On Vertex-Magic Labeling of Complete Graphs

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Suppose G is a graph with vertex-set V and edge-set E. If λ is a one-to-one map from $E \cup V$ onto the integers $\{1, 2, \ldots, e+v\}$, define the *weight* of vertex x to be

$$wt(x) = \lambda(x) + \sum \lambda(xy),$$

where the sum is over all vertices y adjacent to x. We say λ is a *vertex-magic total labeling* if there is a constant h so that for every vertex x, $wt(x) = h$. A graph with such a labeling is a *vertex-magic graph.*

Recently Lin and Miller [3] proved that all complete graphs of order divisible by 4 are vertex-magic. (It had been shown in [4] and [5] that all other complete graphs are vertex-magic.) Our purpose here is to present a simpler proof that all complete graphs are vertex-magic.

In our proof we use the existence of magic rectangles. A *magic rectangle* $A = (a_{ij})$ of size $r \times c$ is an $r \times c$ array whose entries are $\{1, 2, \ldots, rc\}$, each appearing once, with all its row sums, and all its column sums, equal. The sum of all entries in the array is $\frac{1}{2}rc(rc+1)$; it follows that

$$\sum_{i=1}^{r} a_{ij} = \tfrac{1}{2}c(rc+1), \text{ all } j,$$

$$\sum_{j=1}^{c} a_{ij} = \tfrac{1}{2}r(rc+1), \text{ all } i,$$

so r and n must either both be even or both be odd. It was shown in [2] that such an array exists whenever r and c have the same parity, except for the impossible cases when one of r or c is 1, or $r = c = 2$. A very simple construction for all magic rectangles is given in [1].

We also use two $n \times n$ matrices, where n is odd. A denotes the matrix formed from the row $1, 2, \ldots, v$ by back-circulating — $a_{i,j} = a_{i-1,j+1}$, with subscripts reduced modulo v when necessary — so that A is symmetric, and its diagonal entries are all different. If S is the sequence $s_0, s_1, s_2, \ldots, s_{\frac{1}{2}(n-1)}$, then $B(S) = (b_{i,j})$ is the matrix formed by circulating the row

$$s_0, s_1, s_2, \ldots, s_{\frac{1}{2}(n-1)}, s_{\frac{1}{2}(n-1)}, \ldots, s_1$$

— in other words, $b_{1,1} = s_0, b_{1,2} = s_1, \ldots, b_{1,n} = s_1$, and $b_{i,j} = b_{i-1,j-1}$, with subscripts taken modulo v when necessary.

Now define a labeling $\lambda_S(n)$ of K_n by $\lambda(x_i) = a_{i,i} + b_{i,i}$, $\lambda(x_i x_j) = a_{i,j} + b_{i,j}$. It is easy to see that under this labeling, every vertex x has the same weight:

$$wt(x) = s_0 + 2(s_1 + s_2 + \ldots + s_{\frac{1}{2}(n-1)}) + 1 + 2 + \ldots + n.$$

In the case where $S = (0, n, 2n, \ldots, \frac{1}{2}n(n-1))$, every label from 1 to $\frac{1}{2}n(n+1)$ will occur exactly once, so we have a vertex-magic total labeling of K_n. So there is a vertex-magic total labeling of K_n whenever n is odd.

If $n \equiv 2(\bmod 4)$, we write $n = 2v$. We find a vertex-magic total labeling of the union of two copies of K_v. Suppose this labeling has magic constant h. Then we select a magic square m of order v. The edge joining vertex x of the first K_v to vertex y of the second K_v receives label $v^2 + v + m_{xy}$. The result is clearly vertex-magic, with magic constant $h + \frac{1}{2}(3v^3 + 2v^2 + v)$.

To label $2K_v$ we distinguish two subcases. If $v = 4m + 1$, consider the two sequences

$$\begin{aligned} S_1 &= 2mv, 0, 2v, \ldots, (2m-2)v, (2m+3)v, (2m+5)v, \ldots, (4m+1)v, \\ S_2 &= (2m+2)v, v, 3v, \ldots, (2m+1)v, (2m+4)v, (2m+6)v, \ldots, 4mv. \end{aligned}$$

$\lambda_{S_1}(v)$ and $\lambda_{S_2}(v)$ can each be used to label K_v. Each has magic constant $(2m+1)(4m+1)^2$ and between them their sets of labels make up all the integers from 1 to $2\binom{v+1}{2}$. If these labelings are applied to two disjoint copies of K_v, they make up a VMTL of $2K_v$ as required.

In the same way, if $v = 4m + 3$, the sequences

$$\begin{aligned} S_1 &= 2m, 0, 2, \ldots, 2m-2, 2m+2, 2m+5, 2m+7, \ldots, 4m+3, \\ S_2 &= 2m+4, 1, 3, \ldots, 2m+3, 2m+6, 2m+8, \ldots, 4m+2 \end{aligned}$$

can be used to label $2K_v$.

If $n \equiv 0 (\bmod 4)$, say $n = 4m$, we treat K_{4m} as $K_{4m-3} \cup K_3$ with edges joining the two vertex-sets. The copy of K_{4m-3} is labeled using $\lambda_S(4m-3)$, where

$$\begin{aligned} S = {} & 4m, 0, 8m-3, 12m-6, 16m-9, \ldots, (8m-3)+(m-3)(4m-3), \\ & 8m+(m+1)(4m-3), 8m+(m+2)(4m-3), \\ & \ldots, 8m+(2m-1)(4m-3), \end{aligned}$$

yielding constant vertex weight $(2m+1)(8m^2-6m-3)$. The vertices of K_3 receive labels $4m-2, 4m-1, 4m$, and the edges receive $8m-2+(m-2)(4m-3), 8m-1+(m-2)(4m-3), 8m+(m-2)(4m-3)$, in such a way as to give each of the three vertices weight $8m^2-2m+9$. Finally, a magic rectangle R of size $3 \times (4m-3)$ is chosen, and the cross-edge joining vertex i of K_3 to vertex j of K_{4m-3} is labeled $8m+(m-2)(4m-3)+r_{ij}$. The magic rectangle has row and column sums $(4m-3)(6m-4)$ and $3(6m-4)$, so the sum on each vertex of K_{4m-3} of the labels on the cross-edges is $3[(6m-4)+8m+(m-2)(4m-3)]$, and for the vertices of K_3 it is $(4m-3)[(6m-4)+8m+(m-2)(4m-3)]$. Therefore the combined labeling gives constant vertex-weight $16m^3+8m^2-3m+3$. Every integer from 1 to $2m(4m+1)$ is used precisely once, so the result is a vertex-magic total labeling.

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Least common multiples of cubes

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Abstract

A graph G is a least common multiple of two graphs H_1 and H_2 if G is a smallest graph, in terms of number of edges, such that there exists a decomposition of G into edge disjoint copies of H_1 and there exists a decomposition of G into edge disjoint copies of H_2. In this paper we construct a least common multiple of the two cubes Q_a and Q_b for any two positive integers a and b.

Graphs in this paper are assumed to be simple and to have no isolated vertices. We denote the vector space Z_2^n over the field Z_2 by V_n. Let e_k denote the vector with kth component 1 and other components 0. The *n-cube* Q_n is the graph with vertex set V_n and edge set $\{\{v, v + e_k\} : v \in$

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$V_n, 1 \leq k \leq n\}$. It is easily seen that Q_n is n-regular, has 2^n vertices, and has $n2^{n-1}$ edges.

A graph H is said to *divide* a graph G if there exists a set of subgraphs of G, each isomorphic to H, whose edge sets partition the edge set of G. Such a set of subgraphs is called an *H-decomposition* of G. A spanning subgraph of a graph G in which each component is isomorphic to a given graph H is called an *H-factor*. Given graphs H_1 and H_2, a *common multiple* of H_1 and H_2 is a nontrivial graph G such that H_1 divides G and H_2 divides G. A graph G is a *least common multiple* of H_1 and H_2 if G is a common multiple of H_1 and H_2 and no other common multiple has fewer edges.

Several authors have investigated the problem of finding least common multiples of pairs of graphs. The problem was introduced by Chartrand et al in [1] where they showed that every pair of nonempty graphs has a least common multiple. The problem has been studied for several pairs of graphs including cycles and stars [1, 8], paths and complete graphs [7] and pairs of cycles [6]. Pairs of graphs having a unique least common multiple were investigated in [3] and least common multiples of digraphs were considered in [2]. In this paper we make use of a result of Edmonds and Fulkerson [4] on independent subsets in matroids to construct a least common multiple of any two cubes. The following is the vector space version of their result.

Theorem 1. *A subset E of a vector space V can be covered by a family of linearly independent subsets $I_i, i = 1, \ldots, k$, of prescribed sizes $n_i \leq \operatorname{rank} E$ if and only if, for every $A \subseteq E$,*

$$|A| \leq \sum_i \min\{n_i, \operatorname{rank} A\}.$$

If W is a subset of V_n, we denote the complete graph with vertices labeled with elements of W by $K(W)$. If W and X are subsets of V_n with $0 \notin X$, we define $G(W, X)$ to be the subgraph of $K(V_n)$ with edge set $\{\{w, w + x\} : w \in W, x \in X\}$. The following two results allow us to make use of the theorem of Edmonds and Fulkerson [4] to construct least common multiples of cubes. The first is the $k = 2$ case of Lemma 1 in [5].

Theorem 2. *If X is a linearly independent subset of V_n with d elements, then $G(V_n, X)$ is a Q_d-factor of $K(V_n)$.*

The second result that we use, the $k = 2$, $j = n$ case of Lemma 3 in [5], is proved using the theorem of Edmonds and Fulkerson [4].

Lemma 1. *Suppose that $d_1, d_2, \ldots, d_t$ are integers with $1 \le d_i \le n$ for all i and $\sum_{i=1}^{t} d_i = 2^n - 1$. Then the nonzero elements of V_n can be partitioned into linearly independent sets $X_1, X_2, \ldots, X_t$ such that $|X_i| = d_i$ for $1 \le i \le t$.*

We are now ready to construct least common multiples of cubes.

Theorem 3. *Let a and b be positive integers with $a \le b$. Then there exists a graph G with $\mathrm{lcm}(a, b)2^{b-1}$ edges that is a least common multiple of Q_a and Q_b.*

Proof. First note that if G has m edges and is a common multiple of Q_a and Q_b, where a and b are positive integers with $a \le b$, then $m \ge \mathrm{lcm}(a, b)2^{b-1}$. To see this, observe that if Q_b is a subgraph of G then G has at least 2^b vertices and that the degree of each vertex of G is divisible by $\mathrm{lcm}(a, b)$. If follows that G has at least $2^b\mathrm{lcm}(a, b)/2 = \mathrm{lcm}(a, b)2^{b-1}$ edges. Hence, a common multiple of Q_a and Q_b with $\mathrm{lcm}(a, b)2^{b-1}$ edges is necessarily a least common multiple. We now construct such a graph for all positive integers a and b.

We can assume $a < b$, for if $a = b$ then we let G be Q_b. Thus, $\mathrm{lcm}(a, b) \le ab \le b(b-1) \le 2^b - 1$. Let $\mathrm{lcm}(a, b) = bt$. By Lemma 1 with $n = b$, $d_1 = d_2 = \ldots = d_t = b$, and the remaining d_is chosen $\le b$ so that $\sum d_i = 2^b - 1$, we can find t pairwise disjoint linearly independent subsets $X_1, X_2, \ldots, X_t$ of V_b with $|X_i| = b$, $1 \le i \le t$.

Let $G = \bigcup_{i=1}^{t} G(V_b, X_i)$. For each i, $G(V_b, X_i)$ is a Q_b-factor of $K(V_b)$ by Theorem 2, so Q_b divides G. Notice that G is regular with degree tb, and so has $2^b tb/2 = \mathrm{lcm}(a, b)2^{b-1}$ edges.

Let $E = \bigcup_{i=1}^{t} X_i$. We will use Theorem 1 to show that E can be partitioned into s linearly independent sets, each with a elements, where $\mathrm{lcm}(a, b) = as$. It suffices to show that if $A \subseteq E$, then

$$|A| \leq \sum_{i=1}^{s} \min\{a, \text{rank } A\}.$$

If rank $A \geq a$, then this says $|A| \leq sa = bt$, which is clear since $|E| = bt$. Thus we can assume rank $A < a$. Let $A_i = A \bigcap X_i$, $i = 1, \ldots, t$. Since X_i is linearly independent, $|A_i| \leq \text{rank } A$, $i = 1, \ldots, t$. Thus

$$|A| = \sum_{i=1}^{t} |A_i| \leq t \cdot \text{rank } A \leq s \cdot \text{rank } A = \sum_{i=1}^{s} \min\{a, \text{rank } A\}.$$

Now let E be partitioned into s linearly independent sets $Y_1, Y_2, \ldots, Y_s$, each with a elements. Then for each i, $G(V_b, Y_i)$ is a Q_a-factor of $K(V_b)$ by Theorem 2. Thus Q_a divides $\bigcup_{i=1}^{s} G(V_b, Y_i)$. But this graph is also G, since both $\bigcup_{i=1}^{t} G(V_b, X_i)$ and $\bigcup_{i=1}^{s} G(V_b, Y_i)$ consist of all edges of $K(V_b)$ of the form $\{v, v + u\}$ where $v \in V_b$ and $u \in E$. □

We note that the sets $G(V_b, X_i)$ and $G(V_b, Y_i)$ used in the proof of Theorem 3 are Q_b-factors and Q_a-factors, respectively. Thus we have proven that the least common multiples of Q_a and Q_b constructed in Theorem 3 can be decomposed into Q_a-factors and into Q_b-factors.

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