mathrm{Ch}_0(X_K)$ and consequently $1 \in \deg \mathrm{Ch}_0(X)$. Thus, under the assumption $1 \notin \deg \mathrm{Ch}_0(X)$, one necessarily has $J_p(G) = (1)$ and that is why we can assume $J_p(G)$ maximal in the sequel.

We have seen in the previous remark that if $J_p(G)$ is maximal then p must divide the degree of any splitting field of G. Consequently, by [16, Example 3.6]), every projective homogeneous variety under a group of type F_4 or E_8 with maximal $J_p(G)$ ($p = 3$ for the type F_4 and $p = 5$ for the type E_8) is generically split. Then, by Proposition 3.2, the first conclusion of Theorem 1.1 is a direct consequence of the following proposition.

Proposition 5.2. *Let G be a linear algebraic group of type F_4 or E_8 over a field F such that $J_p(G)$ is nontrivial, with $p = 3$ if G is of type F_4 and $p = 5$*

if G is of type E_8. Then, for any extension L/F, the change of field

$$\mathrm{Ch}(\mathcal{R}_p(G)) \longrightarrow \mathrm{Ch}(\mathcal{R}_p(G)_L), \tag{5.3}$$

where $\mathcal{R}_p(G)$ is the associated Rost motive, is surjective in codimension $< p+1$.

Proof. First of all, the homomorphism (5.3) is clearly surjective in codimension 0 since one has $\mathrm{Ch}^0(\mathcal{R}_p(G)_L) = \mathbb{Z}/p\mathbb{Z}$ for any extension L/F. Then, $\mathrm{Ch}^1(\overline{\mathfrak{B}})$ is identified with the Picard group $\mathrm{Pic}(\overline{\mathfrak{B}})$ and is rational since G is of type F_4 or E_8 (see [18, Example 4.1.1]). Furthermore, thanks to the Solomon Theorem for example (see [18, § 2.5]), one can compute the coefficients a_i's in the decomposition (4.4): we get $a_0 = 1$ and $a_1 = \mathrm{rank}(G) = \mathrm{rank}(\mathrm{CH}^1(\overline{\mathfrak{B}}))$. Thus, the isomorphism (4.4) implies that $\mathrm{Ch}^1(\mathcal{R}_p(G)_L) = 0$ for any extension L/F. Therefore, we have already shown that the homomorphism (5.3) is surjective in codimension 0 and 1.

Now we show that it is surjective in codimension 2 and 3 (which proves the proposition for G of type F_4). Since $J_p(G)$ is maximal, one has $\mathrm{Ch}^2(\mathcal{R}_p(G)) = \mathbb{Z}/p\mathbb{Z}$ and $\mathrm{Ch}^3(\mathcal{R}_p(G)) = 0$ by Lemma 4.1. Moreover, since $J_p(G_L) \leq J_p(G)$ for any extension L/F, one has $J_p(G_L) = (0)$ or $J_p(G_L) = J_p(G)$ (i.e is maximal).

If $J_p(G_L) = J_p(G)$ then one has $\mathrm{Ch}^2(\mathcal{R}_p(G)_L) = \mathbb{Z}/p\mathbb{Z}$ and $\mathrm{Ch}^3(\mathcal{R}_p(G)_L) = 0$ by Lemma 4.2 (i) and the homomorphism (5.3) is clearly surjective in codimension 3. Thanks to the decomposition (4.4) and Lemma 4.2 (ii), we see that it is also surjective in codimension 2.

If $J_p(G_L) = (0)$ then on the one hand one has $\mathcal{R}_p(G_L) = \mathbb{Z}/p\mathbb{Z}$ and on the other hand the motivic decomposition given in [16, Proposition 5.18 (i)] implies the following decomposition on Chow groups for any integer k

$$\mathrm{Ch}^k(\mathcal{R}_p(G)_L) \simeq \bigoplus_{i=0}^{p-1} \mathrm{Ch}^{k-i(p+1)}(\mathcal{R}_p(G_L)). \tag{5.4}$$

In particular, one has $\mathrm{Ch}^k(\mathcal{R}_p(G)_L) = 0$ for $k = 2$ or 3 and the conclusion follows.

For G of type E_8, we now prove that $\mathrm{Ch}(\mathcal{R}_5(G)) \longrightarrow \mathrm{Ch}(\mathcal{R}_5(G)_L)$ is surjective in codimension 4 and 5 by showing that one has $\mathrm{Ch}^4(\mathcal{R}_5(G)_L) = \mathrm{Ch}^5(\mathcal{R}_5(G)_L) = 0$ for any extension L/F. By Lemma 4.5, this is true when $J_p(G_L) = J_p(G)$. Moreover, if $J_p(G_L) = 0$ then one has $R_5(G_L) = \mathbb{Z}/5\mathbb{Z}$ and the isomorphism (5.4) implies that $\mathrm{Ch}^4(\mathcal{R}_5(G)_L) = \mathrm{Ch}^5(\mathcal{R}_5(G)_L) = 0$. That completes the proof of Proposition 5.2. □

Finally, using the same notation as in the statement of Theorem 1.1, we want to prove the second conclusion of Theorem 1.1. Since for any generic point ζ of Y, one has

$$1 \notin \deg \mathrm{Ch}_0(X_{F(\zeta)}) \Rightarrow J_p(G_{F(\zeta)}) = (1),$$

by Proposition 3.1 and in view of what has already been done, it is sufficient to prove the following lemma to get the second conclusion.

Lemma 5.5. *Let G be a linear algebraic group of type F_4 or E_8 over a field F such that $J_p(G)$ is nontrivial, with $p = 3$ if G is of type F_4 and $p = 5$ if G is of type E_8. Then one has*

$$\mathrm{Ch}^{p+1}(\mathcal{R}_p(G)) = 0.$$

Proof. Thanks to Proposition 2.3, one can prove the lemma by proceeding in exactly the same way Lemma 4.5 has been proved. □

This concludes the proof of Theorem 1.1.

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