A Note on Defensive Alliances in Graphs

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Abstract

A *defensive alliance* in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ satisfying the condition that for every vertex $v \in S$, the number of neighbors v has in S plus one (counting v) is at least as large as the number of neighbors it has in $V - S$. Because of such an alliance, the vertices in S, agreeing to mutually support each other, have the strength of numbers to be able to defend themselves from the vertices in $V - S$. We prove two conjectures posed by Hedetniemi, Hedetniemi, and Kristiansen in their introductory paper on alliances.

1 Introduction

Alliances in graphs were introduced by Hedetniemi, Hedetniemi, and Kristiansen in [2]. They defined several types of alliances, including the defensive alliances that we consider here. In a graph $G = (V, E)$, a non-empty

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set of vertices $S \subseteq V$ is called a *defensive alliance* if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. In this case, we say that every vertex in S is *defended* from possible attack by vertices in $V - S$. A defensive alliance S is called *strong* if for every vertex $v \in S$, $|N[v] \cap S| > |N(v) \cap V - S|$. In this case we say that every vertex $v \in S$ is *strongly defended.*

In this paper, any reference to an alliance will mean a defensive alliance. Any two vertices u, v in an (strong) alliance S are called *allies* (*with respect to* S); we also say that u and v are *allied.* An (strong) alliance S is called *critical* if no proper subset of S is an (strong) alliance. The *alliance number* $a(G)$ is the minimum cardinality of any critical alliance of G, and the *strong alliance number* $\hat{a}(G)$ is the minimum cardinality of any critical strong alliance of G. For other graph theory terminology and notation, we follow [1].

The following observation was made in [2].

Observation 1 [2] *For the complete graph* K_n, $a(K_n) = \lceil n/2 \rceil$ *and* $\hat{a}(K_n) = \lfloor n/2 \rfloor + 1$.

This observation suggested the following two conjectures:

Conjecture 1 [2] *For any graph* G *of order* n, $a(G) \leq \lceil n/2 \rceil$.

Conjecture 2 [2] *For any graph* G *of order* n, $\hat{a}(G) \leq \lfloor n/2 \rfloor + 1$.

In this note we prove both of these conjectures.

2 Proofs to Conjectures

Let an AB-edge be an edge between a vertex in a set A and a vertex in a set B.

Theorem 2 *If* G *is a connected graph, then* $a(G) \leq \lceil n/2 \rceil$, *and this bound is sharp.*

Proof. Let $\pi = (A, B)$ be a *balanced bi-partition* of $V(G)$, i.e., $|A| = \lceil n/2 \rceil$ and $|B| = \lfloor n/2 \rfloor$, such that the number of AB-edges is a minimum among all such bi-partitions. If either $\langle A \rangle$ or $\langle B \rangle$ is an alliance, then $a(G) \leq$

$\lceil n/2 \rceil$. Hence, assume that neither A nor B is an alliance. Thus, there exist undefended vertices $a \in A$ and $b \in B$ such that $|N[a] \cap A| < |N(a) \cap B|$ and $|N[b] \cap B| < |N(b) \cap A|$. But then $\pi' = (A', B')$, where $A' = (A - \{a\}) \cup \{b\}$ and $B' = (B - \{b\}) \cup \{a\}$, is a balanced bi-partition with fewer $A'B'$-edges than the number of AB-edges of π, contradicting our choice of π. Observation 1 shows that this bound is obtained by complete graphs. □

A polynomial algorithm for constructing an alliance of cardinality at most $\lceil \frac{n}{2} \rceil$ follows directly from the proof of Theorem 2.

Corollary 3 *For any connected graph G, there exists an $O(mn)$ algorithm for finding a defensive alliance of cardinality at most $\lceil \frac{n}{2} \rceil$.*

Proof. Let $\pi = \{A, B\}$ be any balanced bi-partition of $V(G)$.

While neither A nor B is a defensive alliance do

let $a \in A$ satisfy $|N[a] \cap A| < |N(a) \cap B|$;

let $b \in B$ satisfy $|N[b] \cap B| < |N(b) \cap A|$;

let $A = (A - \{a\}) \cup \{b\}$;

let $B = (B - \{b\}) \cup \{a\}$

endwhile

Every iteration of this while-loop decreases the number of edges between A and B. Therefore, there can be at most $O(m)$ such iterations. Each such iteration will involve identifying undefended vertices $a \in A$ and $b \in B$, and looking at every vertex in $N[a]$ and every vertex in $N[b]$. Thus, each iteration takes at most $O(n)$ time. This gives an $O(mn)$ algorithm. □

Next we prove Conjecture 2.

Theorem 4 *If G is a connected graph, then $\hat{a}(G) \leq \lfloor n/2 \rfloor + 1$, and this bound is sharp.*

Proof. Let $\pi = (A, B)$ be a *2-balanced bi-partition* of $V(G)$, that is, $|A| \geq |B|$ and $|A| - |B| \leq 2$, and let the number of AB-edges be a minimum among all 2-balanced bi-partitions. If A is a strong alliance, then $\hat{a}(G) \leq \lfloor n/2 \rfloor + 1$, so assume that A is not a strong alliance. This means that there exists an undefended vertex $a \in A$, where $|N[a] \cap A| \leq |N(a) \cap B|$, that is, a has strictly more neighbors in B than it has in A. But this means that $\pi' = \{A', B'\}$, where $A' = A - \{a\}$ and $B' = B \cup \{a\}$ is a 2-balanced bi-partition having fewer $A'B'$-edges than the number of AB-edges of π, contradicting our choice of π. Thus, A is a strong alliance, and $\hat{a}(G) \leq |A| \leq \lfloor n/2 \rfloor + 1$. Again Observation 1 illustrates that this bound is sharp. □

Note that if $\pi = (A, B)$ is a 2-balanced bi-partition of G with the minimum number of AB edges and $|A| = |B|$, then both A and B are strong alliances. Also if $|A| = |B| + 1$, then A, $A - \{v\}$, B, and $B \cup \{v\}$ are strong alliances. Hence, we have the following corollaries.

Corollary 5 *If G has a bi-partition $\pi = (A, B)$, where $|A| = |B|$ and π has a minimum number of AB-edges among all 2-balanced bi-partitions, then $|V|$ can be partitioned into two disjoint strong alliances, namely, A and B.*

Corollary 6 *If G has a bi-partition $\pi = (A, B)$, where $|A| = |B| + 1$ and π has a minimum number of AB-edges among all 2-balanced bi-partitions, and there exists a vertex $v \in A$ such that $N(v) \cap A = N(v) \cap B$, then $\hat{a}(G) \leq \lfloor n/2 \rfloor$.*

We conclude this note with another corollary to Theorem 4.

Corollary 7 *For any connected graph G, there exists an $O(mn)$ algorithm for finding a strong alliance of cardinality at most $\lfloor n/2 \rfloor + 1$.*

Proof. Begin with any 2-balanced bipartition $\pi = (A, B)$, where $|A| = \lfloor n/2 \rfloor + 1$. If A is a strong alliance, we are finished; otherwise, find an undefended vertex $a \in A$, as in the proof of Theorem 4, and move it to B. In doing so, the number of edges between A and B will decrease by at least one (and maybe more).

Now if B is a strong alliance, we are finished. If not, find an undefended vertex $b \in B$ and move it to A. In doing so, the number of edges between A and B will again decrease, by at least one.

Continue in this fashion, alternating between moving a vertex from A to B, and then from B to A. This process must terminate, because the number of edges between A and B decreases with every move.

Ultimately, either the set A or the set B will be a strong alliance of cardinality at most $\lfloor n/2 \rfloor + 1$.

Since, initially, there can be at most $O(m)$ edges between A and B, and since it takes at most $O(n)$ time to find an undefended vertex, either $a \in A$ or $b \in B$, each of at most $O(m)$ iterations can be carried out in at most $O(n)$ time. Thus, a strong alliance of cardinality at most $\lfloor n/2 \rfloor + 1$ can be found in at most $O(mn)$ time. □

It remains an open problem whether there exists an $O(n^2)$ algorithm for finding an alliance of cardinality at most $\lceil n/2 \rceil$, or a strong alliance of cardinality at most $\lfloor n/2 \rfloor + 1$.

References

[1] G. Chartrand and L. Lesniak, *Graphs & Digraphs: Third Edition.* Chapman & Hall, London (1996).

[2] S. M. Hedetniemi, S. T. Hedetniemi, and P. Kristiansen, Alliances in graphs. Submitted for publication, October, 2001.