Nesting of *already* nested designs

Lucia Gionfriddo
Dipartimento di Matematica e Informatica, Università di Catania
viale A.Doria n.6, 95125 Catania, Italy
lucia.gionfriddo@dmi.unict.it

Abstract. A general definition of *nesting* for a G-design of index λ, without conditions on |V(G)| and |E(G)|, is given in [4]. In this paper we consider the nesting of already nested G-designs and construct all possible nestings of nestings in the case that G is a P_v-design of order a prime n.

1. Introduction

Let G=(V(G),E(G)) be a graph and let λK_n be the complete multigraph on n vertices (every edge is repeated λ times). λK_n is said to be *G-decomposable*, briefly we well write $\lambda K_n \rightarrow G$, if it is union of edge-disjoint subgraphs of K_n, each of them is isomorphic to G. We say, also, that λK_n admits a *G-decomposition* Σ=(V,B), where V is the vertex-set of λK_n and B is the edge-disjoint decomposition of λK_n into copies of G. If $B \in$ B, *B* is called a *block* of Σ. The pair Σ=(V,B) is, also, called a *G-design of order n, block-size* |V(G)| *and index* λ.

Lavoro eseguito nell'ambito del GNSAGA (INDAM) e con contributo del MURST

In [8], Milici and Quattrocchi gave the following definition of *nesting* of a G-design, generalizing the usual nesting for cycle-systems.

Definition 1 Milici-Quattrocchi [8]

Let G=(V(G),E(G)) be a graph and let Σ=(V,B) be a *G-decomposition of λK_n*. A *nesting* of Σ is a triple $N=\{\Sigma,\Pi,F\}$, where Π=(V(K_n),S) is a decomposition of λK_n in m-stars S_m and F: B$\rightarrow$S is a 1-1 mapping such that:

i) for every $B \in$ B, the centre of the m-star $F(B)$ doesn't belong to V(B), all the terminal vertices of $F(B)$ belong to V(B);
ii) for every pair $B_1, B_2 \in$ B, the graphs $B_1 \cup F(B_1)$, $B_2 \cup F(B_2)$ are isomorphic.

It follows, as a necessary condition, that |V(G)| $\geq$ |E(G)|. For |V(G)| = |E(G)|, this definition is equivalent to the usual nesting.

In [4],[5], we gave a new definition a nesting of a G-design in which there are not conditions on |V(G)|, |E(G)|.

Definition 2 [4] [5]

Let G=(V(G),E(G)), H=(V(H),E(H)) be two graphs and let Σ=(V,B) be a *G-design of index λ_1*, briefly $\lambda_1 H \rightarrow G$. A *nesting* N(G,H;$\lambda_1$,$\lambda_2$) of Σ is a triple $N=(\Sigma,\Pi,F)$, where Π=(V(H),S) is an *m-star-design of index λ_2*, briefly $\lambda_2 H \rightarrow S_m$, and F:B$\rightarrow$S is a bijection such that:

i) for every $B \in$ B, the centre of the m-star $F(B)$ doesn't belong to V(B), and V($F(B)$)=V(B) [x vertex of $F(B)$ iff x$\in$ V(B)];
ii) for every pair B_1, $B_2 \in$ B the graphs $B_1 \cup F(B_1)$ and $B_2 \cup F(B_2)$ are isomorphic.

In the case H$\cong K_n$, such a nesting is usually denoted by N= N(G,n; λ_1,λ_2). Observe that N is a G*-*design* of order n, block-size $|V(G)|+1$, index $\lambda=\lambda_1+\lambda_2$, where $G^*=G \cup S_{|V(G)|}$.

If $\lambda_1=\lambda_2=\lambda$, this definition is the same studied in [1],[6],[7]. Consider an example. Let $[x; x_1,x_2, \ldots , x_v]$ be the union-graph between the path P_v of vertices $x_1,x_2,\ldots,x_v$ and edges $\{x_i, x_{i+1}\}$, $i=1,2,\ldots,v-1$, and the star of centre x and terminal vertices $x_1,x_2,\ldots,x_v$, then we can verify that the following design N, defined on Z_5 (the sums are mod 5), having the blocks:

$[j; j+1,j+2,j+3,j+4]$, $[j; j+2,j+4,j+1,j+3]$, for every $j=0,1,2,3,4$

is a nesting $N(P_4,5;3,4)$.

In this paper we consider nesting designs $N(G_i,n;v_1,v_2)$ of Σ_i, where Σ_i is a nesting design $N(G_{i-1},n;\mu_1,\mu_2)$ of Σ_{i-1}, for all admissible i, starting from a *G-design of index λ_1*. We study the case that G is a *path* of order a prime.

In what follows, all the sums will be reduced *mod n*.

2. *k*-nesting and necessary conditions

We give the following definition.

Definition 3:

Let $G=(V(G),E(G))$ be a graph; let $\Sigma=(V,B)$ be a *G-design of index λ_{11} and order n* and let $1\leq k \leq n-|V(G)|$, for an integer n. The 1-*nesting* of Σ is the nesting $N_1=N(G,n;\lambda_{11},\lambda_{12})=(\Sigma,\Pi,F)$, introduced in Definition 2. Let $G_1=G$, $\Pi_1=\Pi$, $F_1=F$. For $k \geq 2$, the k-*nesting* $N_k(G)$ *of the G-design Σ* is a nesting (Σ_k, Π_k, F_k) of $\Sigma_k=$ $=N(G_{k-1},n; \lambda_{k-1}, \lambda_{k-1,2})$.

In what follows, if $B\in$ B and x is the centre of $F(B)$, we will write $(x)\text{-}B=B\cup F(B)$. Now, we see some necessary conditions for the existence of a nested-design $N(G,n;\lambda_1,\lambda_2)$. Some are proved in [4].

Theorem 2.1 [4]: *Let* $G=(V(G),E(G))$ *be a graph and let* $\Sigma=(V,B)$ *be a G-design of index λ_1. A necessary condition for the existence of a $N(G,n;\lambda_1,\lambda_2)$ is that* $\lambda_1 \cdot |V(G)|=\lambda_2 \cdot |E(G)|$

Theorem 2.2: *Let* $G_1=(V(G_1),E(G_1))$ *be a graph and let* $\Sigma_1=(V,B_1)$ *be a* G_1*-design of index* λ_{11}.

i) *a necessary condition for the existence of a* $N(G_1,n;\lambda_{11},\lambda_{12})$ *is that* $\lambda_{11}.|V(G_1)|=\lambda_{12}.|E(G_1)|$;

ii) *a necessary condition for the existence of a* $N(G_k,n;\lambda_{k1},\lambda_{k2})$, *for* $k=1,2,\ldots,n-|V(G_1)|$, *is that for every* $i=1,2,\ldots,k$:

$$\lambda_{i1}.(|V(G_1)|+i-1)=$$
$$=\lambda_{i2}.(|E(G_1)|+(i-1)(|V(G_1)|+(i-2)/2)).$$

Proof: i) It is equivalent to Theorem 2.1. ii) For $i=1$, the statement follows from i) directly. Let $i>1$.
From i):

$$\lambda_{i1}.|V(G_i)|=\lambda_{i2}.|E(G_i)|$$

Since:

$$|V(G_i)|=|V(G_{i-1})|+1$$
$$|E(G_i)|=|E(G_{i-1})|+|V(G_{i-1})|$$

it is:

$$|V(G_i)|=|V(G_1)|+i-1$$
$$|E(G_i)|=|E(G_{i-1})|+|V(G_{i-1})|=$$
$$|E(G_1)|+(i-1)(|V(G_1)|+(i-2)/2).$$

From which:

$$\lambda_{i1}.(|V(G_1)|+i-1)=$$
$$=\lambda_{i2}.(|E(G_1)|+(i-1)(|V(G_1)|+(i-2)/2)).$$

Theorem 2.3 [4]: *Let* $N=(\Sigma,\Pi,F)$ *be a nested-design* $N(P_v,n;\lambda_1,\lambda_2)$. *Necessarily:*

i) $\lambda_1=(v-1).h$, $\lambda_2=v.h$, *for some* $h\in N$;

ii) *if* $n=v+1$, *then either* v *or* h *is an even number.*

3. *k*-nesting of P_v-designs

Now, we prove the following:

Theorem 3.1: *Let v 1 be an integer. Then, for every prime number n, n v, there exists a k-nesting design $N(P_v,n;\lambda_{k1},\lambda_{k2})$, for every k such that* $1 \leq k \leq n-|V(G)|$.

Proof: In what follows, all the sums must be reduced mod n, being n a prime number. Consider the case k=1.

Necessary conditions give: $\lambda_{11}=(v-1)h$, $\lambda_{12}=vh$. It is sufficient to prove the statement for $\lambda_{11}=v-1$, $\lambda_{12}=v$ (all the other cases can be obtained by a repetition of blocks).

Let $\Sigma=(V,B)$ be the P_v-design of order n and index $\lambda_{11}=v-1$, defined on $V=Z_n$ and having the following blocks, for every $i=1,2,..,(n-1)/2$:

$< 0, i, 2i,\ldots, (v-1)i >$
$< 1, 1+i, 1+2i,\ldots, 1+(v-1)i >$
$< 2, 2+i, 2+2i,\ldots, 2+(v-1)i >$
....
....
$< n-1, n-1+i, n-1+2i,\ldots, n-1+(v-1)i >$

where $<x_1,x_2,\ldots,x_v>$ denote the path P_v having vertices $x_1,x_2,..,x_v$ and edges $\{x_1,x_2\},\{x_2,x_3\},\ldots,\{x_{v-1},x_v\}$.
If $G_1=P_v$ and $G_2=(x)-P_v$, we prove that the G_2-design having blocks:

$B_{j,i}^{(1)}$: $(vi+j)-< j, j+i, j+2i,\ldots, j+(v-1)i >$

for every $i=1,2,..,(n-1)/2$,
for every $j=0,1,2,\ldots,n-1$

is a nesting $N_1=(\Sigma,\Pi_1, F_1)$ of Σ. In fact, Σ is a P_v-design in which every pair $\{x,y\}\subseteq V$ is contained in exactly $\lambda_{11}=v-1$ blocks. We can verify that, if $x,y\in V$, $x<y$, and $y-x=h$, then the edge $\{x,y\}$ is exactly contained in the following v-1 blocks:

$<x, x+h, x+2h, \ldots, x+(v-1)h>$
$<x-h, x, x+h, \ldots, x+(k-2)h>$
$<x-2h, x-h, x, x+h, \ldots, x+(v-3)h>$
……………………………………………
$<x-(v-2)h, \ldots\ldots, x-h, x, x+h>$

Further, Π_1 is an S_v-design of index $\lambda_{12}=v$. In fact, if we consider in every block $(x)-<x_0, x_1, \ldots, x_{v-1}>$ of Π_1 the differences $|x-x_u|$, for every $u=0,1,\ldots,v-1$, we can see that:

in all the blocks $B_{j,1}^{(1)}$ these differences are:
$v, v-1, v-2, \ldots\ldots, 1$

in all the blocks $B_{j,2}^{(1)}$ these differences are:
$2v, 2(v-1), 2(v-2), \ldots\ldots, 2$
……………………………………………………………
……………………………………………………………

in all the blocks $B_{j,(n-1)/2}^{(1)}$ these differences are:
$v(n-1)/2, (v-1)(n-1)/2, \ldots\ldots, (n-1)/2.$

If we consider the matrix $M_1[(n-1)/2, v]$ containg these differences, in the same order:

v	$v-1$	$v-2$	………………	1
$2v$	$2v-2$	$2v-4$	………………	2
…	…	…	…	…
…	…	…	…	…
…	…	…	…	…
$v(n-1)/2$	…	…	…	$(n-1)/2$

we can verify that, since n is a prime number, all the possible differences appear exactly one time in every column. This implies $\lambda_{12}=v$.
Let $k=2$. The design N_1 has index $\lambda_{21}=\lambda_{11}+\lambda_{12}=2v-1$. The nesting star-design Π_2, associated with it, should have index $\lambda_{22}=v+1$.
If $G_3=(x)-G_2$, we prove that the G_3-design having blocks:

$$B_{j,i}^{(2)}: \ ((v+1)i+j)-(B_{j,i}^{(1)})$$

for every $i=1,2,..,(n-1)/2$,
for every $j=0,1,2,...,n-1$

is a nesting $N_2=(N_1,\Pi_2,F_2)$ of N_1. We know that N_1 is a G_2-design in which every pair $\{x,y\}\subseteq V$ is contained in exactly $\lambda_{21}=2v-1$ blocks. Further, Π_2 is an S_v-design of index $\lambda_{22}=v+1$. In fact, if we consider in every block (x)-G_2 of Π_2 the differences $|x-x_u|$, for every $u=0,1,\dots,v$, between x and the vertices of the blocks of N_1, we can see that:

in all the blocks $B_{j,1}^{(2)}$ *these differences are:*
$v+1, v, v-1, v-2, \dots\dots, 1$

in all the blocks $B_{j,2}^{(2)}$ *these differences are:*
$2v+2, 2v, 2(v-1), 2(v-2), \dots\dots, 2$

…………………………………………………………

…………………………………………………………

in all the blocks $B_{j,(n-1)/2}^{(2)}$, *these differences are:*
$(v+1)(n-1)/2, v(n-1)/2, (v-1)(n-1)/2, \dots\dots,(n-1)/2.$

If we consider the matrix $M_2[(n-1)/2,v+1]$ containg these differences, in the same order:

$v+1$	v	$v-1$	$v-2$	…………	1
$2v+2$	$2v$	$2v-2$	$2v-4$	…………	2
………	………	………	………	…………	………
………	………	………	………	…………	………
$(v+1)(n-1)/2$	$v(n-1)/2$			…………	$(n-1)/2$

we can verify that this matrix is different from M_1 only for the first column, which is:

$$v+1, 2(v+1), \dots\dots\dots, (v+1)(n-1)/2.$$

Since n is a prime number, this implies that the elements of this column are exactly all the possible differences between (v+1)i+j and the other elements of the P_v-design. This implies $\lambda_{22}=v+1$.

In general, let k be an integer such that $2 \le k \le n-|V(G)|$. The design N_{k-1} has index $\lambda_{k1} = kv+(k^2-3k)/2$. The nesting star-design Π_k, associated with it, should have index $\lambda_{k2} = v+k-1$.
If $G_{k+1}=(x)-G_k$, we prove that the G_{k+1}-design having blocks:

$$B_{j.i}^{(k)}: \quad ((v+k-1)i+j)-(B_{j.i}^{(k-1)})$$

for every $i=1,2,..,(n-1)/2$,
for every $j=0,1,2,\dots,n-1$

is a nesting $N_k=(N_{k-1},\Pi_k, F_k)$ of N_{k-1}. We know that N_{k-1} is a G_k-design of index λ_{k1}.

Further Π_k is an S_v-design of index $\lambda_{k2}=v+k-1$. In fact, if we consider in every block (x)-G_k of Π_k the differences $|x-x_u|$, for every $u=0,1,\dots,v$, between x and the vertices of the blocks of N_{k-1}, we can see that:

in all the blocks $B_{j.1}^{(k)}$ *these differences are:*
$v+k-1, \dots\dots, v+1, v, v-1, v-2, \dots\dots, 1$

in all the blocks $B_{j.2}^{(k)}$ *these differences are:*
$2v+2k-2, \dots\dots, 2v, 2(v-1), 2(v-2), \dots\dots, 2$

……………………………………………………………

……………………………………………………………

in all the blocks $B_{j.(n-1)/2}^{(k)}$ *these differences are:*
$(v+k-1)(n-1)/2, \dots\dots, v(n-1)/2, (v-1)(n-1)/2, \dots\dots, (n-1)/2.$

If we consider the matrix $M_k[(n-1)/2, v+k-1]$ containing these differences, in the same order:

$$\begin{matrix} v+k-1 & \dots\dots & v+1 & v & v-1 & v-2 & \dots\dots & 1 \\ 2v+2k-2 & \dots\dots & 2v+2 & 2v & 2v-2 & 2v-4 & \dots\dots & 2 \\ \dots & & & & & & & \dots \\ \dots & & & & & & & \dots \\ (v+k-1)(n-1)/2 & \dots\dots & (v+1)(n-1)/2 & v(n-1)/2 & & & \dots\dots & (n-1)/2 \end{matrix}$$

we can verify that this matrix is different from M_{k-1} only for the first column, which is:

$$v+k-1,\ 2(v+k-1),\ \ldots\ldots\ldots,\ (v+k-1)(n-1)/2.$$

Since n is a prime number, this implies that the elements of this column are exactly all the possible differences between (v+k-1)i+j and the other elements of the P_v-design. This implies $\lambda_{k2}=v+k-1$.

The proof is completed.

REFERENCES

[1] C.J.Colbourn-M.J.Colbourn, *Nested triple systems*, Ars Combinatoria 16 (1983), 27-34.

[2] C.J.Colbourn-J.H.Dinitz, *The CRC-handbook of Combinatorial Designs*, CRC Press, (1996)

[3] C.J.Colbourn-A.Rosa, *Triple systems*, Oxford Sc.Publ., (1999).

[4] L.Gionfriddo, *New nesting for G-designs*, Congressus Numerantium, 145 (2000), 167-176.

[5] L.Gionfriddo, *On the spectrum of nested G-designs*, Australasian J. of Combinatorics, 24 (2001), 59-80.

[6] C.C.Lindner-C.A.Rodger-D.R.Stinson, *Nesting of cycle systems of even lenght*, J.Comb.Math.Comb.Comp. 8 (1990), 147-157.

[7] S.Kageyama-Y.Miao, *The spectrum of nested designs with block size three or four*, Congressus Numerantium, 114 (1996), 73-80.

[8] S.Milici-G.Quattrocchi, *On nesting of path-designs*, J.Comb. Math. Comb.Comp., to appear.

[9] S.Milici-G.Quattrocchi, *On nesting of G-decomposition of K_v where G has four vertices or less*, to appear.

[10] G.Quattrocchi, *G-design: nesting and embedding problems*, to appear.

[11] D.R.Stinson, *The spectrum of nested Steiner triple systems*, Graphs and Combinatorics, 1 (1985), 189-191.

[12] D.R.Stinson, *On the spectrum of nested 4-cycle-systems*, Utilitas Math., 33 (1988), 47-50.