Bounds on total domination in terms of minimum degree

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Abstract

A set S of vertices of graph G is a *total dominating set*, if every vertex of G is adjacent to some vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. For graphs G with order n and minimum degree δ, we prove that $\gamma_t(G) \leq \frac{1+\ln(2\delta)}{\delta} n$. Furthermore, if δ is sufficiently large then this upper bound cannot be improved to be less than $(1+o(1))\frac{1+\ln(\delta+1)}{\delta+1}n$. As a consequence of our main result, we verify a conjecture of Favaron et al. [4] for all graphs G with minimum at least 8.

Let G be a graph without isolated vertices. A set $U \subseteq V(G)$ is a *dominating set*, if every vertex in $V(G) - U$ is adjacent to a vertex in U. A set $S \subseteq V(G)$ is a *total dominating set*, if every vertex in $V(G)$ is adjacent to a vertex in S. In other words, a total dominating set of G is a dominating set of G that induces a subgraph with no isolated vertices. Every graph without isolated vertices has a total dominating set, since $S = V(G)$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [3], and is now well studied (see [4, 5, 6, 7]).

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The decision problem to determine the total domination number of a graph is known to be NP-complete. Therefore, it is of interest to find good bounds on the total domination number of a graph. Cockayne et al. [3] showed that $\gamma_t(G) \leq 2n/3$ for every connected graph G of order $n \geq 3$. Favaron et al. [4] showed that $\gamma_t(G) \leq 7n/13$ for every graph G of order n and minimum degree at least 3. They further conjectured

Conjecture 1 **([4])** *If G is a graph of order n with minimum degree $\delta(G) \geq 3$, then $\gamma_t(G) \leq n/2$.*

The purpose of this short note is to give a general upper bound on $\gamma_t(G)$ in terms of the order and minimum degree of G, which is asymptotically not far from being optimal. In particular, our result confirms Conjecture 1 for graphs G with minimum degree at least 8.

We prove the following result using a simple probabilistic argument similar to the one used in [1] (see page 6).

Theorem 2 *Let G be graph of order n with minimum degree $\delta > 1$. Then $\gamma_t(G) \leq \frac{1+\ln(2\delta)}{\delta}\, n$.*

Proof. First, for each $v \in V = V(G)$, let us pick an arbitrary neighbor of v in G and denote it by z_v. Let $p = \ln(2\delta)/\delta$. Let us pick, randomly and independently, each vertex of V with probability p. Let X be the (random) set of all vertices picked, and let $Y = Y_X$ denote the set of all vertices in V that do not have any neighbor in X. Let $Z = \{z_y : y \in Y\}$. Note that $|Z| \leq |Y|$, and that X, Y, Z may overlap each other. Clearly, the set $U = X \cup Y \cup Z$ is a total dominating set of G. We show that the expected value of $|U|$, to be denoted by $E(|U|)$, is small.

Let $E(|X|), E(|Y|), E(|Z|)$ denote the expected values of $|X|, |Y|, |Z|$, respectively. Clearly $E(|X|) = np$ and $E(|Z|) \leq E(|Y|)$. We now estimate $E(|Y|)$. Note that $|Y| = \Sigma_{v \in V} \lambda_v$, where $\lambda_v = 1$ if $v \in Y$

and $\lambda_v = 0$ otherwise. For each $v \in V$, the expected value of λ_v is just Prob$(v \in Y)$. Hence, by linearity of expectation, we have $E(|Y|) = \Sigma_{v \in V}$ Prob $(v \in Y)$.

Now, for each fixed $v \in Y$, Prob$(v \in Y)$ = Prob (none of v's neighbors is in X). Since v has at least δ neighbors, each not appearing in X with probability $1-p$, we have $Prob(v \in Y) \leq (1-p)^{\delta}$. Therefore, $E(|Y|) = \Sigma_{v \in V} Prob(v \in Y) \leq n(1-p)^{\delta}$. So, we have

$$\begin{aligned} E(|U|) &\leq E(|X|) + E(|Y|) + E(|Z|) \\ &\leq np + 2n(1-p)^{\delta} \\ &\leq np + 2ne^{-p\delta} \\ &= n(\ln 2\delta)/\delta + n/\delta \qquad \text{(since p} = \ln(2\delta)/\delta) \\ &= \left[\frac{1+\ln(2\delta)}{\delta}\right] n \end{aligned}$$

Consequently, there is at least one choice of $X \subseteq V$ such that the corresponding set $U = X \cup Y \cup Z$ has cardinality at most $\frac{1+\ln(2\delta)}{\delta} n$, yielding a total dominating set of the desired cardinality. ∎

Note that Theorem 2 yields $\gamma_t(G) < n/2$ for a graph G with order n and minimum degree at least 8, which partially verifies Conjecture 1. In general, for large δ, there is not much room for improvement on the linear coefficient of n in Theorem 2 due to the following result of Alon (noting that our upper bound $\frac{1+\ln(2\delta)}{\delta} n$ is less than $\frac{2+\ln \delta}{\delta} n$).

Proposition 3 ([2]) *For large positive integers k, there exist k-regular graphs on $n = k \ln k$ vertices with no dominating set (hence no total dominating set) of size less than $(1+o(1))\frac{1+\ln(k+1)}{k+1} n$.*

Note added in proof

A proof of Conjecture 1 was recently proposed in [8] by Peter Che Bor Lam and Bing Wei.

References

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