An Anti-Ramsey Theorem on Posets

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Abstract

It is known that if P and Q are posets and $*$ is lexicographic product, then (in the Erdős-Rado partition notation), $P{*}Q \to (P, Q)$. It is known that if S and T are trees of rank at most ω, and "$\times$" is Cartesian product, then $S \times T \to (S, T)$.

In this article we exhibit pairs of finite posets P and Q such that $P \times Q \not\to (P, Q)$. In particular, we prove that if B_n is the poset of the power set on n elements, then for each integer $\alpha > 1$, there exists N such that $n > N$ implies $B_{n+\alpha} \not\to (B_n, B_\alpha)$; indeed, we can choose n such that $B_{n+\alpha} \not\to (B_n, B_2)$. We conclude by looking at a few positive results.

1 Introduction

Partially ordered sets seem to be natural objects for Ramseyian investigations. In fact, if one includes the orderings of the real numbers and integers as posets, posets were subjects of Ramsey theory before Ramsey: e.g., the negative result of F. Bernstein ([1], see, e.g., [9, §40.1]) and the later positive result of B. L. van der Waerden ([15], see, e.g., [7, §2.1]). But ever since the seminal paper of Erdős and Szekeres ([4]), most Ramsey theory has been on graphs.

Nevertheless, there are several Ramseyian threads in posets. Perhaps the most important arose from the category-theoretic work of J. Nešetřil & V. Rődl ([12], see also [13]), which generalized the finite Ramsey theorem to posets of given "dimension." In such papers, the authors took a poset P

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and a poset H, and colored all the copies of H in P either green or orange, and looked for monochromatic subposets of P isomorphic to a poset Q.

For example, if we just colored vertices of a poset P, looking for monochromatic copies of a poset Q, we might get some poset Ramsey functions like those exhibited in [5] and [14]. (Indeed, the negative result in this paper will lead to a poset Ramsey function.) Let's first be clear about what our notation and terms mean.

In this paper, we use to the following notation.

Convention 1.1 *Suppose that $\mathfrak{P}$ is a poset, which we regard as a pair $\langle P, <^P \rangle$: P is the set of vertices and $<^P$ is the order relation.*

For simplicity, we will simply refer to the "poset" P directly, taking $<^P$ for granted, and write $p \in P$ for a vertex p of P. A subposet *Q of P will be identified with a set $Q \subseteq P$, where $<^Q = \{(x,y) \in Q^2 \colon x <^P y\}$. Where there is no ambiguity, we will use "$<$" instead of "$<^P$."*

We will need some basic definitions.

Definition 1.1 *Let P and Q be posets. A subposet $R \subseteq P$ is a* copy *of Q if there exists an isomorphism $\pi \colon Q \to R$. In this case, we say that π is an* embedding *of Q into P.*

We may think of a partition $P = G \cup O$ as a *coloring* of P, in which each element of P is colored *G*reen or *O*range. We will then be interested in a green copy of a poset Q or an orange copy of a poset R.

Definition 1.2 *Let P, Q, R be any posets. Then "$P \to (Q, R)$" means the following. For any partition of the elements of $P = G \cup O$, either there is a copy of Q in G or there is a copy of R in O.*

If we imagine that we can color the vertices of P green and orange, so that G is the subposet of green vertices and O is the subposet of orange vertices, then "$P \to (O, R)$" means that for any such coloring, either there is a green copy of O or an orange copy of R.

Perhaps the most basic result for this situation is from H. A. Kierstead & W. T. Trotter ([8]). If P and Q are posets, then the *lexicographic product* "$P * Q$" of P and Q is the poset of vertices $\{(p,q) \colon p \in P \;\&\; q \in Q\}$ with the ordering $(p,q) < (p',q')$ iff $p <^P p'$, or $p = p'$ and $q <^Q q'$.

Proposition 1.1 ([8]) *Let P and Q be any posets. Let $*$ be lexicographic product. Then $P * Q \to (P, Q)$.*

In [10], the less tractible Cartesian product was examined.

Definition 1.3 *Let P, Q be partial orders. Then the* cartesian product *of P and Q is the poset $P \times Q$ of elements (p, q), $p \in P$ and $q \in Q$, where $(p, q) \leq (p', q')$ iff $p \leq p'$ & $q \leq q'$.*

When there is any possibility of confusion, we will use the notations $<^P$, $<^Q$, and $<^{P \times Q}$ to distinguish the three order relations.

The primary result of [10] was:

Proposition 1.2 ([6] and [10]) *Let S, T be trees with no vertices of infinite rank. Then $S \times T \to (S, T)$. However, there exist trees S, T with vertices of infinite rank such that $S \times T \not\to (S, T)$.*

This suggested the following not unreasonable conjecture:

Conjecture 1.1 *If P and Q are finite posets, then $P \times Q \to (P, Q)$.*

But as we shall see, this conjecture is false.

Definition 1.4 *Here is some basic notation that we will use.*

- *For any integer α, let $(\alpha)_0 = 1$ and, if n is a positive integer, let $(\alpha)_n = \prod_{k=0}^{n-1}(\alpha - k)$.*
- *For any positive integer n, $[n] = \{1, 2, \ldots, n\}$.*
- *Given sets A and B, let $A - B = \{x \in A : x \notin B\}$.*
- *Given a set A, let $\mathcal{P}(A)$ be the power set of A.*
- *For any finite set A, let $|A|$ be the cardinality of A.*
- *If A is a set and n is a nonnegative integer, then $\binom{A}{n} = \{X \subseteq A : |X| = n\}$.*
- *If $f : A \to B$ is a function, and $X \subseteq A$, let $f[X] = \{f(x) : x \in X\}$.*

We will be dealing with boolean algebras as posets.

Definition 1.5 *For each n, let $B_n = \langle \mathcal{P}([n]), \subset \rangle$, i.e., the power set of $[n]$ ordered by inclusion. Again, following Convention 1.1, we treat B_n as the set $\mathcal{P}([n])$ and as the partially ordered set $\langle \mathcal{P}([n]), \subset \rangle$.*

Note that if "$\cong$" refers to poset isomorphism, then for any positive integers m and n, $B_m \times B_n \cong B_{m+n}$. Here are some more poset definitions.

Definition 1.6 *Let P be a poset.*

- *If an element p satisfies "for all $p' \in P$, $p \leq p'$," then p is the* bottom *of P.*

- *A subposet Q of a poset P is* upwards closed *in P if, for any $p \in P$ and $q \in Q$, $q \leq p \Longrightarrow p \in Q$.*

- *The* rank *of a minimal element p is* $\mathrm{rank}(p) = 0$*; if p is not minimal,* $\mathrm{rank}(p) = \sup\{\mathrm{rank}(p') \colon p' < p\} + 1$.

- *The* height *of a poset P is* $\sup\{\mathrm{rank}(p) \colon p \in P\}$.

- *A* tree *is a poset P that has a bottom but no copies of B_2 as subposets.*

Notice that in $\mathcal{P}(A)$, the rank of any $X \subseteq A$ is $|X|$.

Now we turn to Conjecture 1.1. Again, observe that $B_{n+1} \cong B_n \times B_1$.

Proposition 1.3 *For each n, $B_{n+1} \to (B_n, B_1)$.*

Proof. Let $\mathcal{P}([n+1]) = G \cup O$ be a partition of $\mathcal{P}([n+1])$ into green and orange vertices. Suppose that O is an antichain in B_{n+1}; we claim that there exists a copy of B_n in G.

We will construct an embedding τ of B_n into G by a recursion, on the cardinality (hence rank) of the sets in $\mathcal{P}([n])$, as follows.

First, if $\varnothing \in O$, then as O is an antichain, $A \neq \varnothing$ implies $A \in G$, and $\{X \cup \{n+1\} \colon X \subseteq [n]\}$ is our green copy of B_n. So, without loss of generality, we can assume that $\varnothing \in G$. Let $\tau(\varnothing) = \varnothing$.

Construct the function $\tau \colon \mathcal{P}([n]) \to G$ by the following recursion on the rank. Suppose that τ has been defined on all sets in $\mathcal{P}([n])$ of cardinality at most m, so that for each $X \subseteq [n]$ such that $|X| \leq m$, we have:

- either $\tau(X) = X$ or $\tau(X) = X \cup \{n+1\}$, and

- if $|X| < m$ and $\tau(X) = X$, then for any $y \in [n] - X$, $\tau(X \cup \{y\}) = X \cup \{y\}$ iff $X \cup \{y\} \in G$.

- if $|X| < m$ and $\tau(X) = X \cup \{n+1\}$, then for any $y \in [n] - X$, $X \cup \{y, n+1\} \in G$ and $\tau(X \cup \{y\}) = X \cup \{y, n+1\}$. Furthermore, for some $Y \subseteq X$, $Y \in O$.

Let $|X| = m+1$. There are two cases.

Case 1. Suppose that for each $x \in X$, $\tau(X - \{x\}) = X - \{x\}$. In this case, two things can happen.

1. If $X \in G$, set $\tau(X) = X$.

2. If $X \in O$, then as the orange vertices form an antichain, $X \cup \{n+1\} \in G$, so set $\tau(X) = X \cup \{n+1\}$.

Case 2. Suppose that for some $x \in X$, $\tau(X-\{x\}) = (X-\{x\}) \cup \{n+1\}$. Then for some $Y \subseteq X - \{x\}$, we have $Y \in O$ and $\tau(Y) = Y \cup \{n+1\}$. As $Y \subseteq X \cup \{n+1\}$, and as O is an antichain, $X \cup \{n+1\} \in G$, and we set $\tau(X) = X \cup \{n+1\}$.

Repeat until $\tau([n])$ is defined. By construction, τ is an embedding, and we are done. ■

The next question is this: if $m, n > 1$, do we necessarily have $B_{m+n} \to (B_m, B_n)$?

Note that this would be the best we could hope for. If $M < m+n$, then we could take B_M and let $G = \{X \subseteq [M]\colon |X| < m\}$ and $O = B_M - G$, and G is too short to accomodate a copy of B_m while O is too short to accomodate a copy of B_n. Thus $B_M \not\to (B_m, B_n)$.

The industrious reader can verify that $B_4 \to (B_2, B_2)$, $B_5 \to (B_3, B_2)$, and $B_6 \to (B_4, B_2)$. The question is how far does this sequence of partition relations hold? In this paper we will find that $B_{17037} \not\to (B_{17035}, B_2)$.

The main result of this paper will give us: *For any $\alpha > 1$, there exists N such that $n \geq N$ implies that $B_{n+\alpha} \not\to (B_n, B_\alpha)$.* The next three sections present lemmas for the main result, which occupies Section 5. Then Section 6 will deal with directions for further research, including the advertized new poset (actually, boolean algebra) Ramseyian function.

2 Pre-Images

We start by looking at some special homomorphic pre-images of posets.

Definition 2.1 *Let P and Q be posets, and let S be a subposet of $P \times Q$. Then S is a* canonical pre-image *of Q in $P \times Q$ if:*

- *For each $q \in Q$, there exists $p \in P$ such that $(p, q) \in S$.*
- *For each $(p, q) \in S$ and $q' \in Q$, if $q < q'$, then there exists $p' \in P$, $p' \geq p$, such that $(p', q') \in S$.*

We can similarly define a canonical pre-image *R of P in $P \times Q$.*

Note that if R is a canonical pre-image of P in $P \times Q$, then the map $R \to P$: $(p, q) \mapsto p$ is an onto homomorphism. Now for a critical fact.

Lemma 2.1 *Let P and Q be posets with bottoms, but at least one of P, Q has no infinite chains. If a poset R is a canonical pre-image of P in $P \times Q$ and if S is a canonical pre-image of Q in $P \times Q$, then $R \cap S \neq \varnothing$.*

Proof. Towards contradiction, suppose that R is a canonical pre-image of P and S is a canonical pre-image of Q, but that $R \cap S = \varnothing$. We will prove that $P \times Q$ admits an infinite chain, with a construction that will produce infinite chains in both P and Q.

We use the following recursion.

The basis of the recursion uses the bottoms p_0 and q_0 of P and Q, respectively. As $(p_0, q_0) \notin R \cap S$, either:

- We have $(p_0, q_0) \notin R$, in which case, as R is a canonical pre-image of P (and q_0 is the bottom of Q), there exists $q_1 > q_0$ such that $(p_0, q_1) \in R$; denote $\vec{r}_0 = (p_0, q_1) \in P \times Q$. As S is a canonical pre-image of Q, there exists a p_1 such that $(p_1, q_1) \in S$; note that $p_1 > p_0$ and denote $\vec{r}_1 = (p_1, q_1)$. Note that $\vec{r}_0 <^{P \times Q} \vec{r}_1$. Or:

- We have $\vec{r}_0 = (p_0, q_0) \in R$, hence $(p_0, q_0) \notin S$, in which case, as S is a canonical pre-image of Q, there exists $p_1 > p_0$ such that $(p_1, q_0) \in S$. Choosing a minimal such p_1, let $\vec{r}_1 = (p_1, q_0) \in P \times Q$. Note that $\vec{r}_0 <^{P \times Q} \vec{r}_1$.

The recursion works as follows. Suppose that we have an upwards chain $\vec{r}_0, \vec{r}_1, \vec{r}_2, \ldots \vec{r}_m$, where, for each $k < m$, $\vec{r}_k \in R$ iff $\vec{r}_{k+1} \in S$ (iff $\vec{r}_{k-1} \in S$ for $k > 0$). For each $k < m$, letting $\vec{r}_k = (p, q)$ and $\vec{r}_{k+1} = (p', q')$, we have:

1. if $\vec{r}_k \in R$, then $p <^P p'$ and $q = q'$, and

2. if $\vec{r}_k \in S$, then $q <^Q q'$ and $p = p'$

Without loss of generality, suppose that $\vec{r}_{m-1} \in S$ and $\vec{r}_m \in R$; we will obtain $\vec{r}_{m+1} > \vec{r}_m$ such that $\vec{r}_{m+1} \in S$, satisfying (1) above for $k = m$. Let $\vec{r}_{m-1} = (p, q)$ and $\vec{r}_m = (p, q')$. As $\vec{r}_{m-1} \in S$ and $q <^Q q'$, by Definition 2.1, there exists $p' \in P, p \leq^P p'$, such that $(p', q') \in S$. As $(p, q') \in R$, we cannot have $p' = p$, so we must have $p <^P p'$. Choose a minimal such p', and let $\vec{r}_{m+1} = (p', q') \in S$. The construction if $\vec{r}_m \in S$ (using (2) above) is similar.

As the construction continues ad infinitum, $P \times Q$ admits an infinite upwards chain (skipping every other pair) $(p_0, q_0) < (p_1, q_1) < (p_2, q_2) < \cdots$, and thus P has an infinite upwards chain $p_0 < p_1 < p_2 < \cdots$ while Q has an infinite upwards chain $q_0 < q_1 < q_2 < \cdots$. ■

3 Power Sets

Since we are working on finite boolean algebras, we want a definition that will specialize the notion of canonical pre-images to boolean posets.

Definition 3.1 *A homomorphism* π: $B_n \to B_m$ *is* faithful *if there is a one-to-one map* τ: $[n] \to [m]$ *such that for each* $a \in [n]$ *and each* $A \subseteq [n]$,

$a \in A$ *iff* $\tau(a) \in \pi(A)$, *i.e.*, $\tau[[n]] \cap \pi(A) = \tau[A]$. *Say that* τ witnesses *the faithfulness of* π.

This notion will allow us to regard copies of B_n inside of B_m as canonical pre-images. To see this, note that for any n-set $C \subseteq [m]$,

$$B_m \cong \langle \mathcal{P}(C), \subset \rangle \times \langle \mathcal{P}([m] - C), \subset \rangle \cong B_n \times B_{m-n}.$$

We will first prove that embeddings are faithful, and then that faithful embeddings produce canonical pre-images.

Lemma 3.1 *If* $\pi \colon B_n \to B_m$ *is an embedding, then it is faithful.*

Proof. We need a function $\tau \colon [n] \to [m]$ to witness π's faithfulness: we claim that any function of the following sort works. As π is an embedding, for each $x \in [n]$, $\pi(\{x\}) \not\subseteq \pi([n] - \{x\})$; for each $x \in [n]$, let $\tau(x) \in \pi(\{x\}) - \pi([n] - \{x\})$.

First, we claim that τ is one-to-one. If $\tau(x) = \tau(y)$, then $\tau(x) \in \pi(\{y\}) - \pi([n] - \{y\})$, so $\tau(y) \notin \pi([n] - \{y\})$. As $z \neq y \Longrightarrow \pi(\{z\}) \subseteq \pi([n] - \{y\})$, this means that $z \neq y \Longrightarrow \tau(y) \notin \pi(\{z\})$. Thus as $\tau(y) = \tau(x) \in \pi(\{x\})$, we must have $x = y$.

We now verify that τ witnesses the faithfulness of π. Let $a \in [n]$ and $A \subseteq [n]$. If $a \in A$, then $\tau(a) \in \pi(\{a\}) \subseteq \pi(A)$. And if $a \notin A$, then as $\tau(a) \notin \pi([n] - \{a\}) \supseteq \pi(A)$, $\tau(a) \notin \pi(A)$. ■

Lemma 3.2 *Let* $\pi \colon B_n \to B_m$ *be a faithful embedding, as witnessed by* $\tau \colon [n] \to [m]$. *Let* $\lambda \colon B_m \to \mathcal{P}(\tau[n]) \times \mathcal{P}([m] - \tau[n])$ *be defined by:*

$$\lambda(X) = (X \cap \tau[n], X \cap ([m] - \tau[n])).$$

Then $(\lambda \circ \pi)[B_n]$ *is a canonical pre-image of* $\mathcal{P}(\tau[n])$.

Notice that as λ is an isomorphism, $(\lambda \circ \pi)[B_n]$ is a copy of B_n.

Proof. First, as λ and π are one-to-one, so is $\lambda \circ \pi$, so as both are isomorphic on their domains and images, so is $\lambda \circ \pi$. Thus $\pi^{-1} \circ \lambda^{-1} \colon (\lambda \circ \pi)[B_n] \to B_n$ is an isomorphism, hence a homomorphism. So $(\lambda \circ \pi)[B_n]$ is a pre-image of a homomorphism onto B_n; we claim that it is a canonical pre-image.

Before continuing, we show that for each $E \subseteq \tau[n]$, $E = (\pi \circ \tau^{-1})[E] \cap \tau[n]$. To see this, note that by faithfulness, $a \in \tau^{-1}[E] \Longrightarrow a \in [n] \Longrightarrow \tau(a) \in \pi(\{a\})$, and repeating for each $a \in \tau^{-1}[E]$, we have $E = \tau[\tau^{-1}[E]] \subseteq \pi(\tau^{-1}[E]) = (\pi \circ \tau^{-1})[E]$. And by faithfulness again, for any $b \in \tau[n]$, $b \in (\pi \circ \tau^{-1})[E]$ implies that $b = \pi(\tau^{-1}(e))$ for some $e \in E$, which implies that $b = e$ as $b \in \tau[n] \Longrightarrow b \in E$. Thus $\tau[n] \cap (\pi \circ \tau^{-1})[E] \subseteq E$.

Now, we claim that $\pi^{-1} \circ \lambda^{-1}$ witnesses the fact that $(\lambda \circ \pi)[B_n]$ is a canonical pre-image of $\mathcal{P}(\tau[n])$. This requires verifying both conditions of Definition 2.1.

- If $E \subseteq \tau[n]$, then $\tau^{-1}[E] \subseteq [n]$. Let $C = ([m] - \tau[n]) \cap \pi(\tau^{-1}[E]) = \pi(\tau^{-1}[E]) - E$. Then $\pi(\tau^{-1}[E]) = (\pi(\tau^{-1}[E]) \cap \tau[n]) \cup (\pi(\tau^{-1}[E]) \cap ([m] - \tau[n])) = E \cup C$. Thus if $E \subseteq \tau[n]$, there exists C such that $(E, C) \in (\lambda \circ \pi)[B_n]$, and $(\pi^{-1} \circ \lambda^{-1})(E, C) = \tau^{-1}[E]$.
- If $(E, C) \in (\lambda \circ \pi)[B_n]$, and if $E \subsetneq E' \subseteq \tau[n]$, we can let $C' = ([m] - \tau[n]) \cap \pi(\tau^{-1}[E']) = \pi(\tau^{-1}[E']) - E'$, and as π and λ are partial isomorphisms, $C \subseteq C'$. Thus $(E, C) < (E', C') \in (\lambda \circ \pi)[B_n]$.

Since both conditions of Definition 2.1 are satisfied, we are done. ■

Thus an embedded image of B_n in B_m can be treated as a canonical pre-image of B_n in B_m. So we conclude by translating Definition 2.1 via Definition 3.1 into a corresponding definition for these posets of boolean algebras.

Definition 3.2 *Let $Q \subseteq B_m$ be a pre-image of B_n under a map κ: $Q \to B_n$. Let $S \subseteq [m]$ have n elements. Then κ is* canonical with respect to S *if:*

- *There is a one-to-one map τ: $S \to [n]$ such that if $A \subseteq S$ and $V \subseteq [m] - S$ and $A \cup V \in Q$, then $\kappa(A \cup V) = \tau[A]$.*
- *For every $A \subseteq S$, there exists $V \subseteq [m] - S$ such that $A \cup V \in Q$.*
- *If $A \subsetneq S$ and $V \subseteq [n] - S$ and $A \cup V \in Q$, then for any A' such that $A \subsetneq A' \subseteq S$, there exists V' such that $V \subseteq V' \subseteq [n] - S$ and $A' \cup V' \in Q$.*

4 A Matching Lemma

Before we construct the counterexample, we need a lemma.

Lemma 4.1 *Fix an integer $\alpha \geq 4$ and an integer $n > 0$ such that $n > 8\alpha!\alpha^3 \ln \alpha$. There is a function σ: $\binom{[n]}{\alpha} \to \mathcal{P}([n])$ such that:*

1. *For each $S \in \binom{[n]}{\alpha}$, $\sigma(S) \cap S = \varnothing$, and*
2. *For each $S, S' \in \binom{[n]}{\alpha}$, $S \neq S'$ implies that $|\sigma(S) - \sigma(S')| \geq \alpha!\alpha$.*

Proof. By Hall's Matching or Marriage Theorem (see, e.g., [2, Theorem 5.1.5]), it suffices to construct a family $\mathcal{F}$ of subsets of $[n]$ such that the following hold:

1. For each $F, F' \in \mathcal{F}$, $|F| = |F'|$.

2. For each $F, F' \in \mathcal{F}$, $F \neq F' \implies |F - F'| \geq \alpha!\alpha$.

3. For each α-set $S \subseteq [n]$, there are at least $\binom{n}{\alpha}$ sets $F \in \mathcal{F}$ such that $F \cap S = \varnothing$.

We construct such a family $\mathcal{F}$.

Choose the maximal m such that:

- First, $m \leq n$, and
- Second, m is a multiple of $\alpha!\alpha$ (so that $m/(\alpha!\alpha)$ is an integer), and
- Third, $(m/(\alpha!\alpha)) - \alpha$ is even.

Note that $\frac{7}{8}n \leq n - 2\alpha!\alpha \leq m$ as $\alpha \geq 4$ (and hence $n > 17{,}034$). Thus $m \geq 7\alpha!\alpha^3 \ln \alpha$.

For the rest of this proof, let $M = m/(\alpha!/\alpha)$.

For each $j \in [M]$, let $A_j = \{(j-1)\alpha!\alpha + 1, \ldots, j\alpha!\alpha\}$. Then let

$$\mathcal{F} = \left\{ \bigcup_{j \in B} A_j \colon B \in \binom{[M]}{(M-\alpha)/2} \right\}.$$

Then:

1. If $F \in \mathcal{F}$, then $|F| = \frac{1}{2}\alpha!\alpha(M - \alpha) = (m - \alpha!\alpha^2)/2$. Thus $F, F' \in \mathcal{F} \implies |F| = |F'|$.

2. By construction, if $F, F' \in \mathcal{F}$, and $F \neq F'$, then there exists j such that $A_j \subseteq F - F'$, and hence $|F - F'| \geq \alpha!\alpha$.

3. It remains to prove that for each α-set $S \subseteq [n]$, there are at least $\binom{n}{\alpha}$ sets $F \in \mathcal{F}$ such that $F \cap S = \varnothing$.

First of all, we observe that any α-set S can intersect at most α sets A_j. Thus each $S \in \binom{[m]}{\alpha}$ satisfies

$$|\{F \in \mathcal{F} \colon S \cap F = \varnothing\}| \geq \binom{M - \alpha}{(M-\alpha)/2} = \frac{(M-\alpha)_{(M-\alpha)/2}}{((M-\alpha)/2)!},$$

and hence

$$|\{F \in \mathcal{F} \colon S \cap F = \varnothing\}| > 2^{(M-\alpha)/2} 2^{\alpha}. \tag{1}$$

The last factor 2^α is possible because $l \leq \alpha$ implies that

$$\frac{1}{2}(M - \alpha) + l > 4l \qquad \text{as} \qquad M > 7\alpha,$$

and thus in the quotient in Formula (1), the last α factors in the numerator are at least four times as large as the corresponding factors in the denominator.

So it suffices to verify that $2^{(M-\alpha)/2}2^{\alpha} \geq \binom{n}{\alpha}$, i.e., that $2^{(M-\alpha)/2} \geq 2^{-\alpha}\binom{n}{\alpha}$. We prove this as follows.

We will need the following inequality:

$$\frac{m}{\ln m} > \frac{2\alpha^2\alpha!}{\ln 2}. \tag{2}$$

To prove this, we let $\beta = m/(\alpha!\alpha^3 \ln\alpha)$, so that $\beta > 7$, and thus $\beta \ln 2 > 4.8$. It follows that $\beta^{-1}\ln\beta < \frac{1}{2.4}\ln 2$ (note that $(\ln 7)/7 < \frac{1}{2.4}\ln 2$, and that the function $x \mapsto (\ln x)/x$ is decreasing for $x > e$). Thus $\ln\beta < \frac{\beta}{2.4}\ln 2$.

Note that as $\alpha \geq 4$, $\alpha\ln\alpha > 5.5$. In addition, as $\beta > 7$, $\alpha\beta\ln 2 > 19$. To prove Inequality (2), we start by bounding $\ln m = \ln\beta + 3\ln\alpha + \ln(\alpha!) + \ln\ln\alpha$. Noting that $\alpha! < \alpha^{\alpha}$, we compute:

$$\begin{aligned}
\ln m &< \ln\beta + 3\ln\alpha + \alpha\ln\alpha + \ln\ln\alpha \\
&< \frac{\beta}{2.4}\ln 2 + 4\ln\alpha + \alpha\ln\alpha \\
&< \frac{\beta}{2.4}\ln 2\frac{\alpha\ln\alpha}{5.5} + 4\frac{\alpha\beta}{19}\ln 2\ln\alpha + \frac{\beta\ln 2}{4.8}\alpha\ln\alpha \\
&= \left(\frac{1}{2.4\times 5.5} + \frac{4}{19} + \frac{1}{4.8}\right)\beta\alpha\ln\alpha\ln 2 \\
&< \frac{1}{2}\alpha\beta\ln 2\ln\alpha.
\end{aligned}$$

Now that we have this bound, we compute

$$\begin{aligned}
\frac{m}{\ln m} &= \frac{\beta\alpha^3\alpha!\ln\alpha}{\ln\beta + 3\ln\alpha + \ln(\alpha!) + \ln\ln\alpha} \\
&> \frac{\beta\alpha^3\alpha!\ln\alpha}{\frac{1}{2}\alpha\beta\ln\alpha\ln 2} \\
&= 2\alpha^2\alpha!/\ln 2,
\end{aligned}$$

proving Inequality (2). Thus $M\ln 2 > 2\alpha\ln m$.

Observe that if $\alpha \geq 4$, α an integer, then $2^{\alpha} \leq (\alpha!)^2$, and thus $\alpha\ln 2 - 2\ln(\alpha!) \leq 0$. Putting this together with Inequality (2), we get

$$M\ln 2 \geq 2\alpha\ln m + \alpha\ln 2 - 2\ln(\alpha!).$$

Thus

$$2^M \geq m^{2\alpha}2^{\alpha}/(\alpha!)^2,$$

i.e.,

$$2^{M-\alpha} \geq \left(\frac{m^\alpha}{\alpha!}\right)^2 \geq \left(\frac{(7n/8)^\alpha}{\alpha!}\right)^2 = \left(\frac{7}{8}\right)^{2\alpha} \left(\frac{n^\alpha}{\alpha!}\right)^2,$$

and hence,

$$2^{(M-\alpha)/2} \geq 2^{-\alpha} \binom{n}{\alpha},$$

and we are done. ∎

5 Constructing the Counterexample

Having traversed three sections of lemmas, we are now ready for the main theorem.

Theorem 5.1 *Fix any positive integer α. For any integer $n > 17{,}034$, if $n > 8\alpha!\alpha^3 \ln \alpha$, then $B_{n+\alpha} \not\to (B_n, B_2)$.*

Proof. We first note that it suffices to assume that $\alpha \geq 4$; notice $8 \cdot 4! \cdot 4^4 \cdot \ln 4 < 17{,}035$. From this we will get, among other things, that $B_{n+4} \not\to (B_n, B_2)$, from which it follows that $B_{n+3} \not\to (B_n, B_2)$, $B_{n+2} \not\to (B_n, B_2)$ and $B_{n+1} \not\to (B_n, B_2)$. Hence we can assume that $\alpha \geq 4$ without loss of generality.

For each α-subset $S \subseteq [n+\alpha]$, we will construct a canonical pre-image (in the sense of Definition 3.2) of $\mathcal{P}(S)$ in $B_{n+\alpha}$. The subposet consisting of the union of these pre-images will be orange; all the vertices outside these pre-images will be green. By Lemma 2.1, this will prevent any green canonical pre-image of any $\mathcal{P}([n+\alpha] - S)$ (and hence any green copy of B_n) from existing. In addition, each pre-image will be a tree, and any two vertices from any two of these canonical pre-images will be mutually incomparable (if x and y are vertices from two of these pre-images, then $x \not< y$ and $y \not< x$): this will prevent any orange copy of B_2 from existing.

Since the hypotheses of the theorem (together with $\alpha \geq 4$) satisfy those of Lemma 4.1, there exists a function σ: $\binom{[n+\alpha]}{\alpha} \to \mathcal{P}([n])$ satisfying: for all S, S', $\sigma(S) \cap S = \varnothing$ and $S \neq S' \Longrightarrow |\sigma(S) - \sigma(S')| \geq \alpha!\alpha$.

Let $S \in \binom{[n+\alpha]}{\alpha}$, and we construct the canonical pre-image of $\mathcal{P}(S)$ as follows.

For each $T \subseteq S$, let P_T be the set of one-to-one functions from $[|T|]$ onto T. For each nonnegative integer $k \leq \alpha$, let

$$P_k = \bigcup_{T \in \binom{S}{k}} P_T,$$

and note that $|P_k| = (\alpha)_k$. If $T \subseteq T'$ and $p \in P_T$ and $q \in P_{T'}$, let "$p < q$" mean that q is an extension of p: $x \in [|T|] \Longrightarrow p(x) = q(x)$. If $T \subseteq T'$ and $p \in P_T$ and $q \in P_{T'}$ and yet $p \not< q$, then for some $x \in [|T|]$, $p(x) \neq q(x)$, so p and q have no common extension and are therefore *incomparible.* Note that if $\alpha \geq 4$, then $\sum_{k=0}^{\alpha} |P_k| = \sum_{k=0}^{\alpha} (\alpha)_k < \alpha!\alpha - \alpha$: $\sum_{k=0}^{4} (4)_k = 65 < 92 = 4!4 - 4$, and $\sum_{k=0}^{\alpha} (\alpha)_k < \alpha!\alpha - \alpha$ implies that

$$\begin{aligned}
\sum_{k=0}^{\alpha+1} (\alpha+1)_k &= 1 + \sum_{k=1}^{\alpha+1} (\alpha+1)_k \\
&= 1 + (\alpha+1) \sum_{k=0}^{\alpha} (\alpha)_k \\
&< 1 + (\alpha+1)(\alpha!\alpha - \alpha) \\
&< (\alpha+1)!(\alpha+1) - (\alpha+1).
\end{aligned}$$

For each $S \in \binom{[n+\alpha]}{\alpha}$, we will construct the pre-image of $\mathcal{P}(S)$ as follows. Let $S' \subseteq \sigma(S)$ have $\sum_{k=0}^{\alpha} (\alpha)_k$ elements, and let γ: $S' \to \bigcup_{k=0}^{\alpha} P_k$ be one-to-one. Then let the pre-image of $\mathcal{P}(S)$ consist of all sets $X \subseteq [n+\alpha]$ satisfying the following condition: if $T = X \cap S$, then choosing $s \in S'$ such that $\gamma(s) \in P_T$, we have

$$X = (\sigma(S) - S') \cup T \cup \{s' \in S' \colon \gamma(s') \leq \gamma(s)\}.$$

Call $\gamma(s)$ the *ordering* of X, and note that each such $\gamma(s)$ fixes an ordering of selecting a subset $T \subseteq S$: imagine that $\gamma(s)$ tells us which elements were selected first, and in what order. (Notice that $|\{s' \in S' \colon \gamma(s') \leq \gamma(s)\}| = |T| + 1$.) Let Q_S be the poset of these sets X, ordered by inclusion. Observe that Q_S is defined from S, S', and γ only. We now verify that Q_S is a canonical pre-image of B_α, that it is a tree, and that any two vertices from two such pre-images are incomparable.

Before launching into this verification, we should note that as $S \cap \sigma(S) = \varnothing$, if $X = (\sigma(S) - S') \cup T \cup \{s' \in S' \colon \gamma(s') \leq \gamma(s)\}$, we can unambiguously define $\kappa^*(X) = X - \sigma(S) = T$.

First, this Q_S is a canonical pre-image of B_α. Recalling Definition 3.2, notice that:

- Letting τ: $[\alpha] \to S$ be one-to-one, then we have the onto map $\kappa = \tau^{-1} \circ \kappa^*$: $Q_S \to B_\alpha$ such that for any $(\sigma(S) - S') \cup T \cup U \in Q_S$, $\kappa((\sigma(S) - S') \cup T \cup U) = \tau^{-1}[T]$.

- For any $T \subseteq S$, choose $\gamma(s) \in P_T$ and $V = (\sigma(S) - S') \cup \{s' \in S' \colon \gamma(s') \leq \gamma(s)\}$ gives us $T \cup V \in Q_S$.

- If $(\sigma(S)-S')\cup T\cup U\in Q_S$, where $|T|<\alpha$ and $U=\{s'\in S'\colon \gamma(s')\leq \gamma(s)\}$ for some $s\in\gamma^{-1}[P_T]$, then for any $t\in S-T$, let $s^+\in S'$ be such that $\gamma(s^+)=\gamma(s)\cup\{(|T|+1,t)\}$. Then $(\sigma(S)-S')\cup T\cup U\subseteq (\sigma(S)-S')\cup(T\cup\{t\})\cup(\{U\cup\{s^+\})\in Q_S$.

As Q_S satisfies Definition 3.2, it is a canonical pre-image of B_α.

Second, each of these canonical pre-images Q_S is a tree. Let $X_1=(\sigma(S)-S')\cup T_1\cup U_1$ and $X_2=(\sigma(S)-S')\cup T_2\cup U_2$ be incomparable: the respective orderings $\gamma(s_1)$ and $\gamma(s_2)$ admit $x\leq|T_1|,|T_2|$ such that $\gamma(s_1)(x)\neq\gamma(s_2)(x)$. Then the orderings $\gamma(s_1)$ and $\gamma(s_2)$ are incomparable. Then for any $T\subseteq S$ such that $T_1,T_2\subseteq T$, if $X=(\sigma(S)-S')\cup T\cup U\in Q_S$, then the ordering $\gamma(s)$ cannot be an extension of both $\gamma(s_1)$ and $\gamma(s_2)$, and hence either $X_1\not\subseteq X$ or $X_2\not\subseteq X$. Repeating for all incomparable pairs in Q_S, we see that Q_S is a tree.

Third, if $S_1,S_2\in\binom{[n+\alpha]}{\alpha}$, where $S_1\neq S_2$, then we claim that for any $X_1\in Q_{S_1}$ and $X_2\in Q_{S_2}$, X_1 is incomparable to X_2. It would suffice to prove this for X_1 maximal in Q_{S_1} and X_2 minimal in Q_{S_2}. Then $X_2=\sigma(S_2)-S_2'$, and for some U of cardinality $\alpha+1$, $X_1=(\sigma(S_1)-S_1')\cup S_1\cup U$. As $|X_1|-|X_2|=2\alpha+1$, $X_1\not\subseteq X_2$. And as $|X_2-X_1|\geq|(\sigma(S_2)-S_2')-(\sigma(S_1)\cup S_1)|=|(\sigma(S_2)-\sigma(S_1))-(S_2'\cup S_1)|>\alpha!\alpha-[(\alpha!\alpha-\alpha)+\alpha]=0$, we have $X_2-X_1\neq\varnothing$, so $X_2\not\subseteq X_1$. ■

6 Excelsior

So there exist finite posets P and Q such that $P\times Q\not\to(P,Q)$. Where does this leave us? There seem to be two ways to go (besides improving on the 17,035 or the $8\alpha!\alpha^3\ln\alpha$).

First, there is a consolation prize, which we can get by twiddling F. Galvin's [6] proof of the finite version of Proposition 1.2.

Proposition 6.1 *Let P and Q be finite posets with bottoms, and let $P\times Q=O\cup G$ be a partition of $P\times Q$. Then either there is a canonical pre-image $R\subseteq O$ of P, or a canonical pre-image $S\subseteq G$ of Q.*

To prove this, we need a little lemma.

Lemma 6.1 *Let P_1, P_2, and Q be finite posets with bottoms. Let P_1, P_2 be upwards closed subposets of a poset P. Let R_1 be a canonical pre-image of P_1 in $P_1\times Q$ and let R_2 be a canonical pre-image of P_2 in $P_2\times Q$. Then $R_1\cup R_2$ is a canonical pre-image of $P_1\cup P_2$ in $(P_1\cup P_2)\times Q$.*

Proving the lemma is just a matter of checking the criteria for canonical pre-images in Definition 2.1.

Proof of Proposition 6.1. The proof is by induction on height(P)+ height(Q). For the basis, if height(P) + height(Q) = 0, then P and Q are both single vertices, and the Proposition is true. In fact, it is true if height(P) = 0 or height(Q) = 0, so for the rest of this proof, assume that height(P), height(Q) > 0.

Here is the inductive step. For a positive integer $m > 0$, suppose that for any posets P' and Q' such that height(P')+ height(Q') $< m$, the Proposition is true for P' and Q'. Suppose that height(P)+ height(Q) $= m$. Let p_0 be the bottom of P and let $p_1, \ldots, p_r$ be the vertices of rank 1 in P, and for each $i \in [r]$, let

$$P_i = \{p \in P\colon p \geq p_i\}.$$

Similarly, let q_0 be the bottom of Q, let $q_1, \ldots, q_s$ be the vertices of rank 1 in Q, and for each $j \in [s]$, let

$$Q_j = \{q \in Q\colon q \geq q_j\}.$$

Notice that each P_i is upwards closed in P, and that each Q_j is upwards closed in Q, thus enabling us to use Lemma 6.1.

By the induction hypothesis,

- for each $i \in [r]$, if $P_i \times Q = O' \cup G'$, then either there is a canonical pre-image $R_i \subseteq O'$ of P_i, or a canonical pre-image $S \subseteq G'$ of Q, and
- for each $j \in [s]$, if $P \times Q_j = O' \cup G'$, then either there is a canonical pre-image $S_j \subseteq G'$ of Q_j, or a canonical pre-image $R \subseteq O'$ of P.

Thus if $P \times Q = O \cup G$, there are three possibilities:

1. For some $i \in [r]$, there is a canonical pre-image S of Q in $G \cap (P_i \times Q)$, and hence in G.
2. For some $j \in [s]$, there is a canonical pre-image R of P in $O \cap (P \times Q_j)$, and hence in O.
3. Both (1) and (2) fail. Hence for each $i \in [r]$, there is a canonical pre-image R_i of P_i in $O \cap (P_i \times Q)$. And for each $j \in [s]$, there is a canonical pre-image S_j of Q_j in $G \cap (P \times Q_j)$. Suppose, without loss of generality, that $(p_0, q_0) \in O$. Then by Lemma 6.1, $\{(p_0, q_0)\} \cup \bigcup_{i \in [r]} R_i$ is a canonical pre-image of P_i in O.

From these three cases, the Proposition follows. ■

There is another direction we can take to get positive results. This direction is towards a new Ramsey function.

Definition 6.1 *For each pair of positive integers* m *and* n, *let* $\mathrm{br}(m,n)$ *be the least integer* r *such that* $B_r \to (B_m, B_n)$.

In this paper we have established that for any integer $\alpha > 0$, $n > 17{,}034$ such that $n > 8\alpha!\alpha^3 \ln \alpha$, $\mathrm{br}(n,2) > n + \alpha$. However, we can prove:

Proposition 6.2 *For all positive integers* m, n, $\mathrm{br}(m,n) \leq mn + m + n$.

In particular, for each n, $\mathrm{br}(n,2) \leq 3n+2$.

Proof. By Proposition 1.1, if $*$ is lexicographic product, then for any m, n, $B_m * B_n \to (B_m, B_n)$. So it suffices to prove that $B_m * B_n$ can be embedded in B_{mn+m+n}.

We define an embedding τ: $B_m * B_n \to B_{mn+m+n}$ as follows. Let $[mn+m+n] = [m] \cup \bigcup_{k=0}^{m} A_k$, where, for each $k \in \{0,1,\ldots,m\}$, $A_k = \{m+kn+1,\ldots,m+(k+1)n\}$. Then for each $C \subseteq [m]$ and $D \subseteq [n]$, let

$$\tau(C,D) = C \cup \left(\bigcup_{k=0}^{|C|-1} A_k\right) \cup \{m + |C|n + d \colon d \in D\}.$$

We claim that τ is an embedding of $B_m * B_n$ into B_{mn+m+n}.

First, if $C \subsetneq C' \subseteq [m]$ and $D, D' \subseteq [n]$, then

$$\begin{aligned}
\tau(C,D) &= C \cup \left(\bigcup_{k=1}^{|C|-1} A_k\right) \cup \{m+|C|n+d \colon d \in D\} \\
&\subseteq C \cup \left(\bigcup_{k=1}^{|C|} A_k\right) \\
&\subsetneq C' \cup \left(\bigcup_{k=1}^{|C'|-1} A_k\right) \cup \{m+|C'|n+d' \colon d' \in D'\} \\
&= \tau(C',D').
\end{aligned}$$

Second, if $D \subsetneq D' \subseteq [n]$, then

$$\begin{aligned}
\tau(C,D) &= C \cup \left(\bigcup_{k=1}^{|C|-1} A_k\right) \cup \{m+|C|n+d \colon d \in D\} \\
&\subsetneq C \cup \left(\bigcup_{k=1}^{|C|-1} A_k\right) \cup \{m+|C|n+d' \colon d' \in D'\} \\
&= \tau(C,D').
\end{aligned}$$

Third, if C and C' are incomparable, then for any $D, D' \subseteq [n]$, $\tau(C, D) \cap [m] = C$ and $\tau(C', D) \cap [m] = C'$, so $\tau(C, D)$ and $\tau(C', D')$ are incomparable. Fourth and finally, if D, D' are incomparable, then similarly as $\tau(C, D) \cap A_{|C|}$ and $\tau(C, D') \cap A_{|C|}$ are incomparable, so are $\tau(C, D)$ and $\tau(C, D')$.

From the four observations, we see that τ is an embedding. ■

Observe a suggestion of sharpness about this result. The height of $B_m * B_n$ is $(m+1)(n+1) = mn + m + n + 1$, which is also the height of B_{mn+m+n}: if $k < mn + m + n$, $B_m * B_n$ cannot be embedded in B_k. So if we want to improve on Proposition 6.2, we will have to try a different proof.

Here are two conjectures about the Ramsey numbers br. In spite of the result of this paper, we conjecture that:

Conjecture 6.1 *For any fixed k,* $\lim_{n\to+\infty} \frac{1}{n}\mathrm{br}(n, k) = 1$.

Conjecture 6.2 *For any fixed $x \in (0, 1)$,* $\lim_{n\to+\infty} \frac{1}{n}\mathrm{br}(xn, (1-x)n) = 1$.

It may even be possible that for any fixed $x \in (0, 1)$, $\mathrm{br}(xn, (1-x)n) = n$ for sufficiently large n.

We conclude with a question arising from Conjecture 1.1:

Question 6.1 *What can be said for minimal R (or maximal P, Q) such that $R \to (P, Q)$?*

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